Jerzy Mioduszewski

GEORG CANTOR - ON DEDEKIND, KRONECKER AND ON HIMSELF

In re mathematica ars proponendi questionem pluris facienda est quam solvendi
- Georg Cantor - Thesis

T h e a u t h o r. If it had been Georg Cantor - the story would have been more or less the same as that presented here.

T h e s e t s. I see sets as formless whirling clouds of sand, particles of which are indiscernable [1]. So, can we do somewhat with them? Is it possible to count their elements? Can we take them one by one? Which of them is the first if the elements are not enumerated? But I want to regard sets as primary to numbers. So, I should restrict in searching them to qualitative tools. In order to compare the sets I should try to exhaust the elements of one of them by elements of the other. To do that I could try to embed one into other. If so, there is a need in some geometrical tools, but these should be also excluded, as depending of a given in advance mathematics. It is not obvious how to halve the set [2] if it is deprived of any form. Must I agree with Dedekind who claims [3] that the sets are always given in a context from which they inherit their forms and even the dynamics?. Kronecker says that it is nothing to do in the absence of formula. I do not agree to any of these restrictions. I regard sets as primary to other mathematical notions.

I have no fear of infinite sets. Moreover, I think that infinity is in the true nature of sets. Bolzano, about whom I have heard recently a lot, accepted infinity of sets, but he feels a fear of paradoxes which can appear when several different aspects of a set are considered in a given reasoning. Like Galileo, he hesitated to accept equality of sets when their elements are in one-to-one correspondence. But I feel force to look for this kind of comparison. However, contrary to the views of people being far from mathematics, in the search of infinity is not so much of poetry. The questions concerning infinity are produced by our thoughts, many of them are like unwanted guests, with a little bit of charm. It is our task to give them form, individuality and dynamics.

Things, named by Kant “the things in themtselves”, do not demand such a care from us. They present a form and beauty on their own. A stone falls along the line, geometry creates circles and the living beings the spirals, without alien will. We can contemplate these events being free from the need of creating mahematics to understand them, the nature puts this all just before our eyes. Although nature has the number in itself, the need of counting is in us. Animals

– our younger brothers - do not count. The Earth is not interested in knowing how many circulars were made in its way around the Sun.

T h e   n u m b e r s. Are the numbers human creations? They were our servants in everyday life. However, they quickly revealed their amazing nature to us. No matter that we regard numbers as creations of our thought, in the highest parts of their world we are in the position of Pygmalion feeling them to be subjected to their own laws, which seem to be independent of us.

The number, not always warmly welcomed, enters geometry and physics, enlarging enormously the natural area of these sciences as well as its own scope of possibilities. During last centuries the geometry has been radically rebuilt and at our time the same concerrns analysis. We do not know what the aim of such extended mathematics is, being only in free connexions with nature.

Is this Pygmalion's future destined also for sets? At the beginning there is no cause for predictions. The sets are tabula rasa. Dedekind says that t h e y a r e u s e d, for instance in that algebra which had been elaborated by him. The question is f r o m w h a t   t h e y a r e? They were born in the world of our thoughts. Is this world fully subordinated to us?

M o s e s   M e n d e l s s o h n . Dedekind says that there is no need to evoke to space or time in searching the origins of numbers. His views in this respect are close to those of Moses Mendelssohn [4], who being in opposition to Kant, searched with great care the mathematical aspects of philosophy. Mendelssohn was doubtful concerning the mathematical character of geometry. Geometry considers things "in concreto", he wrote. The geometric figures just a r e, meanwhile the true mathematics considers things "in abstracto", having in view several exemplifications of a given mental situation. The mumbers are alien to figures. The numbers, being abstract, not only a r e,   they also u s e us, according to Dedekind's golden phrase. Mendelsohn used to name arithmetics "the other science".

S e t s   b e f o r e   n u m b e r s. Dedekind maintains that sets can be used to explaining the emergence of the idea of number in our thoughts regarding the sets and operations on them as primary. According to the saying by Kronecker, numbers are God's creations, however Dedekind goes much more deeper into the matter. As a key to further reasonings he took into consideration the operation on sets called the one-to-one correspondence, situated at the same level of primarity as sets themselves. This operation   corresponds to a mental act of associating to the elements of a set uniquely determined elements of another given set. In the case of one-to-one correspondence no element is asigned to two or more distinct elements. Dedekind distinguishes finite sets as those which admit no one-to-one correspondence with their proper subsets. The notion of the number does not appear in that definition!

From our trips along the Alps I remember his further speculations. He took into consideration the set called by him "the world S of our thoughts". This world is like a stream. For each thought there is the thought about that thought, so he concluded that the stream is not finite. Choosing a minimal substream initiated at our consciousness 1, he get an inductive system of natural numbers 1, 2, ...!

I do not oppose this beautiful story looking as if taken directly from Schopenhauer. Dedekind's search for the sources of arithmetics reminds me the corresponding search of Helmholtz on physics, as well as Riemann's on geometry. Contemporary mathematics is embedded in the stream of ideas going from philosophers with plenty of gold sentences. Many of them are scholastic in character like that by Weierstrass which claims that the least upper bound of a continuous function defined on a closed segment is a value of this function at a point. The scholastic phrases overcome contemporary analysis.

T h e l a d d e r o f n u m b e r s. When I counted in my childhood dreams the numbers 1, 2, ... till infinity, I restated that reaching the heaven I can count further 1, 2, ... Although the ladder to the heaven is infinite, only the crossing the barrier makes a problem, the further trip along the heaven is similar to that along the earth. This childish speculations about t r a n s f i n i t y are near to those which one can hear from theologians. Nonetheless, I had similar feelings too in my mathematical reasonings with trigonometrical series, deleting step by step from line sets their parts consisting of isolated points. There are sets that after an infinite number of steps remain not exhausted, the remainder is plenty of points and the procedure may be performed again. I heard that Du Bois Reymond noticed the same phenomenon comparing degrees of growths of functions, and this led him to an astonished conclusion that no sequence of criteria of convergence of series can be universal.

C o n t i n u u m. It is the second pilar on which mathematics is built. However, contrary to numbers its mathematical character is not entirely clear not only to me, but also to Dedekind, although we gave them an arithmetical description. It was Aristotle who denied the possibilty to regard continuum as a set, although it was used later by Newton as a playground for Eudoxos's proportions ordered according to their geometrical magnitudes. Nevertheless, most of mathematicians, also in our days, regard continuum rather as a geometrical or even a physical object. Some properties postulated by Gauss are sufficient to use them whith full rigour in mathematical reasonings. According to Gauss there is no need regarding them as an object which should be constructed using much more primary tools.

Having in view these geometrical and physical motivations, we should think that the continuum was rather explained by us, not constructed. For instance, we explain how to fill the gaps, and these were intuitively known since Newton, or even Eudoxos. We did that, and the matter used for this purpose seems to be irrelevant.

Fig. 1. Continuum

Although Dedekind agrees often with this modest view, but he says that we can regard our constructions as completely free from geometry and physics. Firstly, we can regard fractions as creations of our thoughts, viewing them as ordered pairs of natural numbers, abstracted from common factors. Later, we operate just with logical abstracts. In my setting, these logical abstracts are are classes of sequences of fractions. Completing the set of fractions by these abstracts, and extending the ordering, we get an ordered set without gaps, thus an object ordered as physical *rigid body*. No matter how complex this contruction is, only numbers, sets and logic are used for that purpose.

While talking with people I refer many times to the above argumentation. Now, I feel some faults in it. The fractions m/n and p/q are regarded as equal if mq = np, and here a physical motivation linked with weighing bodies is hidden. In the pure arithmetical motivations we should rather pay attention to divisibility [5]. Moreover, the distance |m/n - p/q| means geometrical, even physical, nearness. Thus we were not free of physical and geometrical ideas in our constructions, at least in motivations. Thus, the continuum does not belong to the scope of pure sets.

Pure sets. I used to say that in order to get a set, which I call *pure*, I should, having a concrete set, abstract firstly of the nature of its elements, and secondly of its structure, for instance of its ordering.

However, some difficulties are hidden in this assumption. We stand before the ghosts of the dead sets! Their elements are nothing more than the shadows of the preceding ones. Do they remember their previous state? But, let us leave these troubles, since the same troubles concern also the pure numbers which are the ghosts of apples and pears. Even the greatest philosophers do not oppose this uncertain situation.

Let us be allowed to treat the elements of pure sets like w h i t e b i l l s, which are indiscernable each to other. This would contradict Leibniz's views in the case of more than one bill, but the white bills really exist. Moreover, there are many collections the elements of which are indiscernable at the first moment, but they are discerned in the run of the reasonings. So, let us ignore Leibniz's doubts. Our mind is able to think even about a d o t which breaks out into two dots indiscernable each of the other and of the mother dot. Continuing this process we get a set which is an example of pure creation of the mind, the nature of whose elements is irrelevant for further purposes.

Look at the numbers. They, being free from any designations, make us no obstacles in counting them. We have no problem h o w to produce them, and h o w to divide. Note, however, that we get from mathematics no advice w h e n we should produce and w h e n to divide. This is a general problem of applicability, and we cannot omit that problem searching pure sets.

The further abstraction, namely the abstracion from the s t r uc t u r e, is more essential. If a figure has a shape, some places of it can be treated as points, or symbols representing the situation of that place on the figure. Recall Euclid's observation that "the line is in the same position with respect to its points". Was this observation the cause for which Euclid had no motivation regarding the line as the set of points?

Nonetheless, I do not reject the idea of p u r e sets, even if they were deprived any structure. The sets of white bills are objects of our thoughts, so we must make an attempt to search them.

(1872) M e e t i n g a t I n t e r l a k e n. We met by accident: "Professor Dedekind, I pressume? - this was more or less so. It was a meeting of two theories of irrationals. Dedekind is truly proud of his theory, which corresponds exactly to expectations of Ancients. It allows in a short and rigorous way to demonstrate that square root of 2 times square root of 3 equals square root of 6, he said. He says that that idea came to his head during his lectures at the Polytechnics at Zurich. He remembered that this happened on November 25, 1858. I knew this date from his booklet [6], and I was wondering why he did not add that this happened at 9 o clock in the morning. He said that he regarded his construction only as an interesting exercise and that he neglected to prepare it for the publication until last year, when Kossak's theory of real numbers was published. That one is not valued by him highly, as being deprived of mathematical spirit and beauty. It is not a secret for me that Dedekind dislikes Berliners, and that the theory is in fact Weierstrass'. My theory was accepted friendly by him. It is the theory which should be done by Cauchy, he said, but Cauchy felt a fear of higher levels of abstracion and bumped himself into a vicious circle in his definition of real number. Our continua are equivalent as ordered sets due the property of c o n t i n u i t y of the orderings, he said, and this is the meaning of the word continuity which appeared in the title of his booklet. Due to this property the constructed object might be named t h e l i n e. The significance of continuity of the line was also noted by Bolzano in his rigorous proof of the "Zwischenwertsatz".

It is hard to say that I was talked to Dedekind. It was Dedekind who was speaking. He treated me as a beginner. Please write me about your concepts - he said as a farewell.

Dedekind is one of our g r e a t f o u r. The meeting with him interrupted my loneliness which I felt in Halle when my search on trigonometrical series was completed. I spent some student years in Zurich, but these were years when Dedekind came back to Brunswick, to take the chair of mathematics which had been offered to him at Polytechnicum, into which the old Carolinum had been transformed.

(1874) A m o d e s t b e g i n n i n g. There is no wonder that Dedekind was not astonished at my remark in the style of Gallileo that the fractions can be viewed as a sequence. This might be done in several ways.

He concerned more seriously my proof that cannot be done with the set of points of the line. As a response he sent me his own proof, in fact the same as mine. The continuity of the ordering plays the crucial role in our proofs. The argumentation goes parallelly to the one I used in my proof concerning the coefficients of trigonometrical series.
Sending his own proof, Dedekind deprived me of feeling satisfaction coming from discovery. Is this a flaw of his character? Can I suppose that he inherited it from Gauss, who was not able to be openhanded toward young Bolyai? Had he been aware of this question and had he the proof before me? I can not exclud this, hearing about his known everhasting "Treppenverstand", being admitted to him with a smile.

I agree that his proof is somewhat simpler. Should I express to him my thanks for such a small detail? It would be embarassing to both of us. Besides, the proof is in fact the same as mine.

P r o o f. Take points a_1, a_2, ... on the line. Take an interval I_1 omitting a_1, and an interval I_2 omitting a_2 and lying with its ends in I_1. Continue. The common point of all these intervals, whose existence is assured by the continuity, is distinct from each of a_n.

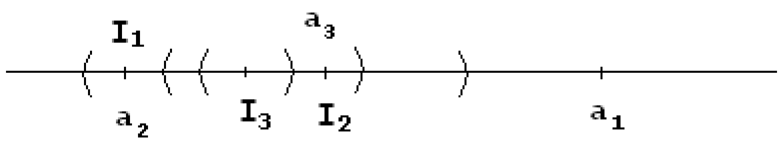

Fig. 2. The proof.

I should have noted that in this reasoning I had to deal with mathematically concrete sets, with argumentations being motivated geometrically, even physically. Would I find appropriate tools if I were standing face to face with sets without any structure?

Dedekind mantains that my result is worth publishing together with a comment that the set of algebraic numbers is countable, the proof of which is analogous to my proof of countability of fractions. Both these claims acknowledge the existence of transcedent numbers .

Did I discover what had been done by Liouville? Without indicating any transcedent number? This is just vicious circle, as I proved the existence of numbers which were created by myself! I feel fear and I hear my heart throbbing.

T h e c r y o f B e o t i a n s. Why was Gauss so anxious in front of the crowd of Beotians? Mathematicians form a community like an order. The laws are not expressed in writing, however the professional canon has been known from the antiquity. No apostasy is anticipated, and this Pytagorean principle has been in force till our days. The matter from which the results have emerged must be solid. This was observed by Gauss. The matter in which he carved his works were numbers, inaccessible to prophans, for whom was nothing else to do but to kneel before him. His other love was geometry, close to geodesy and physics. He had the first and the last word there. He took care of things of the highest importance and this ensure him the name of princeps mathematicorum. Many results remained only in meager notes. He enjoyed knots, but the elaboration of the theory he left for Listing. He would regard as a waste of time to write mononographs like those of Cauchy, with a careful codification of notions like derivative, integral and continuity. However, the other side of his highest esteem was loneliness.

Being the discoverer of theorema egregium, he was far as no one else in these times from the acceptation of Euclid’s postulat on parallels. But he hesitated to discuss that problem with Beotians, who knew that question only in simplified vague forms. Perhaps, he had great Newton in mind whose big idea of reconstructing the changing quantity from its intensivity of change was reduced by prophans to dy|dx, depriving mathematics of beauty for centuries.

The Beotians – how many of them in my surrounding – accepted my sets in their simplest form. Later, being owners of my idea, they became my correctors. Did I make a mistake by presenting my ideas in statu nascendi?

K r o n e c k e r a n d t h e B e r l i n e r s. Although my paper was admitted in „Crelle", I am not free from feeling of anxiety. There are no doubts that I am observed in Berlin. Dedekind, no matter that he holds himself in Brunswick, is one of the Berliners. I heard that the worst opinions on my interest in sets are demonstrated by Kronecker. I felt them extremally bitter, as I was very close to him in Berlin, and he had always been friendly towards me there. He is a fanatic of arithmetization, but in a classical fashion. According to him the numbers are primary to other mathematical notions. Explaining them by other terms would be a desacralization. However, he treated the Kossak's construction indifferently. He swallowed this bitter pill being reassured by Weierstrass that he had only in view the purposes of rigorization. Also Dedekind holds himself at a distance from regarding our arithmetical construction as a true final step to arithmetization of mathematics, being convicted that the arithmetical methods are irrelevant to the most essential parts of geometry. Arithmetization is no more than a tension of our thoughts, forced to count and order everything. Meanwhile, analysis is scholastical from its very beginnings, and as a basis of its mathematical status the physical continuum suffices, although we know that only our aritmetization turned the thruts postulaled by Newton into theorems, among them the most important: $f' = 0$ implies $f = \text{const}$.

All mathematicians in Berlin belong to the line of goettingenian algebraists the tradition of which goes to Gauss and Dirichlet. Although I hear no critical opinions towards me, but their indifference is equally depressive for me. They treat my mathematics as marginal and admit it with tollerance as harmless "Spitzfindikkeiten".

The algebraic theory of numbers, initiated by Gauss and known to me from the Dedekind's "Supplement 11", rose in works of Kronecker and Kummer to the enormous level of abstraction. The notion of divisibility of numbers and the property of being prime must be redefined. In order to save the uniqueness of decomposition into primes the ideal elements were introduced. I am overpowered by this great mathematics.

I am alone with my pure sets. I left my research in trigonometrical series. It was a surprise to my colleagues that the convergence of the series on an interval, which can be as small as we want it to be, suffices to infer convergence to zero the coefficients. This allow me to enter into the core of Riemann's and Adamandus Schwarz's reasonings and ultimately obtaining the theorem of the uniqueness of trigonometrical development. I should add that I was warmly encouraged in my work by Kronecker. Now, as a prodigal son I am feeling his piercing sight over me.

In the review in "Jahrbuch uber die Forschritte der Mathematik" Netto reduced my proof to two short sentences.

Being in Berlin I asked Dedekind what he mantained about the eventual equipollence between the line and the plane. He said that some time ago he had considered that question but he left it. I asked other people but they seemed indifferent to the question.

(1877) T h e p l a n e a n d t h e l i n e. I got a quick response from Dedekind on my proof that the set of points of the plane and on the line are in one-to-one correspondence. He regarded the result as interesting, but his inquiry into the details was unbearable. Reshuffling the decimal representations of the coordinates x and y of the ponit p on the plane I get the point f(p) o the line. The transformation f is obviously one-to-one. But Dedekind says that not all points can be the values of f, indicating points on the line those which have alternatively cipher 0 in their decimal representations, reminding me that I excluded in advance the representations ending on 0-es. I was irritated by my oversight [7].

For a while, I might be convinced that I had proved even more than equipollence of line and the plane, because I established the equipolence between the plane and a part of the line! From this fact the equipollence between the plane and the line should follow a direct way, having in view that the line is a part of the plane But there is no evidence concerning such a rule.

The same evening I sent to Dedekind the proof based on reshuffling representations by continuous fractions, where the uniqueness of representations in the area of irrational numbers is assured without exclusion. I got the equipollence between the set of irrationals and the set of pairs of irrationals. This suffices for obtaining the previous result, as the set of irrationals is equipollent with the whole set of real numbers, as they differ only by countable set of rationals.

However, this statement was somewhat embarassing. I got many proofs, but I was not satisfied with them as they seemed to me far from the mathematical perfectness. Finally I reduced them finally to a lemma asserting the equipollence between the segment and the segment deprived of a single point. Clearly, it suffices to consider the segment (0, 1] with the end 1 removed. Dedekind approved my considerations being glad with the proof of the lemma, which after some changes looks as follows [8]:

Take on the segment (0, 1] the set S of dyadic fractions 1, ½, ¼, ... Let f(x) = x/2. The transformation which equals f on S, and the identity otherwise, serves as the desired equipollence.

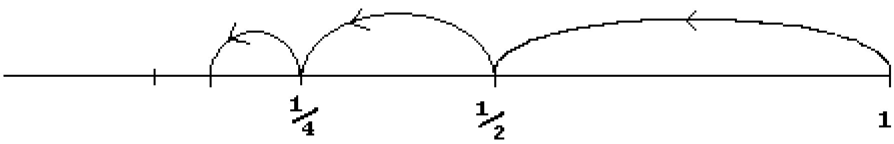

Fig. 3.

Although I was not completely happy with my reasonings, I expected from Dedekind much more attention and friendship. Instead, I heard some cool advices, even warnings, especially against linking my result with the problem of d i m e n s i o n to which the one-to-one correspondence is irrelevant if it is not continuous. I agree with this comment and it was always clear for me that the correspondence based on reshuffling of ciphers, no matter what the kind of development is, must be far from continuity. Nonetheless, there is a bit of cold distance in this comment. But looking once more at the letter, I can see many polite words and expressions, which may hardly be called warnings, they are rather "advices".

Dedekind is extremally just in his opinions, discerning the mathematical value of the result from any kind of empathy. He give me advice to be far from philosophy. He regarded the philosophy of mathematics as his own field. What else than philosophy is presented in his "Was sind und was sollen die Zahlen? ", the book which hsd been growing up in his mind for years accordinng to the rules known only to him? "Please, leave the philosophy to Greeks" – he wrote in one of the letters.

A t h e o r e m w h i c h s h o u l d b e p r o v e n. I am not satisfied with my proofs which seem to me accidental. They concern sets which are are mathematically ready. Can I apply my reasonings to sets which I call pure?

I came back to my first attempt of the proof of equipollence between the plane and the line. I showed that the plane can be viewed as equipollent to a subset of the line. On the other hand, the line is contained in the plane. It schould be true, in general, that if a set B is contained in the set A, and the set A is in one-to-one correspondence with a subset of B, then there exists a one-to-one correspondence between A and B. This seems a general law of the theory of sets, in fact, of the theory of pure sets. But the lrules, as well as hypotheses, are not provided in the theory of pure sets. I can see that statement as a theorem of my theory, the first fairly formulated theorem which concerns pure sets.

Meanwhile, I am overjoyed by Dedekind's approval of the reasoning with equipollence between the segment and the segment deprived of its end.

I had been waiting for almost three months for the letter from Crelle. I asked Dedekind what the reason for the delay would be. His reply was somewhat vague. Perhaps the delay was caused by the difficulty with the evaluation of the result. Then, I asked about the possibility of publication the result as a separate dissertation, but he advised patience.

The paper was finally published, I felt that there were some disturbances though. Weierstrass accepted, but can I be sure that he did so without hesitation?

I n t h e D e d e k i n d' s s h a d o w. Dedekind's letters became unpleasant. Formerly, he warned me against abusing the meaning of dimension. It was a misunderstanding. It had always been clear for me that in the problem of dimension the continuity of the one-to-one correspondence was essential. When I sent him my proof of the impossibility of continuous one-to-one correspondence between spaces which differ in dimension his response was typical of a profesional teacher who with a sharp-eyed delight tracks even the smallest mistakes of the pupil, also these which are without any importance. I lost any intention to exchange my ideas with him. It would be funny to ex[plain his attitude towads me in terms of rivalry. I should rather admit that he was always unpleasant towards my interests in infinity. He regarded them as an unimportant playground. On the other hand, I see that these cool relations may have had a source in his character, and may have been independent of me personally as well as of the subject matter.

Now, I have an intention to go further toward the actual infinity, and I lost the will to colaborate with him. I think that this would be devastating for me. Even in the case of his tolerance, I would feel myself as forced into dependence of him. Critical Kronecker's opinions, which reach me, are not as depressive as Dedekind's cold polite letters. I must be free from his shadow. I must find my own way in mathematics.

I was truly tormented during the exchange of the latest letters with him concerning the final version of my proof about the non-existence of continuous one-to-one correspondences between E^m and E^n if m and n differ. The proof runs by induction with respect to m and n. Dedekind clings to a small gap in the description of a map of a secondary importance, which is undefined in a finite, perhaps countable, number of points, not wanting to see that the gap can be easily filled up.

I know that he has in mind his own proof, and that he exchanges letters with Thomae and Netto on the subject, and that he has a hope in Netto's good idea.

That is why I sent my proof to "Goettinger Nachrichten" without waiting for Dedekind's approval [9].

G e t t i n g o l d w i s e m a n. He is getting old, although being only in his late forties. One can say "das ewige Misantroph" of him. The native Brunswick and the father's villa are enough for his needs. He rejected the invitation to Halle, as only Berlin might be corresponding to his ambitions. But now also Berlin would be unpleasant in view of animosities which had been arosed with years. He felt wiser than all the Berliners, however he got this position during years step by step, never being the first. He was in Goettingen in the shadows of great descendants of Gauss, firstly of Dirichlet and then of Riemann, and now he is drowned out by the sound around Kronecker and celebrations around Weierstrass, who is changed into a kind of a mathematical idol. We are used to reading about tragic fate of young geniuses, but the fate an learned men getting old is also tragic. Brunswick is our Beotia. Nothing important came out from there. It is at equal distance from Berlin and Goettingen, but also from Hamburg and Halle. Would this town be famous for the fact that it was the life place of Dedekind? No! Because Gauss was born there and the monument will be devoted to him! Perhaps also Gauss felt himself a Beotian because of his Brunswick's difficult years in childhood. Would the Beotia be a real prophecy dismissing of life? However, in the nearest distance from Brunswick the smooth Harz mountains are situated, and the charming Harzburg. But, the poetic Weimar, the residence of Goethe, cannot be omitted. As a bitter truth we know that he never invited Gauss to his home, disregarding Gauss's great fame. Gauss often visted Weimar buying there glasses for his instrumentaria [10].

G r e y m a t t e r o f s e t s. The aim of my „Mannigfaltigkeitslehre" is to evoke my childish dream of extending the notion of the number beyond the scope of natural numbers. But before doing that I should take many attempts toward the search for sets lying on the line, being of the interest in itself as well as the matter for further reasonings. I do that in order to make my theory selfcontained. I wade through ordinary properties, accessible even to prophans, of describing the positions of points in sets. The points might be isolated. Otherwise, they accummulate to other points of the set, to which these points may belong or not. I have been familiar with these notions since my work with trigonometrical series. But now, I must list systematically all the details. The idea of the transfinity which overhelmed me is far from this wearing job.

There are however exceptions among these trifles. These are sets which are obtained from a segment after deleting open subsegments so that no full subsegment remains. No matter, how small such a set is with respect to the longitude, it remains equipollent with the full segment. I found a nice arithmetical formula describing one from these sets [11]. However, it became known to me that these sets are familiar to people working in the theory of the integral.

Are all the sets on the line, excluding finite sets and sequences, equipollent with full line?

Searching the subsets of the line I feel no essential resistance in overcoming the difficulties Moreover, I do not feel myself fully as a mathematician. I feel here a grey dead matter. What a difference with the living matter of trigonometrical series! These are embeddeed in the realm of arithmetics, supported by the rythmus getting step by step from the numbers. For Kronecker, to whom I am indebted so much, the numbers are the living heart of mathematics. I slop before him my head. I understand that being an oldfashioned mathematician he must be against the theory of sets. He is open in his criticism and he does no hide this critisism also in talks with me. I compare him to Cavalliere de Mere, famous from his animosy to Blaise Pacal. He is the oldest from our great four.

I sent again a letter to Dedeknd about my idea of transfinite numbers. I got no answer.

T o w a r d t h e f r e e d o m o f m a t h e m a t i c s. Dedekind noticed the obvious property of natural numbers, that in each set of numbers there is the least one. I observed that my sequence of symbols

1, 2, ... , + 1, + 2, ...

also enjoys this property. Thus, I see no obstacles to call my symbols the n u m b e r s. To what purposes will they be u s e d? I have some idea for that. But meanwhile they just a r e! They belong to mathematics as all other things which appear in our thoughts and are free of contradictions. Mathematics forecasts no barriers in its development. The essence of mathematics is in its f r e e d o m and I am not obliged to give explanations concerning the aplicability of my new system of numbers. Now, writing my “Memoire”, I devote some pages to manifest my convictions toward f r e e m a t h e m a t i c s [12].

How will I be welcomed by my mathematical colleagues in the role of philosopher or even a prophet? Such people are disliked and even excommunicated by other mathematicians. Perhaps, it would be better to publish dry results firstly, to wait and see the opinions. Behind any manifestation the people will detect the lack of full conviction for the value of the work which is done. On the other hand, I feel an inner necessity to demonstrate my beliefs which have been so long suppressed by myself.

However, even in this free mathematics, I do not feel as much freedom as I expected before. Making a step up the ladder of my numbers, I restate that it was foreseen by mathematics. When I try to choose among possible solutions, I met so many necessary conditions, that the result is uniquely determined. Where is that promissed satisfaction which should accompany the free creation? We are able discover only the everexisting forms!

But at that moment I have only a vague insight into them. The extended sequence of numbers is in some aspects subjected to the rules known from the set of integers, for instance with respect to its ordering. However, the addition of a new element has no impact on increasing the quantitative size of the collection. The sets I obtained in the first steps of a rather long procedure, are always equipollent with the set of integers. But here a much more primary question arises: can I regard these collections as well formed entities and calling them s e t s? If so, it depends on my decision! So, which one is the value of my pure creation for the learned community?

I am not free of these irresolutions also at my home. In my notes, which are open for my guests, no sign of integral or summation of series appears. In order to keep the balance I wrote a paper concerning the algebraic integers. I had a hope that it would be commented by Dedekind, but it was not the case. I know that Gertrude and Elsa are aware of my uneasy silences fall between us and at some moments my cry of irritation without any apparent cause.

My illness – I must use this word – consists of seeing all in grey colours. The sets deprived of mathematical contexts are deprived also of energy for stimulating thoughts, what I remember from the time of working on trigonometrical series and from my early papers when I manipulated with numbers. The main source of depression I feel with sets on manifolds. They are unshaped, disordered and being almost deprived of individual features. Only a sound of sudden iritation make me free of of these dark thoughts. Also a full of irritation review I wrote on Frege’s book. I wrote that that the notion of the number cannot be reduced to the common property of sets being equipollent each to other. It is worthless or only a temporary substitute. The numbers should be defined before, and the only possible way to get them is to distinguish them among ordinal numbers.

I try to stop my emotions. I expected that an exchange of letters with His Eminenz will be sufficient for this purpose, but I got disappointed. The theologians expect to find a road to the heaven in my numbers. They are living in a parallel world.

T h e w e l l - o r d e r e d s c a l e. My numbers form hardly a collection. According to my initial declaration, a collection of elements can be accepted as a set if there is an idea which describes it as well-formed entity, which allows to decide if a given thing is a member of the collection or not. Although this vague condition is sufficiently clear for collections appearing in mathematical contexts, but in the case of my pure sets causes a serious problem. Collections of my numbers cannot be regarded in any way as a b s t r a c t e d from sets which are r e a d y in mathematics, in particular as those white bills which I am ready to tolerate. They are introduced inductively, accepting each new set as the set of elements of previously accepted sets. Is this iterative procedure in consensus with my previous declarations for forming sets? The situation will be somewhat clearer when I take into consideration the set of all of those my symbols for which the sets of their predecessors are countable. Such a collection is well motivated, so I decide to regard it as set. I call this set t h e II c l a s s of transfinite numbers [13]. I proved that this set is not countable.

T h e p r o o f. Suppose that its elements can be arranged into a seqence a_1, a_2, ... Let A_n be the set of predecessors of a_n. These sets are well ordered thus are contained in each other as initial segments, so their union, is countable, as the sets A_n are countable. This set, being well ordered and countable, is one of A_n for some n. Thus, the set of numbers of the II class would be similar in its well ordering to one of its initial proper segments. A contradiction I denote by the set of numbers of the I class. The same symbol I use for denoting the number in my ladder of symbols, the first after all the numbers of the II class.

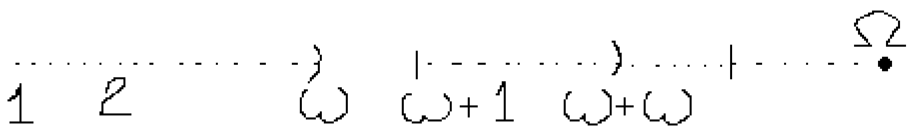

Fig. 4. The number

Being uncountable, the set of numbers of the II class is not an element of itself.

Is the set of numbers of the II class equipollent with the set of points of continuum? This question arises naturally, but on the other hand, it seems to be exotic as these sets belong to so different mathematical contexts!

Continuum once again. I come back to my first talk with Dedekind concerning the physical character of continuum. After this talk I expressed my doubts about the possibility to include continuum into the system of notions based only on numbers and sets. Now, the question appears if it is possible to exhaust the continuum iteratively point by point, making it well ordered. No matter how well performed, this ordering will be in no connexion with the natural, dense in itself, geometrical ordering .

I regard my arithmetical continuum a test of my theory. To be true, continuum does not belong to my theory formally. It is an outsider and foreign for the theory of pure sets, in particular for the idea of transfinity. It enter Platonian mathemathics through the kitchen door from the praxis with measuring of fields in the valley of Nile. Nevertheless, its existence in mathematics became a fact. Moreover, there are no obstacles in viewing it as a set! The last was established, contrary to views of Aristotle, by Dedekind and myself!

The old aporia of Zeno concerning the flying arrow is renowned and still without explaining. Our description is static and irrelevant to the motion of variable. But let us leave that question, and let us pay attention to the subsets of the continuum, the diversity of which is enormous. I have already mentioned the supposition that the infinite subsets on the continuum which are not equipollent with the integers are equipollent with the whole entity of continuum. This is true for irrationals and even for some sets with a non dense position in continuum. However, my search is rather far from being complete. The acknowledgement of this statement would follow if the continuum would occur to be equipollent with the set of tranfinite numbers till , since for that set the statement is true.

For a while, some much more general questions have become my obsession. Are there sets besides those which are subsets of continuum? It is hard to believe that continuum serves as a universal container for sets. If so, it should be as big as the world of our thoughts! But, what do we mean saying an arbitrary set? A hundred years ago a controversion between Euler and d'Alembert took place concerning the meaning of the arbitrary function.

Mittag – Leffler. I learned that singular sets lying on the line, as well on the plane, are in the focus of interests in the theory of functions. Singularities of analytic funtions are not necessarily isolated, they can accumulate at points, and even form continua. Gustave Mittag-Leffler, who has had quite a lot of publicity lately, made a small inacuracy in a reasoning with accumulation of points. I wrote a comment to him. His reply was very polite. He wrote that he read my papers and that they were interesting also for his colleagues in Sweden. Mittag-Leffler was educated in Berlin under Weierstrass. Recently, he founded a very prestigueous journal “Acta Mathematica” in Stockholm. He spent his last year in France.

Encouraged by him, I sent my paper concerning singularities of sets on the line, plane and other manifolds to “Acta”, which are close to his interests. He is a rich man, full of charm and energy in organization of mathematical life, being very influential in Berlin. On his behalf Sophie Kowalewska has received veniam legendi in Stockholm.

Our correspondence has become with months systematic. I am informed from his letters about recent news from mathematics. We exchange opinions being in consensus concerning the views on Kronecker. It is clear for me that his comments are not in direct connection with my recent results concerning transfinity. In the Weierstrass’ analysis transfinity do not appear, although some limit sets in complex analysis might be very curious. I learned from him that my theorem on the decomposition of sets into two parts, one dense in itself and the other which is countable, was obtained also by his collegue Bendixson, and that Phragmen in Finland obtained an important result about the sets which cannot disconnect the plane. I become now much more quiet hearing these news. I feel not so alone in my interest in sets.

However, the mathematics is rather in the far background in our letters. They give me an opportunity to express my views. In response I got careful acceptance and delighted words for my ideas on the transfinity. I agree to Mittag-Leffler’s proposal to publish in “Acta” the French translations of my previous papers, in particular those on “linear Mannigfaltigkeiten”.

“V e r s o h n u n g b r i e f “ t o K r o n e c k e r. It was not Kronecker, but just I myself, who was a cause of my fatal state of health. It is my fault that I left without words our common interests in the theory series. He was my doctor father and I feel myself as a prodigal son before him. My correspondence with Mittag-Leffler became inessentional, deprived of any valuable content. I cannot say that I feel coldness, however there is nothing of warmth in our relations, simply exchanging plain compliments, as these of his last visit in Halle. He agrees with me in many matters. I understand that he is against Berliners, who accept Fuchs closing the door before him. He is full of personal gossip, but being silent about my papers which I sent to “Acta”.

Meanwhile, Kronecker who declared openly his opposition to sets talked with me sincerely, even wholeheartedly. He never omitted me in his polemics. I got news that he is preparing a polemical article in “Acta”. However, I am not able in my my recent state of health to participate in any public dispute. I decide to write to my old Professor a letter expressing my views and true feellings.

K r o n e c k e r’s r e p l y. I did not expect so much attention from Kronecker, His response was truly sincere and friendly. He has wroten that he is truly against my views, but he added that we ought to possess views. Even his warning at the end of the letter I can fully understand. He wrote that the creation of new notions is not necessary for mathematics. They are for him in the far background. In mathematics only the concrete patterns are of highest value, and these are contained in formulas. Just f o r m u l a s - as we learn from history - are everlasting! Which concerns several theories – here even Lagrange is not the exception – time is merciless. Meanwhile his r e s o l v e n t remains!

I recognize in these words my thoughts which I have had many times. But each of us choose his own path along mathematics, often by accident. Was this accident Interlaken? Our paths along mathematics run unicursally, backwards are not anticipated. Not only we, mathematicians, are subjected to these crude laws, but also mathematics itself. These thoughts might seem pessimistic, but as never before, I feel stable and quiet, being free of my inner hesitations. I am thankful to Kronecker for his true and wise words.

M i t t a g - L e f f e r’ s r e p l y. I did expect such a reply. The letter begins with plain compliments. He highly evaluates my ideas and the style of thinking, but admitted that not all mathematicians might have the same views. After some further plain words, getting in the role of a menthor, he suggests that it would be better to close the article with concrete conclusions, no matter that my ideas presented in the paper are so important. Finally, he mildly suggests to publish the results in the form of a separate article. In reply I withdraw my paper from “Acta”. I recognize that our friendship ended. But was it really a friendship? Can I use this word for a relation being no more than acquitance with some appropriate amount of courtueasy? But the most depressive for me is the fact that I agree with his comments. I should accept the fact that my efforts to confirm my hypothesis concerning continuum are hopeless.

The tools which I prepared, such as some lemmas concerning ordered sets, which demanded so enormous work from me, seem to be useless for getting my “promised theorem”. I lost the belief that the arithmetical continuum can be located on my scale of transfinite numbers. But let us imagine that I really got my “promissed theorem” and that I sent it to “Acta”. Could I expect from Mittag-Leffler more than kind words: “thank you for an interesting result”? The experience of mathematics sems to be alien to him. What a difference from me and Dedekind!

P u s h k i n. I remember my father's talks on Pushkin. We came here from Petersburg, where all the people have their own view on the famous controversy of the great poet with the overthere cavaliere de Mere. According to a common opinion neither cavaliere de Mere, nor the Emperor, was responsible for the fate of Pushkin, but the poet himself. The rules of the Emperror's court were well known to everyman but not to the great poet. The people treated small wickednesses on the court as events which are plain and just destined for that place. But not Pushkin. He lived in the world created by himself in his thoughts [16]. It is the fault of his mind that he was unable to put out of sight even the smallest inaccuracies.

Is my nature the nature of Pushkin? What were rational reasons for expecting that my transfinite numbers will be enthusiastically welcomed in "Crelle"? Why do I expect more than two sentences in a review by Netto? My collegues are rarely welcomed by nice and spontaneus quotations in print. On the other hand, the publicity of some people is in most cases falsely based. It is not clear for me that Weierstrass is really a leader. The uniform continiuty was invented by Seidel and Gudermann. It was Heine who explained when we can integrate term by term.

Did Pushkin have the right to see himself as a leader? He wrote that he expects "exegi monumentum". In this respect he was right. The community is used to such posthumous gestures, the cost of them is not high. But he was convicted that his poems gave him the right to express the judgements on behalf the whole nation. Would the authorship of poems overweight the merits of professionals? To express curious opinions is a customary privilege of a jester on courts. Like Minotaurus, Pushkin had to see himself in a double role, seeing himself in the mirror of his own thougths and in the reflected light of strangers. A loud sigh of Tsar, a triffle of a young captain, silence in the hall, the Natalie's smile and other small details strained him. In order to reject dark thoughts he behaved as a hero even in the simplest everyday situations. Also his inner mirror remembered not only the big poems written by him. There were unnecessary triffles inscribed into the notice books of charming dames and which should never be published pamphlets. During one of such dark moments he overheard a well known whisper ... He knew that his grandfather had came there from Ethiopia in a deputation of the King-of-Kings, and he was proud of him. His grandfather was not alien here. But his grandson was!

These reminiscences of Pushkin are true so much that there is no need for any comments. But, does the last of them have also somewhat in common with me? My grandfather came from Copenhagen to Petersburg where my father was born and baptisted as a lutheranian. I have never attributed importance to the details of my life story and I have never asked been asked by anyone in Petersburg and here about that. However, I have heard that among folk people this is not the case.

D o n o t a c c u s e f o l k p e o p l e. The young Felix Klein [17], who is so popular now, expressed words which are foreign to university communities. He said that the character of mathematics depends on the spirit of nation. I agree that our mathematics changed due to Gauss, becoming more conceptual and free from the French mode of calculations. But I cannot agree with Klein's simplicity. In his "Erlangen program" he reduces mathematics to the search of invariants of transformations, und classifies mathematics according to hierarchy among these. Gauss, being the greatest geometer, located the core and beauty of mathematics in the theory of numbers, which by no means is subjected to Klein's invariants. Is the theory of numbers not northern? There are some words about sets in his "program" which express only his incompetence. He has writen that continuity of a transformation means that it transmits adjacent infinitesimaly small into adjacent infinitesimaly small. Are the infinitesimally small, this relict of the eighteenth century mathematics, in the spirit of northern mathematics? I dislike infinitesimals, having my own reasons for that, but now even more, thanks to the prophet of the new faith! Infinitely small is a v a r i a b l e tending to zero, not a mathematical object like number or figure.

Is Felix Klein alone? Helmholtz in his "Rede" [18] claims that the strenght of the state depends on the development of science. Science supports moral values. He classifies nations according to the merits for the human culture. It would be hard to oppose these wise words, but we can see with our own eyes how they are brought into action. Also Dedekind in the foreword to the third edition of his "Supplement 11" used the words in the spirit of Helmholtz, no matter how far the mathematical rings and corps are from their much better known exemplifications.

Pride of the enourmos bloom of science and culture in their country overhelmed Germans. It is hard to believe that at the time of young Gauss only the Frenchmen were mathematicians. But since 1810 every inhabitant have become a citizen of Prussia. No less than thirty years ago the doors of gymnasia were opened before every young man. One says that about fifty universities are situated in German speaking couutries. We see ourselves as Greeks, or more often as Germans from the times of the decline of Roman Empire. But more often as those from the Niebelungen saga, which begins with "Es war in Bur1gunden solch edel Magdelein ...". The same spirit, but in the form of the Beethoven's quartets, can be observed at quiet evenings celebrated in university communities.

Something is boiling and approaching our doors. There is a publicity around the book by Heine [19], who warned that besides overhelming Beethoven’s music and our great philosophy a great wave arises, the force of which is enormously greater than that which pushed the enlighted men of the last century into the whirl of revolution. The prejudices spread among folk people. They do not omit the walls of universities. The spiritistic seanses serve the famous physicist Zoellner for searching the fourth dimension, the idea of which he borrowed from Klein.

I allow myself to join that chorus of nonsenses introducing the bold Hebrew letter sleph - - to denote the classes of sets equipollent to each other [20].

(1890) M e e t i n g i n H a l l e. All the Berliners were against the Meeting being at meetings permanently. Kronecker was against too, but against meetings at all. He said that in the nature of our contacts with mathematics is to stand before it alone. I agree with him in this respect. I also dislike spectacles. However, there are many issues around mathematics which should be decided by the whole community. Yet in Heidelberg I convinced the majority of colleagues to get together in order to keep the matters in our hands. They have decided that the first meeting of the Association will be held in Halle and I was elected to be the organizer. We expect a great deal from mathematicians. Despite his previous polite letter Kronecker will not participate because of illness of his wife [21].

But Klein, Hilbert and Minkowski, thus the prospective Goettingen, will be all present. Mittag-Leffler declared his arrival. I asked him how many bedrooms I should prepare. There were a lot of such triffles, but I recognized myself as a quite good organizer. Most time I spent expelling letters. Besides, I prepare the scientific program of the meeting and the project for the future ”Jahresbericht DMV”.

The idea of new Goetingen came from Klein and his collegue Althoff – the director in Ministerium in Berlin. They both formight together the battle in France. Klein’s aim is to enter Goettingen by the strong foot. He wants to introduce “mathematisches Regiment” there, pulling from Koenigsberg the young, in persons of Hilbert and Minkowski. Before our eyes stands the “pontifex maximus mathematicus” of the Second Reich. Even Weierstrass has some fears and will be absent in Halle. There is no need to accent national features in science, although I agree with the modest views of Helmholtz. I agree that our meeting will be an important step for the unity of profesional mathematicuans in our country. However, after the meeting I intend to contact mathematians in France and Russia in order to be in cooperation. The first will be Vassilieff in Kazan. My young years which I spent in Sankt Petersburg will be an advantage for me to write to him.

S h a k e s p e a r e a n d B a c o n. Recalling the days of the meeting in Halle, I restate again that in everyday affairs, no matter how difficult they are, I was really quite good. Now I understand that people of other professions do not feel so much fatigue, even if the work is enormous. The effects of their work can be seen almost at once. Not so long do they wait for acknowledging the value of the job. Even the errors are not so depressive, as they can can be explained as the errors of the imperfect matter. In mathematics the theatrum of events is the word of our thoughts. Thus just our thoughts are guilty of errors, but my thoughts means m e ! We are in a permanent state of seeing ourselves in the mirror of our judgement. What a relief we feel if we can escape into every day work from that shadow of our inner Minotaurus. I could see how many even clever people escape to administrative positions taking in hands the task of organization in order to be in the true whirl of life ...

The work of a historian and a researcher of lterature, even an explorer seems to me rather quiet. In order to escape from my emotional mathematics, I spent five years solving the mystery of Shakespeare works, searching the hypothesis - not my own – obout the authorhip of Francis Bacon of the core of these works. In that work I have never met problems to which the answer would be “yes” or “no”, the answers were stated in the convention “it depends”. So the work in libraries, no matter how hard it was and how much it exhausted my forces, I felt as somewhat which took place beyond me, not destroying the state of my thoughts and my nervous system. Nonetheless, I do not envy historians and literature scientists that excess of quiety.

A v a l a n c h e o f s e t s. Functions f defined on a set X and assuming the two values m and w are in an one-to-one correspondence with subsets of X, consisting of points where the function assumes the value m and the value w elsewhere.

The set of these functions can not be enumerated by the elements x of X. Assign namely to such an x the function f_x . I can construct once more function of this kind being different from each of f_x. Take namely a function f whose value f(x) at x is w if $f_{(x)} = m$, but $f_{(x)} = w$ if f(x) = w if $f_x (x) = m$. The function f differs from f_x just at x.

This means that the set of these functions cannot be embedded into the set of points of the domain, even in the simplest case when only two values are allowed. Thus, the set of functions is of a greater p o w e r than that of the domain, having in view that the domain can be obviously embedded into the set of these functions by representng each a from X with the function assumimg the value m at a and the values w elsewhere [22].

Thus, according to the comment made at the beginning, the set of subsets of X is of greater power than X itself. There is a well known fact from the combinatorics that $2^n > n$. Now, extending the symbols, we get $2^X > X$ for the arbitrary power X. The powers of sets are endless!

In the particular case of the set N of natural numbers I get $2^N > N$. But there exist subsets of the aritmetical continuum having the power 2^N[23]. Thus, for the power c of the continum I get c > N, too. So I get a new proof of the uncountability of continuum, performed in the area of pure sets.

No one is now against the sets. My new paper hs been published in "Mathematische Annalen". It distinguishes from the other papers in the volume by the bold Hebrew alephs. For some people only the alephs will be worth noticing ...

Since the powers of sets are endless, the ladder of my transfinite numbers gets a new meaning. It never ends! The infinity of my ladder of symbols is a b s o l u t e. The segments of the ladder represent sets of the same power. But, is there a segment, representing the power of continuum?

The whole ladder should be free of representing any kind of the mathematical reality. In my review of Frege I was against the vague notion of power as the common symbol for sets equipollent with a given set without indicating a representative among them. At that time only the naturaI numbers, the set of natural numbers and the set of my numbers of the II class I regarded as the candidates of such representatives. Now I have infinitely many of candidates among other initial segments of my ladder of numbers, forming an ascending transfinite sequence of sets, which are distinct in power. I think that to each well defined set the power in this restricted sense should be assigned. But, I have always doubts concerning the c o n t i n u u m in this respect, viewing it aa a troublesome souvenir to my theory of numbers. However, I suggest that the collection of a l l my numbers, as well a set of all sets, are not well defined, and that they cannot be regarded as sets. Thus, not all collections should be invited to the dance.

V a s s i l i e f f. At last I found a time to write a letter. I do not know much of about Vassilieff [24]. My experience in Russian customs makes it easier for me to write the first words. I wrote about the future Congress. But, I was not able to deny myself a pleasure to recall some reflections of my childhood years in St. Petersburg. I left Petersburg in my fifteen year. I remember the clouds over the great and grim Neva river. I agree with the saying “the North attracts”. The father’s affairs were ended there and he decided to spend the decline of his life in Germany. He was in fact a stranger in Petersburg coming there from Copenhagen, where my grandfather had a strong position in the Jewish community. I remember Petersburg as an attractive multicultural town. Only after the arrival to Germany did I learn about mathematical life there. Vassilieff is interested in international contacts and in organization of the Congress, too.

H e r mi t e. My interests in Bacon brought me nearer to Hermite, who manifests many interests also beyond mathematics, which happens often among mathematicians. Our views coincide on many fields. Why are mathematicians, much more often than other people, in permanent run for acceptance, looking always for new results? I know this from my own experience. Hermite, whose efforts are really great, can reject that mirror at which I am looking at myself permanently. His authority allows him to express influential opinions. He is always calm and his inner cool transfers to me. In our letters the sets are absent. I know that he is against them. The subject of our letters goes beyond mathematics.

The phantoms of prejuditions and superstitions go around Europe. The freemasonry and ocultism are spreading over France. In Germamy the cult of nation and state is overhelming. No matter that these trends are apparently opposite. People escape from Christian traditions, which have always moderated the thougts and prevented the conflicts.

Not long ago I was asked about my opinion on the appointment of a position in Freiburg. I indicated young Husserl, who was known to me as a student of mathematics. I was not so much for him, but I thought that he was the best among the seven candidates. But Husserl was rejected. There is no doubt that his Jewish origin was the obstacle. I wrote in my opinion about his close affection to the lutheranian tradition. Obviously, Husserl is no more than a deist, panteist, and in fact somewhat undefined. I defended him by some circular arguments, viewing in his deismus nothing dangerous, no matter that all these “isms” join somewhere together.

The signs of prejudice, which we have first observed among folk people, are now being forced into science, where they gather dangerous colours. Views foreign to science have entered the walls of universities and science is viewed as the playground for religious and national rivalry. Grey-headed Helmholtz claimed that development of science makes the society better. Meanwhile, we see that science is used not necessarily for weal, but rather for satisfying political ambitions. Will the future mathematical congresses be battle fields?

At its very beginning Christianity stood versus Judaism, but one can say that it is the highest level of the former. I can not exchange such thoughts with every man, but I can do that with Hermite, asking for instance, who was in fact Josef of Arhimatea. We are liberated men, but at the same time closely connected to religious tradition, which gives us the right to a bit of heresy. But in our times religion serves often as a sign of identification. Not the strenght of your faith is of importance, but the question "were you baptisted just after being born?" This would be not understandable for my father, whose letter full of thoughts on God, I preserve to this day. It was written to me when I began my studies in Zurich, and which was up to today the mainstay for my views. In the letters to Hermite I often use apparently free-thinking, even heretic phrases a la Renan. I like jokes, not so many people know me from this side.

T h e F i r s t C o n g r e s s [25]. Much effort from on my side was made toward the organizing the Congress. However, I was not intended to be overthere officially. In one of the lectures the sets appear, but in the kitchen form, as I am used to saying.

E m b a r r a s d e r i c h e s s e. Now, I have no obstacles to see my ladder of numbers as endless. Hilbert says that this leads to controversies, as the ladder itself is well-ordered and therefore it should be one of its initial segments. However, this is no more as sophistic controversy which can be removed if we find a property discerning the whole ladder from its segments. I had yet a similar controversion with the II class of my numbers, but I was able in a direct way to show that in view of its uncountability it cannot be an initial segment of itself.

I introduce my pure sets iteratively. I begin from sets which are in mathematics r e a d y, extending their scope by accepting as sets unions of sets accepted before, if the sets can be numbered by elelements of a se0t already accepted. I suggest that the family of subsets of a set which is already accepted should be accepted, too. I show Hilbert a proposed list of operations to create acceptable sets from sets already accepted. But I have a fear that he will regard it as a list of axioms.

Among my rules forming new sets I cannot find such one which allows to accept the collection of my numbers as a set. I stood before such a question in the case of the numbers of II class. But, now the decision of accepting the whole ladder as a set leads to a contradiction. So, I restrain myself of doing that. With respect to this matter I am free.

The most embarrasing are sets which I call r e a d y. They are ready in mathematics but they can be foreign in my iterative theory of sets. In the iterative theory the sets are c r e a t e d. We are in another situation if a set is a b s t r a c t e d from a set which accepted as ready in mathematics. I accept it as a set of p u r e u n i t s, thus in a w h i t e b i l l convention. The white bill convention is more liberal. I concern the c o n t i n u u m as belonging to it. The ordering in the continuum is not described in terms of relation “belonging to”. The same trouble concerns the description of its subsets. However, without these foreign objects my theory will be without mathematical content. In the case of sets defined iteratively, only their structure is taken into consideration, and in the description only the membership relation suffices.

During the talk with Hilbert in Harzburg, Dedekind joined us for a while. Although he was interested in our discussion, he left us quickly, as it was the time of the last stage-coach to Brunswick that evening.

I expressed Z w i s c h e n m e n g e n s a t z almost twenty years ago. Now, it was proved by young Felix Bernstein. He took into consideraration a grasp with the segment and the segment from which the end is removed. I and Dedekind overlooked that this grasp can be easily generalized. The proof is perhaps the most interesting among those which belong to the white bill convention in the theory of sets.

Most of the problems concerning sets will be now dissolved without me. I do not oppose consider the theory of sets as a common property of mathematicians. I raised and formulated the problems. Now I recall as a prophecy the words from my Thesis: t h e a r t of p r o p o n e n d i q u e s t i o n e m ... I remember my fear of Beotians from my years when I was involved into complex proofs of secondary theorems. Now, my theory becomes mature and I should be free of anxiety about its future.

T h e l e t t e r s f r o m N e r v e n k l i n i k. The bad thoughts were always my own. It was Mittag-Leffler who told me that thruth in one of his letters. But we can also read in Evangelia that all the evil comes from our inner self, so in order to avoid the evil we should go to people. I felt the lack of Kronecker, the truly openhearted man, with whom I would like to discuss freely the problems which are troubling me. In my talks with Dedekind I was not so free, as he always overhelmed me with his wisdom. Being about fifty, I observed that I am getting older differently from Dedekind, who is in a permanent contact with younger people. I never promoted doctors. The contact with young Hilbert is rather superficial and rather cool. He felt himself independent in mathematics. I see him to be subjected to a doctrine of axiomatization of everything.

Now, I am writing to Hilbert joking about the place in which I am. It is the University Nervenklinik. Biographers will admit that I was serene also in difficult situations. I believe that they will be, as my theory, mainly thanks to my opposers, reached the level of devotion.

Was Dedekind among my opposers? His philosophy concerning natural numbers, published later in “Was sind und was sollen die Zahlen” had already been was known to me before our meeting at Interlaken. I regret that I restricted in my talks with him to arithmemtics. My experience in philosophy was at that time insufficient to understand his deep ideas. Now, I am able to raise serious pholosophical questions concerning sets and numbers. But, when we met incidentally in Harzburg this summer, our talk was broken after a few words. I wrote him a letter some days later asking about the sets which I called well-defined. His responnse was very polite. He wrote that this is a serious question, but rather far from his recent interests, and that he would be in the role of a dilettante discussing that question. Is that simply an uwillingness to a discuussion with me or an escape from the question? I asked him if he accept the system of the natural numbers as a set. I think that it is no stronger than the acceptance of the transfinite. I recall my last letter to him from the eighties with my first announcement on the transfinite numbers, which he admitted without enthusiasm. His mathematics is restricted to notions which are carefuly defined and eleborated to the last detail. He met with criticism my ideas of pure sets, but I heard that my number cłasses are accepted by him as the candidates for representing powers.

I came back to our letters from the seventies. It was only my illusion that we understood each other. In these letters he was in the role of a school teacher to whom the last sentence belongs. He was interested only in the correctness of proofs. We did not exchange ideas. I presume that I was not an exception. Was he in contact with Dirichlet when the “Supplement” to the Dirichlet ideas was written. These were in fact his own ones, elaborated through years. The exchange of ideas was performed only in his own head. The same was with the idea from “Was sind und was sollen die Zahlen?”. The properties of the system of the natural numbers become not surprising if they are deprived of his deep motivations concerning the selfreferences of thoughts. In the Peano axiomatical setting they became even trivial. I think that Dedekind’s idea of the stream of thoughts has never been discussed among mathematicians. To support his beliefs he discovered Bolzano. Being embedded in the world of his own ideas, he felt no need to be close to other mathematicians. Now, it is easier for me to understand his “Treppenverstand”.

Now he exchanges polite letters with Felix Bernstein. No wonder that after some years a sheet of paper will be found on his shelf with his short sketch of the idea of the proof.

The politeness in his contacts with people seems to be superficial, deprived of empathy. He is mild and he never uses unpleasant words. Thus he enjoyed the opinion of a man friendly to people. In other words, he is not able to be unfriendly. In this expression the signs plus and minus are in a modest equilibrium.

In the past I was sadly experienced by his rapid change in our relations which for years were friendly and were without words interrupted by him. To avoid the embedding in the state of depression I created Kronecker as the substitute of the cause of my inner fears. At the same time I put Dedekind in the unpleasant role of getting in old wise man deprived human feelings. This fault of character is often observed among mathematicians. Was the crudeness of Gauss a pattern for him?

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L e t u s s a y s o m e w o r d s f o r C a n t o r. Cantor spent the summer of 1900 in the Nervenklinik. He did not participate in the Paris Congress. Was he aware that the aim of Hilbert to include set theory into his list of problems? At that time these problems were not yet famous and there were troubles to include them into a plenar session. Although Cantor granted to Hilbert a bit of affection, he could not avoid a bitter comment that the "goettingener mathematishes Regiment" declares its aspirations to the leadership in matematical community. Was Cantor glad that the continuum hypothesis was included into the list of worldwide problems? Or did he see himself as a needless personage among the problems which are his own?.

The news came to Cantor softened by crossing the filter of the Nervenklinik walls. Not all of them reached Cantor. The young Erno Jurgens published a paper in which he declared that Cantor's proof that the Euclidean spaces which differ in dimension cannot be in one-to-one continuous correspondence, is not "stichhaltig". Jurgens remembered an old proof of Luroth, which was disrespected by Cantor, concerned the dimensions two ond three, presenting also his own. Cantor never mentioned his unfortunate "proof", removing it from his memory.

The continuum hypothesis would be rejected if we would know that continuum cannot be represented as an aleph. On the Congress in Heildelberg 1904 he presented a subtle proof of the promised condition. Cantor was on the session and was deeply impressed by Koenig's claim. But, the next day Zermelo presented the proof that every set can be well-ordered. These two results were in contradiction with each other [26].

There were no doubts concerning the correctness of Zermelo's proof. However, there were discussions concerning its value. In the proof a grasp was used, called now the p r i n c i p l e o f c h o i c e. Zermelo proved a well-ordering of set is determined by a selection of points from the subsets of the set. The value of the principle might be questioned, as the existence of such a selection is only postulated, not always possible to construct effectively.

Dedekind's shadow occurs to be extremely long. According to a comment by Zermelo [27] his famous proof was based on an old idea of Dedekind from "Was sind und was sollen die Zahlen" allowing him to represent his finite sets on the scale of natural numbers.

None the less, due Zermelo's discovery, Cantor's ideas can be viewed now in full light. His theory was born again. His scale of transfinite numbers, which was treated before as a play with symbols, becomes now the role of mathematical absolute.

The meeting at Interlaken made Cantor take a dramatic decision to choose his own path through mathematics. The cost of leaving the path destined for the majority of mathematical professors was enormously high. He was alone with his problems, with no encouragement from the mathematical community. The relations with Dedekind were not only cool, but even more, he saw him during many long years in the role of crude God Father, if not just his own father, who disuaded him mathematics as the object of studies.

However, Cantor's last years are not the years of recollecting the past. He is now strewing by honors. His theory becomes popular in England. He has never floodered with the mathematics of Englanders, but England itself was always located by him on the top of civilized world. The membership of the London Mathematical Society and an honorary doctorate at St. Andrews allowed to him to visit the island of his dreams. His old problems concerning placing powers of sets on the scale of transfinite numbers are now renowned by Hardy [28]. Discussions around the Russells antinomies were troublesome for Cantor. He regarded them no more than a result of logical misunderstanding.

There were honors from Christiania and Kharkov. But not from Berlin! A surprising news came to Cantor about a stupendous Poincare's assault against the theory of sets which took place at the Congress in Roma 1908. He had heard about Poincare' as of French replique of Klein before. Now, he recognized that it might go further than known to him “pontifex maximus” in unrefined expressions. In the letter written to Hermite he wrote that French Klein atributes himself the right to pass judgements on behalf all mathematicians, being at the same time primitive and plain, expressing opinions without no knowlegde on the subject. However, much more astonishing that Poincare’ himself was the applause from congressmen for his publicistic arguments, including some bitter alusions to this German theory including the sneers from German words like “menge” ! But now Berlin invited him to be a member of the Academy!

To the patient of university Nervenklinik such news came in a rather mild form. Hilbert, Bernstein, Schwarz and many others came to Halle to the occasion of the seventienth Cantor’s anniversary. Dedekind was living in Brunswick. The war propheted by folk people was in the full run.

In the year 1908 Ernst Zermelo, forced by Hilbert, formed theory of sets into an axiomatic system. This idea seemed alien to Cantor.

Notes

[1] One can hear an read that Cantor came up with the idea of sets incidentally searching trigonometrical series. Although this sems to be a simplification, it can serve as an explanation of Cantor’s neophitic faith to the idea of sets about which he had never hear before.
[2] The possibility of halving infinite sets, i. e. presenting its power m in the form m+m, seems to be obvious, but the proof, given by G. Hessenberg, 1916, based on the principle of choice, is not so easy. see W. Sierpiński, Cardinal and ordinal numbers, Warszawa 1958, p. 416. This concerns also the proof that after addding a new element the infinite power remains the same. However, there are only minor difficulties for obtaining results mentioned above for powers of many several concrete sets, for instance for the power c = continum, getting c = c + 1 and presenting c as c + c.
[3] One can assume that Dedekind’s idea from “Was sind was sollen die Zahlen” was known to Cantor, being implicitly expressed in Dededinds “Supplement 11”, 1863.
[4] Moses Mendelssohn (1720 – 1786) – philosopher - the author of important for mathematical sciences work “Uber die Evidence in metaphysischen Wissenschaften”, 1753, Polish translation, Wrocław 1999.
[5] In 1900 Kurt Hensel constructed another completion of rational numbers, based on divisibility properties of numbers, the so called p-adic numbers.
[6] The date mentioned by Dedekind can be found in the introduction to his “Stetigkeit und irrationale Zahlen”, 1872.
[7] As was noted by Julius Koenig (1904), Cantor‘s proof can be saved if instead reshuffling ciphers the blocks of zeroes ended by non zero-cipher, i. e.

blocks like 00005, will be shuffled. To this end also some special kind of special continuos fractions can be used (Hurwitz, Sierpiński, Tietze).
[8] This Cantor's proof can be easily generalized to the following reasoning, being a proof Cantor- Bernstein theorem.

Let B be a subset of A. Let f be an one-to-one map of A onto a aubset f(A) of B. Let a be an element of A do not belonging to B. Move under f by one place to the right the orbit (a, f(a), f(f(a)), ...) of a. We get the subset (f(a), f(f(a)), ...) of B being in one-to-one correspondence with the former. Extend this map to the whole of A to be the identity outside the orbit of a. Thus, we get one-to-one correspondence between A and B.

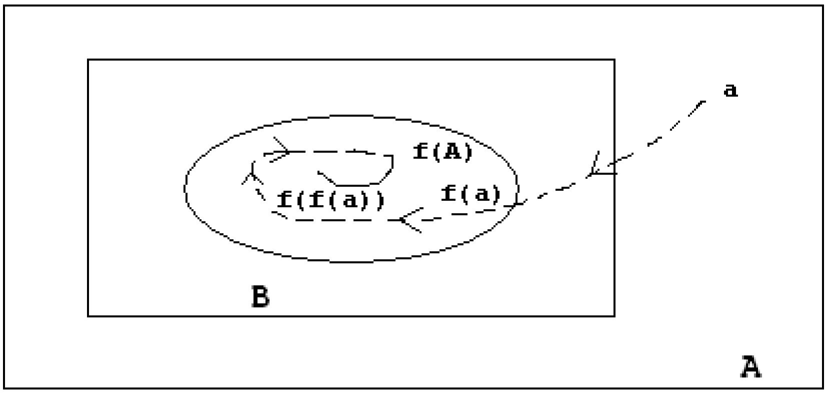

Fig. 5.

[9] Cantor's proof concerning the dimension was not only erroneous but even incorrigible, as was declared twenty years later by Erno Jurgens. But, at the same time as Cantor's,   a correct proof of the non-existence of continuous one-to-one correspondence between the space and the plane was given by Jacob Luroth. See for the details to the article by Dale M. Johnson. The full solution of the problem was given by L. E. J. Brouwer (1911). The key of the proof was the theorem on the invariance of the domain, discussed with Dedekind by Johannes Thomae. See for the details to the article by Dale M. Johnson (1979).
[10] About reserve between Goethe und Gauss the author knows from the article "Gothe und Gauss" by K. R. Birmann.
[11] The famous Cantor's ternary set was described in "Memoire Nr 5" in a small note.

[12] “Memoire Nr 5” from “Uber unendliche lineare Punktmannigfaltigkeiten” (1883) can be regarded as the most important for Cantor’s works o sets. However, this made no obstacle for Jean Cavailles to write about some passages of this work as “verbosity”.
[13] Introducing the set of numbers of II class Cantor was in an antinomial situation like that which occured later with the set ot a l l ordinal numbers.
[14] The fact that is the least uncountable power demands a proof. The dedefinition asserts only that it is the least among countable ordinal numbers.
[15] The known “Versohnungbrief” can be found in the collection “Georg Cantor. Briefe”, Springer.
[16] Jurij Lotman, “Aleksander Puszkin”, Polish translation, Warszawa 1990.
[17] Felix Klein (1849 – 1925) - „Odczyty o matematyce“, Warszawa 1899.
[18]0 Hermann von Helmholtz (1821 – 1894) – „Uber die tatsachliche Grundlagen der Geometrie“.
[19] Heinrich Heine, „Filozofia niemiecka“, Polish translation
[20] The symbols of alephs were treated by Cantor rather anecdodically.
[21] Kronecker died a year later.
[22] The reasoning quoted here belongs in fact to Paul Du Bois Reymond, see Hardy, “Degrees of infinity”.
[23] For instance Cantor’s ternary set.
[24] Alexandr (?)Vassilieff (1853 – 1929), professor in Kazan in the times of Cantor.
[25] Congress in Zurich, 1997.
[26] At the first moment no fault in Koenig’s proof was found. About the role of Felix Hausdorff in founding the fault see Walter Purkert’s comments in the volune II of “Felix Hausdorff. Werke”, Bonn 2003.
[27] Zermelo indicates “Satz 159 in “Was sind und was sollen die Zahlen“.
For the details see the book by G. H. Moore, for instance, comment 1 on p. 144.
[28] G. H. Hardy came back to the problem representing powers of sets on the scale of ordinal numbers. He proved that if there is at least one initial segments on the Cantor’s scale which cannot be emebedded in an one-to-one manner into a given set, then the power of this set is a power of some initial segment of the scale. Later (1916) Hartogs proved that for sets accepted in Zermelo’s system the assumed by Hardy condition might be omitted.

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A d d e d a f t e r t h e t e x t h a d b e e n f i n i s h e d. It has always the author wonder that in most books devoted to the development of the theory of sets a big omission was made. This omission is Richard Dedekind. Among mathematicians who created set theory, only Ernst Zermelo attributed the crucial role of Dedekind in solving the big questions of the theory as well as in creating the ideas of imperishable value.

After a long search through literature at last two works hsve been found by the author. One is the essay "Mysteries of Richard Dedekind" by David McCarty, who gave a broad insight into Dedekind's ideas and their influence on contemporary mathematics. The other is a historical essay by Jose Ferreiros "On the Relation between Georg Cantor and Richard Dedekind", Historia, Mathematica 20 (1993), 343 – 363, where Dedekind's role was presented accordingly to the facts and to the inner truth of mathematical events. None the less nothing will be changed in the story written here, even if some advantageous corrections might be made. The convention in which the story has been written gives the author a bit of freedom in choosing among facts and colours. However, let be alowed to the author at the end to quote words chosen by Jose Ferreiros as a motto, which are implicitly contained also in the author's text: "The break in our relations was very painful to me, as for all these long years I used to present my inner mathematical beliefs to your's fully-grown judgement" – from the letter Cantor to Dedekind (1882).