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Urysohn Lemma or Lusin-Menchoff theorem?

"Leave to the Moscovitians their inner quarrel, let they lead them among themselves" - a paraphrase from Pushkin [1].

The article concerns the two theorems mentioned in the title which are logically separated, the only one link between is a mathematical p a t t e r n. Original version - in Polish [2] - was published in 1996. Some changes were made during the translation.

I. U r y s o h n \ L e m m a \ a n d \ n o r m a l i t y \ o f \ t o p o l o g i c a l \ s p a c e s

The theorem called the \ U r y s o h n \ L e m m a \ claims that having a closed subset F and an open subset U of a normal space such that $F \subset U$, there exists on that space a continuous function f into the closed interval [0, 1] of reals such that $f(x) = 0$ for x in F and $f(x) = 1$ for x outside of U.

A topological space is called \ n o r m a l \ if

(1) between closed subset F and open subset U always a closed subset K can be put containing F in its interior; $F \subset \text{int } K \subset K \quad U$, in symbols.

Not always normal spaces were described in this way. T i e t z e [3], who systematically discussed \ s e p a r a t i o n \ c o n d i t i o n s, \ beginning from T_0 through T_2 to the normality, thought normality as a condition T_4 allowing to separate each two disjoint closed subsets by disjoint. Although the equivalence of \ T i e t z e' s \ conditions with the condition (1) is a simple logical exercise, the fact of passing to the form (1) was a significant stimulus for further reasonings with normality. This reformulation appeared in \ U r y s o h n' s \ (first) posthume 1925 paper [4] the main purpose of which was the theorem asserting that the cardinality of connected normal spaces (having more than one point) is continuum. The paper was finished in August 1924, As we can read from \ P. \ S. \ A l e x a n d r o v ' s \ comments in "Trudy" [5], the collection of U r y s o h n ' s \ works (pages 177 - 218.), the paper was finished in August 1924, three days before \ U r y s o h n' s \ tragic death. .

Although in the introductory part of the paper \ T i e t z e's \ form of normality was used, but in the preparatory part an auxiliary lemma is proved asserting that the \ T i e t z e' s \ condition T_4 implies the condition (1). Thus, we see that this fact was for \ U r y s o h n \ significant, even a novelty.

The operation given by formula in (1) was in [4] iterated, and \ U r y s o h n \ obtained a well known to topologists chain of inclusions,

/(2) $F_r \subset \ldots\ \text{int}\, K_r \subset K_r \ldots \subset U$,

where r runs over the dyadic fractions $k/2^n$ of 1. We could expect that the well known continuous function will be written. But it was not the case, as the chain (2) of sets was sufficient for the proof of the theorem concerning cardinalities. None the less, the \ U r y s o h n \ f u n c t i o n \

(3) $f(x) = \inf\{ r: x \in K_r\}$.

appeared in the third annex of the paper, page 208 in "Trudy ". However, it remained without applications, treated as an end in itself. There is only a comment that the function gives an answer to a known at those times question of \ M a u r i c e \ F r e c h e t, \ who asked if it is possible to define on general topological spaces real-valued non constant continuous functions. The problem was of great general importance for set topology as a mathematical discipline, however, without any impact onto purely mathematical questions.

However, in the"Additional remarks" at the end of the paper \ U r y s o h n \ wrote that ".., the theorem of paragraph 25 is significant for the problem of metrization, and my aim is in the nearest future to publish a paper, in which I shall show that each normal space with a countable base is homeomorphic to a metric space". Let us note an emotional \ A l e x a n d r o v' s \ comment many years later in "Trudy" (1951) (Comment [6] on page 216). We read: "In fact, in these lemmas - here, formulas (1) and (2) - ... a key to the proof the metrization theorem is contained".

Only in the \ s e c o n d \ posthume \ U r y s o h n 's \ paper [6], where the Lemma was explicitly stated and proved again, the theorem on the metrizability of normal spaces with countable bases was stated and proved in a well known manner with the use of the Urysohn function. This second posthume \ U r y s o h n' s \ paper, as we can read from \ A l e x a n d r o v' s \ comments to the paper in "Trudy", was elaborated almost entirely (excluding the introductory paragraph) by \ A l e x a n d r o v \ from dispersed \ U r y s o h n' s \ notes. Being aware of \ A l e x a n d r o v' s \ further results in set topology, we can include \ P. \ S. \ A l e x a n d r o v \ to the fathers of the Lemma, too.

II. L u z i n - M e n s h o v \ t h e o r e m

In the theory of real functions we can hear and read without quotation to any concrete paper, about \ L u z i n - M e n s h o v \ t h e o r e m. \ The theorem concerns the sets of density points of measurable sets on the real line. It has in its \ a c c u s t o m e d \ version the following form:

(4) If F is a perfect subset of a measurable subset U consisting exclusively of

its points of density, then there exists a perfect set K lying between F and U such that the set F is contained in the set K^* consisting of the density points of K ; in symbols, $F \subset K^* \subset K \subset U$.

Recall, that p is a \ d e n s i t y \ p o i n t \ of a measurable set A if the quota of the measure of set A in the intervals [p - h, p + h] with respect to the longitude 2h of these intervals tends to 1 if h tends to zero. According to \ L e b e s g u' e, \ almost all points of any measurable set are its density points. The operation of passing from a measurable set A to its measure interior A^* has the same formal properties as the operation of the interior in topological sense. The sets U consisting exclusively of their density points stand in analogy to topologically open sets : call them here m e a s u r e o p e n.

Topological open sets are measure open. However, there are measure open sets not open topologically; for instance, the set of irrationals. Henceforth, not every measure closed set, for instance, the set of rationals, is closed topologically, can be put as F in the formula in (4). Thus, Luzin - Menshov theorem does not assert the normality of the measure topology.

Only about fifty years later this measure topology appeared in real analysis again [7]. It is called recently the density topology. We do not claim that the density topology was in the \ L u z i n' s \ and \ M e n s h o v' s \ intentions. The notion of topological space as the s e t \ a n d \ t o p o l o g y \ seems to be far from the interests of both mathematicians.

The theorem was stated, but - as we can suppose - remained without proof. The proof was presented by \ V e r a \ B o g o m o l o v a, \ who established it in her 1924 paper [8]. Motivations of \ B o g o m o l o v a' s \ paper went beyond the statement (4), and the theorem was stated and proved in the paper in a \ s p e c i a l \ form allowing to get a common key to the known constructions of singular everywhere differentiable functions, for instance of that one with densely situated set of intervals of constancy constructed by \ M a z u r k i e w i c z [9]. The problem was suggested by \ L u z i n \ via recent interests to \ D e n j o y' s constructions and via his talks with \ S i e r p i ñ s k i \ in Moscow in the years 1915 - 1918.

B o g o m o l o v a \ wrote that "the theorem was proved by \ N. \ N. \ Luzin \ and \ D. \ E. \ M e n s h o v. I did not know their method, I got later another proof, which I am presenting in this paper". The proof of Luzin-Menshov theorem, given by \ B o g o m o l o v a \ is far from obviousness. The author is indebted to Dr \ I w o n a \ K r z e m i ñ s k a \ who read that paper and delivered him comments of essential value. The proof, according to the manner in \ L u z i n' s school, was effective. In order to allow the mentioned above applications the set K from (4) had in this paper some additional properties. Moreover, it was shown that the measure of the set K can be given in advance. So, the essential result (4) is surely proved. Having (4), an iteration rule was applied which results a chain of inclusions

(5) $F \subset \ldots \subset K^*_r \subset K_r \subset U$,

where r runs over dyadic fractions on [0, 1]. Then, the function

(6) $f(x) = \inf (r: x \in K^*_r\}$

was defined, which occurred to be approximately continuous. Being bounded, its Lebesgue indefinite integral is an everywhere differentiable function. The integral of f occurs to be, depending of the manner of specialization, the mentioned above \ M a z u r k i e w i c z \ function, or nowhere monotone everywhere differentiable function, attributed by \ B o g o m o l o v a \ to \ D e n j o y \. One point set can be put in (5) into the place of F, thus the existence of the functions (6) implies \ c o m p l e t e \ r e g u l a r i t y, \ of the measure topology (a property between regularity and normality).

III. \ C o m m e n t s.

Let us begin with a resume. The patterns of functions written by \ B o g o m o l o v a \ and \ U r y s o h n \ are the same. The Luzin - Menshov theorem serves as a pattern for the notion of normality in \ U r y s o h n' s \ research.

It seems that the Luzin - Menshov theorem became known to mathematicians by oral communication, and was treated as an existing result even before \ B o g o m o l o v a' s \ paper. However, it remained for many years without continuation. The earliest quotations to this theorem, known to the present author, are the papers by \ Z y g m u n t \ Z a h o r s k i \ published about twenty years later. The theorem was quoted there under the names \ L u z i n \ and \ M e n s h o v, without indicating any particular paper. In the first of these papers, published in Tohoku (1941) [9] a proof of the theorem was given on about five pages in print. The paper by B o g o m o l o v a was not quoted. Theorem was applied for smoothing parametizations of of curves having the tangent everywhere. In the same year \ Z a h o r s k i \ published in Matemaiceskij Sbornik a paper [11], where theorem was applied in a construction of everywhere differentiable function with given in advance a G-delta set (of measure zero) where the derivative is infinite, solving a problem raised by \ V. J a r n i k (Tohoku 1933); B o g o m o l o v a \ is quoted as the author of the theorem.

To the problems discussed by \ B o g o m o l o v a \ returned \ K a p l a n \ and \ S l o b o d n i k \ (1977) [12], Luzin - Menshov theorem was expressed in thi paper in its essential form (4). It is shown that this form is sufficient for getting all \ B o g o m o l o v a' s \ results on everywhere differentiable singular functions.

Mazurkiewicz function was one of many other functions of this kind of singularity.

The first was \ K o e p c k e, \ who about 1890 constructed an everywhere differentiable nowhere monotone function (called by \ B o g o m o l o v a as \ D e n j o y \ function). In 1907 \ P o m p e j u \ constructed a strictly increasing function whose everywhere existing derivative vanishes at the dense set of points. Motivation for these constructions came from the theory of integral. The derivatives of these functions are not integrable in the Riemann sense, and they served as examples showing insufficiency of the Riemann integral for restoring the function from its derivative. The constructions were made according to individual methods. \ Z a h o r s k i ' s \ 1941 papers initiated a method of a reparametrization of the domains of functions of bounded variation, by means of a homemomorphisms constructed with the aid of Luzin-Menshov theorem, making these functions everywhere differentiable, what was done by \ Z a h o r s k i \ in his 1950 paper [13] . Thus, for instance, the function of the \ M a z u r k i e w i c z\ type can be obtained by a reparametrization of the domain of the well known Cantor- Lebesgue function.

This \ Z a h o r s k i ' s \ approach was developed into a general procedure in 1970-es in the book by \ B r u c k n e r \ [14] . In this book a contemporary proof of Luzin - Menshov theorem (in the form (4)) is given. No comment on the source of the theorem is given.

Independently of great mathematical value of the results based on \ L u z i n \ - \ M e n s h o v \ theorem, these results as well as the theorem itself are treated as a kind of mathematical art. The theory of real function never pretended to be a dominating one among mathematics.

The fate of the sisterly result, called the Urysohn - Lemma, was quite different. We cannot imagine the set topology, thought by \ T i e t z e \ in his 1923 paper, without non constant continuous real valued functions. Such a theory would have no link to number spaces. However, besides this fundamental reason, the \ p a t t e r n \ to which the Urysohn function underlies is of no less importance. Diversity of situations to which they can be applied, gives to topologists a great deal of tools sewing set topology with geometry. In the posthume \ U r y s o n' s \ paper, prepared for publication by \ A l e x a n d r o v, the first such an application appeared, namely the metrization theorem for normal spaces with countable bases. It is not the aim of this article to describe other famous theorems in this direction. Let however the mappings into the nerves of coverings be mentioned as being a tool for approximation of spaces by polyhedra in the dimension theory.

According to a comment of \ P, \ S. \ A l e x a n d r o v \ to the second posthume \ U r y s o h n' \ s \ paper (in "Trudy" pages 214 - 218), \ U r y s o h n \ presented his final result at the Moscow Mathematical Society in May 1924. The paper by \ B o g o m o l o v a, accepted in Mat. Sbornik in May 1923, was probably just printed at that time. The coincidence of patterns of functions from these two authors must have been obvious for the Moscow community of mathematicians. Thus, it is difficult to explain the absence of quotations of \ B o g o m o l o v a \ in the \ U r y s o h n' s \ posthume paper, the more in the second posthume \ U r y s o h n \ paper, elaborated by \ A l e x a n d r o v. \ Can we accept as a justification that in June 1923 both mathematicians went to Gottingen - see page 11 in the book [15] by \ M, \ B e c v a r o v a \ and \ I . N e t u k a - being probably far from Moscow events?

The silence around the sources of the Urysohn lemma is embarrassing. There is a lack of any quotation to \ U r y s o h n \ in \ T o p o l o g i e \ I \ written in 1935 by \ A l e x a n d r o v \ together with \ H e i n z \ H o p f \, although the Lemma is formulated and applied. Let be noticed the fact that in \ A l e x a n d r o v' s \ and \ U r y s o h n' s \ mathematical CV's the brevity overwhelmed the history of the discovery of this so important result in the set topology. Reading the mentioned posthume papers by \ U r y s o h n \ and \ A l e k s a n d r o v' s comments | one have a feeling that the pattern of normality and the pattern of function appeared there as a kind of novelty.

V. B e y o n d \ m a t h e m a t i c s

The author has no right to discuss the causes of the absence in \ U r y s o h n' s papers of quotations to \ L u z in \ and \ M e n s h o v . \ Perhaps this absence of quotations shows no more than cold relations between both known topologists and their mother center in Moscow and a break of contacts. Also the astonishing silence around this important result in the years of the so called \ L u z i n \ a f f a i r \ can be explained as a consensus with the facts, which it is not worthy to return to.

The years of the story coincide with the decline of the famous \ L u z i t a n i a, \ a group of young mathematicians gathered around \ L u z i n. \ It was caused by ambitions of young people, to whom both "PS-es" belonged, to be independent in choosing the problems. But, let us quote \ P.\ S. \ A l e x a n d r o v' s \ words uttered years later: " The key of \ L u z i n's \ tragic fate was his personality, concentrated on his own, distant from people, in his not easy, also for his disciples, complicated psychology". Add to these words the passion to mathematics and not always moderate behavior of his students.

In the late 1920s these mathematical events were included into the stream of affairs from beyond mathematical background. They began with the general plan of the restoration of Academy of Sciences. \ L u z i n \ was removed to the philosophical branch of the Academy. There must have been political reasons for that decision, but not only these. As we know, even \ D m i t r i j \ E g o r o v \, the father doctor and close \ L u z i n' s \ friend, was far in mathematical quarrels from supporting \ L u z i n \ The \ t r a v l a \ around \ L u z i n \ came to the apogeum in middle 30's, when in the press there appeared many anonymous accusations, among them accusations of weak doctoral dissertations having been made under Luzin's supervision. Although \ L u z i n \ did not share the fate of \ E g o r o v ,\ who was exiled to Kazan where he died, these events moved \ L u z i n \ outside the active mathematical life. We do not know if these accusations had their source in mathematical community. It seems that the mathematical community was involved in that affair without their own will. The events were described with care by \ A. \ P. \ Y u s h k e v i c h \ [16] in a broad political context.

Recently there were published the materials from the session of the Academy commission discussing the "Luzin affair" [17]. The author was astonished that no word was said there about the situation around the Urysohn Lemma, although the events, such as those described here, suggest an evident cause of conflict. Perhaps the cause of the indifference of mathematicians to that question can be explained by the fact that the Luzin-Menshov theorem was not in the center of interest at that time, as well as the Urysohn Lemma, which, being so simple in itself, became famous much later, only when the forming of notions of general topology was finalized. This resembles the situation around many other famous theorems and notions the authorship of which was left to the free choice of the future communities of mathematicians. Still another cause we can see in the situation into which both sides of conflict were involved unwillingly. If this was the case, they had no interest in extending the quarrel beyond the questions initiated by official accusers.

R e f e r e n c e s

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