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Everywhere oscillating functions

This article is divided into two parts. In the first one a well known lemma is stated and proved. In the second one an application is given to obtaining functions from the title of the article.

1. The two climbers.

The lemma in its original form concerns two continuous functions from the closed interval onto itself having no interval of constancy. Let f and g be such two functions transforming onto itself the unit interval $[0,1]$. Consider for convenience only the case when these functions transform 0 into 0 and 1 into 1. In 1952 T. Homma [1] proved that there exist functions a and b satisfying the mentioned above conditions at ends and such that

$$f(a(t)) = g(b(t)) \quad \text{for each } t. \tag{1}$$

This lemma can be stated in the following anecdotical form.

There are two climbers starting from the places on the level 0 and climbing, one along the slope $f(x)$ and the second along the slope $g(y)$ to the top on the level 1 - see the fig. 1. Can they choose their paths in such a way that at each time t they would be on the same level?

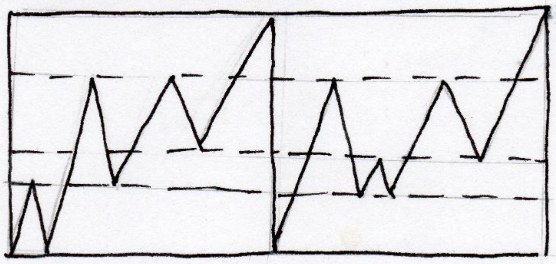


Fig. 1. The two climbers

The assumption that there are no intervals of constancy cannot be removed, what can be seen when the function f is constant on an interval and the function g has on this interval infinitely many proper minima and maxima oscillating

[1] T. Homma, *A theorem on continuous functions*, Kodai Mathematical Seminar Reports 1 (1952), 13 - 16.

around the value of constancy of f. This cannot be happen if the functions are piecewise linear.

The lemma is helpful in solving many interesting topological problems - let us mention K. Zarankiewicz [2] with his proof of a theorem of Dyson asserting that a continuous function defined on the sphere assumes the same value at the vertices of a square inscribed into a great circle [3].

The essence of the lemma - together with possible applications - lies in its discrete arithmetical version which will be stated and proved below. This version has obvious translation to the case of functions which are piecewise linear.

2. A r i t h m e t i c a l l y c o n t i n u o u s f u n c t i o n s.
We shall consider functions h between initial segments $k = [1, ..., k]$ and $l = [1, ..., l]$ of the set of natural numbers [4] which are continuous in the following meaning:

$$\text{(2)} \qquad \text{if} \quad |r - s| \leq 1, \quad \text{then} \quad |h(r) - h(s)| \leq 1.$$

Call such functions *arithmetically continuous.* Arithmetically continuous functions transform neighbour numbers into neighbour ones; equality $h(r) = h(s)$ is not excluded if r and s differ by 1. For convenience - and what suffices for the purposes of this article - we shall consider only the case of functions which transform 1 into 1 and k into l.

3. U n i f o r m i z a t i o n L e m m a .
Let $f : n \to m$ and $g : n \to m$ be arithmetically continuous functions satisfying the described above conditions at ends. There exist an initial segment p and arithmetically continuous functions $a : p \to n$ and $b : p \to n$ satisfying the mentioned additional conditions such that

$$\text{(3)} \qquad f(a(t)) = g(b(t) \quad \text{for all t from the segment } p.$$

P r o o f. Let us consider the lattice $n \times n$ consisting of (ordered) pairs (x, y) of natural numbers x and y. We can imagine the lattice as a chessboard $n \times n$; see the fig. 2. Label the fields (x, y) black, if $f(x) = g(y)$. Label the remaining fields white. According to the asumptions concerning the values of f and g at

[2] K. Zarankiewicz, *Un théorème sur l'uniformisation des fonctions continues et son application à la démonstration du théorème de F. J. Dyson sur les transformations de la surface sphérique*, Bull. Acad. Polon. Sci. 3 (1954), 117 - 120.

[3] According to the knowledge of the author, the theorem on climbers was discovered by Zarankiewicz independently of other authors on the way of the proof of Dyson's theorem.

[4] The same symbol is used for the initial segment of the set of natural numbers and for the number of elements in it.

ends, the fields $(1,1)$ and (n,n) are black. The difference $f(x) - g(y)$ is positive or zero for the pairs (x,y) with $y = 0$ and for the pairs (x,y) with $x = n$, i. e. for the fields lying on bottom side and on the right side of the board. From the same reasons, this difference is negative or zero on the left and on the top sides.

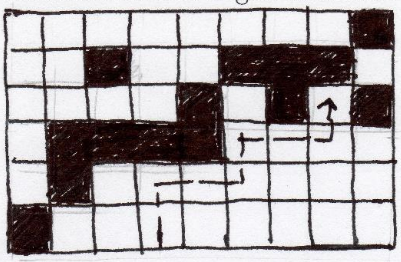


Fig. 2. The Chessboard Theorem

Consider the rock situated on a white field on the bottom or right sides of the board. Such a field exists and the difference $f(x) - g(y)$ for it is negative. There is no path for the rock through white fields to a place situated on the top or on the left side of the board. In fact, at each step the coordinates change at most by 1, moreover - as it is the rock step - only one coordinate changes. So, the difference between values of f and g change - see (2) - at most by 1. Because at the end of the path - which lies on the top or on the left side of the board - the difference should be negative, the rock must be pass by the place where this difference is zero, thus it must be at this moment on the black place.

The non-existence of a path for the rock through white fields from the bottom-right sides to the top-left ones implies the existence - accoording the known *Chess Board Theorem* - a path for king trough the black fields from $(0,0)$ to (n,n).

Let $(a(t), b(t))$ be an arithmetically continuous pametrization of this path. The existence of such parametrization follows from the fact that if the king changes the field for an adjacent one, the values of a and those of b, change at most by 1. We may assume that the king starts at $a(1) = b(1) = 1$ and that he finishes the path at the end p of the segment with $a(p) = n$ and $b(p) = n$. So, we obtain a pair of arithmetically continuous functions a and b satisfying the mentioned above conditions at ends, for which (1) holds, i. e. there is $f(a(t)) = g(b(t))$ for all numbers t in the segment p.

Clearly, we have $m \leq n \leq p$ in the above theorem.

4. R e m a r k s. Our proof depends on the *Chessboard Theorem* which was told as well known. There are many sources of this theorem. The author knows it from Hugo Steinhaus' *Mathematical Snapshot*, where it was mentioned without proof and was thought there as a tool in the proof of the Brouwer's Fixed Point Theorem for the plane square. According oral informations of Professor Steinhaus the theorem was discovered in thirties by Wodzimierz Stoek. A proof

of this theorem can be found in the book of Shashkin (1989) [5]. Another proof was given by W. Surówka (1993) [6].

Note, that the chessboard in this theorem is not necessarily square.

Another story about the *Chessboard Theorem* is given by Steven Gail (1979) [7], who attributed this theorem to Danish physics and mathematicians from the period of the II World War, who prefered the hexagonal chessboard instead of usual one.

5. A g e n e r a l i z a t i o n.

The uniformization theorem easily generalizes to arbitrary finite collections of arithmetically continuous functions.

Let S be a finite collection of arithmetically continuous functions from a segment n onto a segment m satisfying mentioned before additional conditions concerning the values at ends. Clearly, $m \leq n$. Under these assumptions there exist a number p and a collection of arithmetically continuous functions $a_f : p \to n$, such that the values $f(a_f(t))$ for any given number t from the segment p are the same for all f from S.

To prove this, it suffices uniformize firstly arbitrary two function from S, and then uniformize the result with arbitrary third function from S, and continue this procedure step by step. After finite numbers of steps we get the desired result.

E x a m p l e 1. Let S be the set of *all* arithmetically continuous functions (additional conditions as above) from the segment n onto the segment m. Each function $h : p \to n$, obtained as a result fa_f of uniformization of all functions f from S call a *majorant* of arithmetically continuous functions for the pair (m, n) of segments. Obviously, if h is a majorant, then each composition ha, where a belongs to the class of arithmetical functions considered here, is still a majorant.

E x a m p l e 2. Consider the set of arithmetically continuous functions of a segment n onto the segment m (the same as in the preceeding example) consisting of functions $f_{i,j}$ defined for each pair

$$(i, j), \quad \text{where} \quad |i - j| \geq 4, \tag{4}$$

of numbers from the segment m as follows. The function $f_{i,j}$ runs from the value 1 at 1 monotonically till to a place where it assumes the value i, and then monotonically to the value m at n from the place where it assumes the value

[5] Ju. A. Shashkin, *Nepodvishnyje toczki*, Popularnyje lekcji po matematikie, vyp. 60, Nauka, Moskwa 1989.

[6] W. Surówka, *A discrete form of Jordan curve theorem*, Annales Mathematicae Silesianae 7 (1993), 57 - 61.

[7] Steven Gale, *The Game of Hex and the Brouwer Fixed Point Theorem*, The American Mathematical Monthly 86 (1979), 818 - 827.

j, having on the segment between mentioned above places an oscillation of the shape of letter N as on fig. 3, the amplitude of which is $|i - j|$.

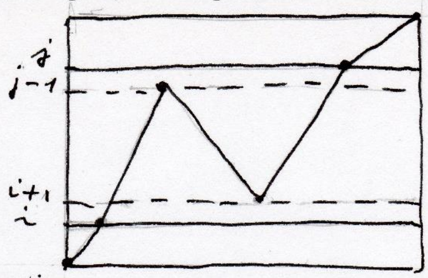
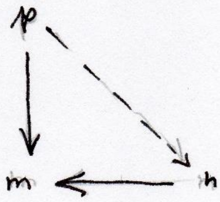


Fig. 3. An N-function

Call these functions *N-functions.* The number of N-functions is equal to the number of pairs (i, j) on the segment m which satisfy the condition (4). Call each function g obtained as a result fa_f of uniformization of all N-functions from n onto m an *N-majorant* attributed to the pair (m, n) of segments.

The role of the number 4 (other possible are 5, 6 etc.) in the condition (4) will be clear from applications.

6. M a j o r a n t s a s e v e r y w h e r e o s c i l l a t i n g f u n c t i o n s. If $h : p \to m$ is a majorant of the set S of all arithmetically continuous functions for the pair (m, n) of segments then if $f : n \to m$ is an arbitrary function belonging to S, then there exists an arithmetically continuous function $a : p \to n$ such that $h = fa$. In other words, the pair of functions h and f can be completed to a commutative diagram



In fact, the function a_f from the section completes the diagram.

The functions g which are N-majorants - see the Example 2 - have the same property but only with respect to N-functions. Call *N-like* the run of an arithmetically continuous function r from a subsegment U of the set of natural numbers onto a subsegment V if the domain U can be divided into three adjacent segments S', S'' and S''' such that $r|S'$ transforms S' onto the initial subsegment of V from which its end element is removed, $r|S''$ transforms S'' onto the subsegment of S from which both its end elements are removed, and $r|S'''$ transforms S''' onto the subsegment of V from which the first element is removed - see fig. 5; call *N-like* also the runs in opposite direction, i. e. from the end to the first elements of V.

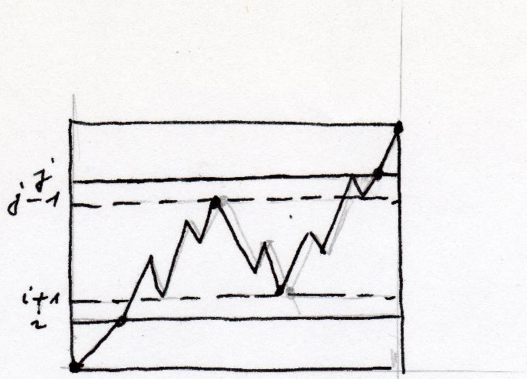


Fig. 4. An N-like oscillation

The N-functions $f_{i,j}$ have N-like runs on the preimages of segments (i, j). The run remains still N-like if we reparametrize the domain by an arbitrary arithmetically continuous function .

Let $g : p \to m$ be an arithmetically continuous function. Call g *everywhere oscillating* if for each subsegment (i, j) of n such that $|i - j| \geq 4$ it has an N-like run on each maximal subsegment contained in the preimage of (i, j) which is transformed onto (i, j) under g.

The property of being everywhere oscillating may be expressed colloquially as follows: if the value on the level l is reached by the function, then before reaching the level $l + k$, where $k \geq 4$, the function should run down to the level $l + 1$, see fig. 5.

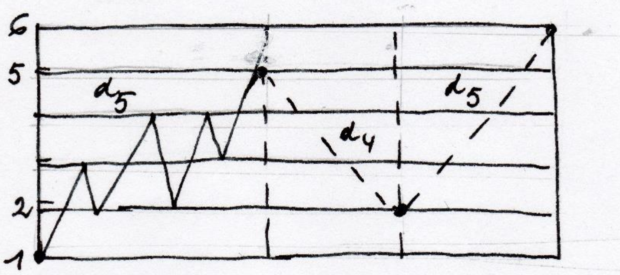


Fig. 5. An example of everywhere oscillating function for $m = 5$. If we repeat the run of this function from the opposite side - from the level 6 to the level 2 - and complete the run by putting into the middle segment the evereywhere oscillatin run for $m = 4$, we get evereywhere oscillating function for $m = 6$.

7. T h e o r e m. *The N-majorants are everywhere oscillating.*

P r o o f. Let $h : p \to m$ be an N-majorant. Let (i, j) be a subsegment of the segment m with $|i - j| \geq 4$. Let U be a maximal subsegment of p contained in the preimage $h^{-1}((i, j))$ of the segment (i, j). The function h assumes at ends of U values i and j. For convenience, we assume that $i < j$, that $a < b$ are the ends of U and that $h(a) = i$ and $h(b) = j$.

Let $f_{i,j}$ be an N-function associated with the subsegment (i, j) of m. Represent the majorant h - according to the property (2) of majorants - as a composition $f_{i,j}a$, where a is a function from p onto n.

To get the preimage $h^{-1}((i,j))$ of (i,j) under h, look firstly for the preimage of (i,j) under the N- function $f_{i,j}$. This preimage is equal to the subsegment U of n outside of which the function $f_{i,j}$ assumes values greater than j or less than i - see the description of N-functions in section 4 . In order to get maximal subsegments of p which are is transformed under h onto (i,j), we should take maximal subsegments V of p which are transformed onto U under the function a. The run of the function h on such subsegments V is N-like, since h restricted to V is formed from an N-like run by a reparametrization of its domain V.

8. F u r t h e r r e m a r k s.
Everywhere oscillating functions - called *very crooked* - served as a tool in the description of hereditarily indecomposable continua. They were independently discovered by E. E. Moise and R. H. Bing in 1948, and appeared implicitly in paper by B. Knaster, 1922, where the first hereditarily indecomposable continuum was constructed.

Bing's construction of everywhere oscillating functions [8] was illustrated on fig. 6. The everywhere oscillating functions constructed by this rule have minimal domain n for a given range m. Denote the number of elemnts in its domain by d_n. We have $d_3 = 3$, $d_4 = 6$ and the recurrent formula $d_{m+1} = 2d_m + d_{m-1}$. We get the exponential growth with the ratio $1 + \sqrt{2}$ for the sequence d_m, and the exact formula for d_n can be found [9].

Let m be given. According to the rule presented in this article [10], in order to get everywhere oscillating function with the range m, it suffices to take n three times greater than m. We will have then in the set of all functions from n to m all the N-functions for the range m, thus the majorant for the set of all arithmetically continuous functions from n to m is everywhere oscillating, by Theorem from the preceding section. For the obtained on this way function the equation $h = fa$ from section 6 has a solution a for all arithmetically functions from n to m. Everywhere oscillating functions have - in general - this property only with respect to N-functions. There arises the question how large are the domains p of majorants h for the set of functions from n to m when m and n are arbitrarily given.

[8]R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J. 15 (1948), 729 - 742.

[9]See A. I. Markushewitch, *Vosvratnyje posledovatelnosti*, Popularnyje lekcji po matematikie, vyp. 1, Moskva - Leningrad 1951.

[10]In fact, from the author's paper *A functional conception of snake-like continua*, Fund. Math. 51 (1962), 179 - 189.