c-Removable sets: Old and new results

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(joint with Dušan Pokorný)

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Motivation: The talk will be formula-free. Relaxing stuff.

Setting: $f\colon U \overset{\text{open}}{\subseteq} \mathbb{R}^n \to \mathbb{R}$ continuous.
Question: What closed sets in $\mathbb{R}^n$ are negligible in terms of convexity of f? ("Removable for convexity".)

Definition (J. Tabor & J. Tabor (2009))

They call $A \subset \mathbb{R}^n$ intervally thin (IT) if it is "essentially transparent" (in all directions). (Fig. 1—blackboard)

Recall: A function f is locally convex on an open set $U \subseteq \mathbb{R}^n$ $\overset{\text{def.}}{\Longleftrightarrow}$ each point of U has a convex nbhd in U where f is convex.

Theorem (J. Tabor & J. Tabor; slightly simplified)

Let $F \subseteq \mathbb{R}^n$ be closed IT, and
$f\colon \mathbb{R}^n \to \mathbb{R}$ be continuous and locally convex on $\mathbb{R}^n \setminus F$.
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IT sets vs. c-removable sets

Definition (2013; M.R. & Dušan Pokorný)

A closed set $F \subseteq \mathbb{R}^n$ is c-removable $\stackrel{\text{def.}}{\Longleftrightarrow}$ every f cts. on $\mathbb{R}^d$ and loc. convex on $\mathbb{R}^d \setminus F$ **is convex**.

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Theorem (J.T. & J.T.): IT $\Longrightarrow$ c-removable.

Problem (2009; J.T. & J.T.)

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Answer (2013; D.P. & M.R.)

NO. The *Holey Devil's Staircase* is c-removable & not IT. (Fig. 2)

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Fact (2009; J.T. & J.T.)

(i) $\mathcal{H}^{n-1}(A) = 0 \Longrightarrow A$ is IT ($\Longrightarrow$ c-removable).

(ii) $U \subset \mathbb{R}^n$ open and connected, $A \subset U$ closed IT $\Longrightarrow U \setminus A$ connected.

(iii) Closed IT sets form an ideal of sets.

Theorem (2013; D.P. & M.R.)

Let $K \subseteq \mathbb{R}^n$ be compact and IT in n LI directions. Assume that for a dense set of line segments $L \subseteq \mathbb{R}^n$, the intersection $K \cap L$ is at most countable. Then K is c-removable.

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Examples (summary):

(i) A line segment in $\mathbb{R}^2$ is not c-removable. Nakreslit! (Fig. 3)

(ii) A DC-curve in $\mathbb{R}^2$ is not c-r. ($f : \mathbb{R} \to \mathbb{R}$ DC $\Leftrightarrow$ f AC and f' BV.)

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(v) The Cantor's dust (C^2 where C is the classical Cantor Ternary Set)?? We don't know (and are quite sad about it).

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Let $M \subset \mathbb{R}^n$ be a set which does not contain any continuum of topological dimension $(n-1)$. Then M is c-removable.

Theorem (Pokorný, 2012)

There is a 0-dim compact set in $\mathbb{R}^2$ which is not c-removable.

Corollary

Pasqualini made a mistake. (Or we did.)

Note that the last statement also follows from the fact that Pasqualini's proof contains a mistake which cannot be repaired. In addition, it is a consequence of example

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Is there a Lebesgue null 0-dim compact set $K \subseteq \mathbb{R}^2$ which is NOT c-removable? (All the examples above have positive measure.)

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We now basically know the answers.

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Is it the Koch Snowflake?

Maybe, but we have a better example:

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Definition: DC function $\equiv f_1 - f_2$ with f_1, f_2 convex functions.

Theorem (DP & MR, 2018)

Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be Lipschitz. Then the graph of f in $\mathbb{R}^n$ is c-removable $\iff f|_V$ is DC for no open $V \subseteq \mathbb{R}^{n-1}$.

Corollary

The graph of any Lipschitz nowhere DC function is a nontrivial example of non c-removable continuum (with the correct dimensions).

Conjecture: (almost done) There is a function $f : K \to \mathbb{R}$ (where K is 0-dim, cpt) such that the graph of f in $\mathbb{R}^2$ is not c-removable.
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Conjecture: (almost done) There is a function $f : K \to \mathbb{R}$ (where K is 0-dim, cpt) such that the graph of f in $\mathbb{R}^2$ is not c-removable.
It would follow: An example of a Lebesgue null non c-removable discontinuum.

Do I have 5 minutes? NO $\Rightarrow$ skip; YES $\Rightarrow$ continue.

Theorem

Let $K \subset \mathbb{R}^2$ be compact and IT in two different directions. Assume that for a dense set of line segments $L \subset \mathbb{R}^2$ the cardinality of $K \cap L$ is at most countable. Then K is C-removable.

- Take a non-convex function $f : \mathbb{R}^2 \to \mathbb{R}$ which is loc. convex on $\mathbb{R}^2 \setminus K$ (for a contradiction).
- Observe that this function has to be convex on all lines in the directions of interval thinness of K.
- WLOG f is separately convex and is not convex on the line L given by $y = x$. Assume also that L happens to have countable intersection with K.
- Lemma: A separately convex function cannot have a concave angle.
- $L \cap K$ is a countable compact set, so it is scattered. Deduce that f is convex on L.

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Theorem (DP & MR, 2018)

Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be Lipschitz. Then the graph of f in $\mathbb{R}^n$ is c-removable $\iff f|_V$ is DC for no open $V \subseteq \mathbb{R}^{n-1}$.

Dziękuję za uwagę.

Thank you for your attention.

Děkuji za pozornost.