The Matkowski–Wesołowski problem – Update

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Question

Jacek Wesołowski and similarly Janusz Matkowski asked whether the identity on $[0,1]$ is the only increasing and continuous solution $\varphi\colon [0,1] \to [0,1]$ of the equation

$$\varphi(x) = \varphi\left(\frac{x}{2}\right) + \varphi\left(\frac{x+1}{2}\right) - \varphi\left(\frac{1}{2}\right)$$

satisfying

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi(1) = 1.$$

Motivation

Theorem (Misiewicz–Wesołowski)

Assume that X, Y are symmetric, independent random variables having strictly positive continuous densities on $\mathbb{R}$. Let $(W, Z) = \mathcal{W}_2(X, Y)$ (doubling the angle). Assume that W is symmetric.

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Sketch

Random variables R, Θ:

$$\begin{aligned} X &= R\cos 2\pi\Theta, \\ Y &= R\sin 2\pi\Theta, \\ W &= R\cos 4\pi\Theta, \\ Z &= R\sin 4\pi\Theta, \\ \Xi &= 2\Theta - \lfloor 2\Theta \rfloor, \end{aligned}$$

leads to

$$f_{R,\Xi}(r,\xi) = \frac{1}{2}\left(f_{R,\Theta}\left(r,\frac{\xi}{2}\right) + f_{R,\Theta}\left(r,\frac{\xi+1}{2}\right)\right).$$

Sketch II

Playing with the densities (symmetry and evaluating at specific points)

$$f_{R,\Theta}(r,\xi) = \frac{1}{2}\left(f_{R,\Theta}\left(r, \frac{\xi}{2}\right) + f_{R,\Theta}\left(r, \frac{\xi+1}{2}\right)\right).$$

Sketch III

Manipulating: Cauchy equation.
What about if there are no densities?

How to find solutions

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- Looking at the connection to systems of the form

$$\varphi\left(\frac{x}{2}\right) = p\varphi(x)$$
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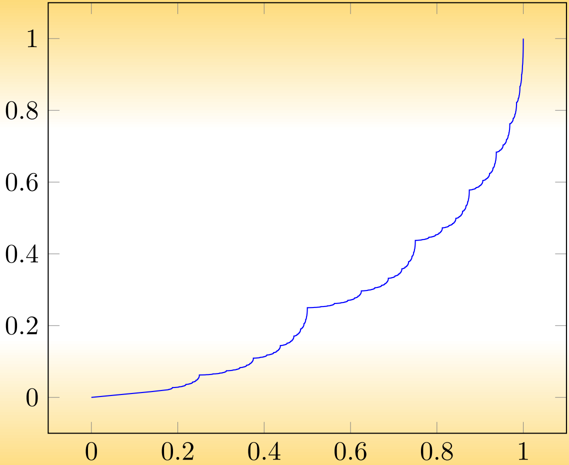
  for $x \in [0,1]$.
- Using Hata's work (see later).

Example of a solution



Absolute continuity

Assume φ is an absolutely continuous solution. Then

$$\varphi'(x) = \frac{1}{2}\varphi'\left(\frac{x}{2}\right) + \frac{1}{2}\varphi'\left(\frac{x+1}{2}\right)$$

almost everywhere and $\varphi' \in L^1([0,1])$.

Going to $\mathbb{R}$

$$\varphi(x) = \frac{1}{2}\varphi\left(\frac{x}{2}\right) + \frac{1}{2}\varphi\left(\frac{x+1}{2}\right) \quad x \in \mathbb{R}.$$

- periodic
- in L^1
- monotone increasing, $\lim_{x\to-\infty}\varphi(x) = 0$ and $\lim_{x\to\infty}\varphi(x) = 1$.

Periodic uniqueness (Hata)

Let $p \in \mathbb{N} \setminus \{1\}$, $\lambda, \mu \in \mathbb{R}_+^*$.

$$\frac{1}{p}\left\{f\left(\frac{x}{p}\right) + \cdots + f\left(\frac{x+p-1}{p}\right)\right\} = \lambda f(\mu x),$$
$$f(x+1) = f(x), \quad x \in \mathbb{R}$$

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- $\lambda > \frac{1}{\mu p}$ $\mu \in \mathbb{Z}$, $\lambda \neq 1$. If $f \in \mathrm{BV}$: $f = 0$.
- $\lambda = \frac{1}{\mu p}$ with $\mu \in \mathbb{Z}$. If $f \in \mathrm{AC}$: $f = 0$.

Periodic non-uniqueness (Hata)

Let $p \in \mathbb{N} \setminus \{1\}$, $\lambda, \mu \in \mathbb{R}_+^*$.

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- $\lambda = \frac{1}{\mu p}$ with $\mu \in \mathbb{Z}$. If $f \in \mathrm{BV}$: many continuous solutions.

L^1: Nikodem

Let

$$Ph(x) = \frac{1}{2}h\left(\frac{x}{2}\right) + \frac{1}{2}h\left(\frac{x+1}{2}\right)$$

and $g \in L^1([0,1])$. Then

$$f(x) = \frac{1}{2}f\left(\frac{x}{2}\right) + \frac{1}{2}f\left(\frac{x+1}{2}\right) + g(x)$$

has a solution $f \in L^1([0,1])$ if and only if

$$\sum_{k=0}^{\infty} P^k g \in L^1;$$

moreover $f = c + \sum_{k=0}^{\infty} P^k g$ for some $c \in \mathbb{R}$.

ε-invariant measure

Assume m is σ-finite measure on some space X.
We assume that $T\colon X \to X$ is measurable and $m(T^{-1}(N)) = 0$ whenever $m(N) = 0$.
μ is called *ε-invariant* if

$$|\mu(T^{-1}(A)) - \mu(A)| \leq \varepsilon m(A)$$

for all measurable A.

Nikodem

Let P_T be the Frobenius–Perron operator of T:

$$\int_A P_T f \, dm = \int_{T^{-1}(A)} f \, dm$$

for all $f \in L^1$ and A measurable.
μ finite absolutely continuous with respect to m. Then μ is ε-invariant under T if and only if its Radon-Nikodym derivative $f = d\mu/dm$ satisfies

$$|P_T f - f| \leq \varepsilon$$

almost everywhere.

Generalization

$$\varphi(x) = \sum_{j=0}^{N-1} \varphi(f_j(x)) - \sum_{j=0}^{N-1} \varphi(f_j(0)).$$

If φ is absolutely continuous:

$$\varphi'(x) = \sum_{j=0}^{N-1} f_j'(x)\varphi'(f_j(x)).$$

Some variant

$$\psi(x) = \sum_{j=0}^{N-1} |f_j'(x)| \psi(f_j(x)) + g(x).$$

What about different factors?

$$\psi(x) = \sum_{j=0}^{N-1} h(x)\psi(f_j(x)) + g(x).$$

Can be handled by a change of variable formula and the Banach fixed point theorem (if h is small).

Absolutely continuous and Steinhaus property

We say that μ has the classic Steinhaus property if $A - A$ has nonempty interior for any set A with positive measure.
Artur Bartoszewicz, Małgorzata Filipczak, Tomasz Filipczak: X is a polish group. Then μ is absolutely continuous with respect to the Haar measure if and only if μ has the classic Steinhaus property.

Why strange limits

monotone increasing $\lim_{x\to\infty} \varphi(x) = 0$ and $\lim_{x\to\infty} \varphi(x) = 1$: φ is like a distribution function of a random variable.

Bernoulli convolution

Let $\lambda \in (0,1)$ and $(X_n)_n$ be a sequence of independent random variables: $P(X_j = -1) = P(X_j = 1) = 1/2$. Further, let

$$S_\lambda = \sum_{j=0}^{\infty} X_j \lambda^j.$$

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The distribution function F_λ satisfies

$$F_\lambda(x) = \frac{1}{2}\left[F_\lambda\left(\frac{x-1}{\lambda}\right) + F_\lambda\left(\frac{x+1}{\lambda}\right)\right].$$

Some results

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- Erdős: $\lambda \in (1/2, 1)$ and λ^{-1} is a Pisot number: ν_λ singular
- Shmerkin: $\{\lambda \in (1/2, 1) : \nu_\lambda \text{ is singular}\}$ has Hausdorff dimension 0.

Why Pisot numbers?

Fourier transform: $\widehat{\nu}_\lambda(t) = \prod_{j=0}^{\infty} \cos(2\pi\lambda^j t)$.

$$|\widehat{\nu}_\lambda(\lambda^n)| \geq \prod_{j\in\mathbb{Z}} |\cos(2\pi\lambda^j)|.$$

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$$\alpha^n + \sum_{k=2}^{m} \alpha_k^n \in \mathbb{Z}.$$

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If $|\alpha| \geq 1$ and $|\alpha_k| < 1$, then α^n approaches integers quickly.

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