Subgroup Lattices of Groups

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Martin-Luther-Universität Halle-Wittenberg

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Summer School of the institute of Mathematics
University of Silesiain Katowice
Brenna

What are groups?

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Definition

Let G be a set and $\cdot : G \times G \to G$ be a binary operation.
Then $(G, \cdot)$ is a group if and only if

- $\forall a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$,
- $\exists e \in G \ \forall a \in G : a \cdot e = a$, and
- $\forall a \in G \ \exists a^* \in G : a \cdot a^* = e$.

Let's go back in time.

Origin:
https://pixabay.com/de/zeit-portal-time-machine-reisen-2034990/

Origin: https://de.wikipedia.org/wiki/Évariste_Galois

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Nowadays: permutational group theory or geometrical group theory.

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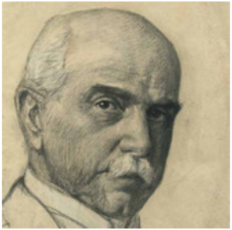

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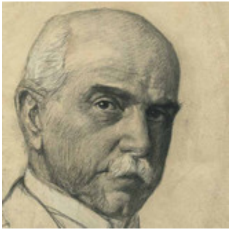

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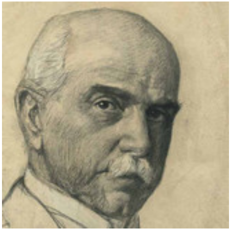

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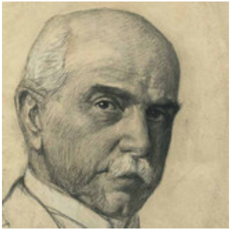

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Nowadays: arithmetical or combinatorial group theory.

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In the rest of the talk, we say that G is a group and denote by 1 the neutral element and by g^{-1} the inverse element of g, for every $g \in G$.

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Task: Understand G via $L(G)$!

Example S_3.

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Arithmetic interpretation

$\cdot$	id	(12)	(13)	(23)	(123)	(132)
id	id	(12)	(13)	(23)	(123)	(132)
(12)	(12)	id	(123)	(132)	(13)	(23)
(13)	(13)	(132)	id	(123)	(23)	(12)
(23)	(23)	(123)	(132)	id	(12)	(13)
(123)	(123)	(23)	(12)	(13)	(132)	id
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(13)	(13)	(132)	id	(123)	(23)	(12)
(23)	(23)	(123)	(132)	id	(12)	(13)
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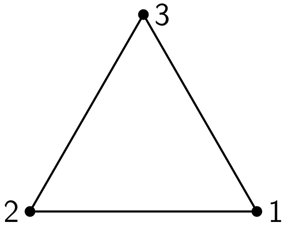

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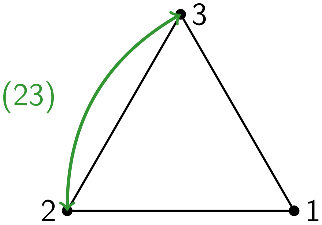

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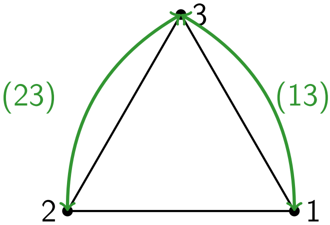

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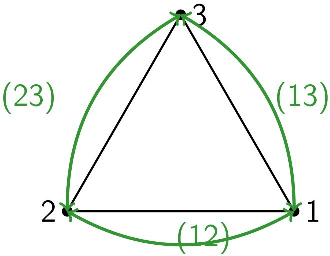

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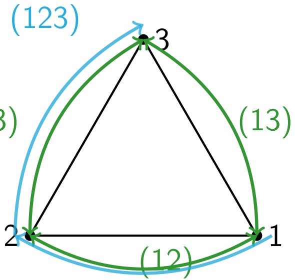

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(13)	(13)	(132)	*id*	(123)	(23)	(12)
(23)	(23)	(123)	(132)	*id*	(12)	(13)
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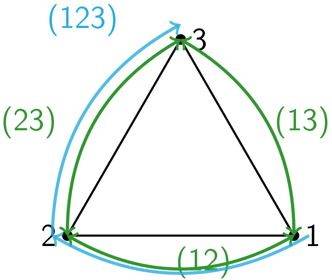

Subgroup structure

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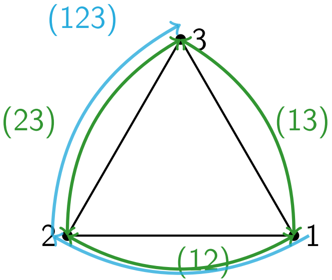

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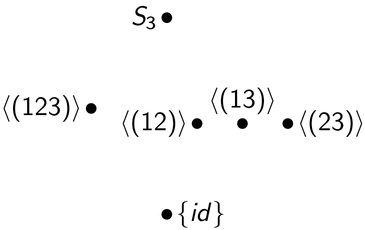

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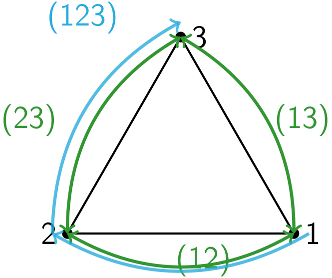

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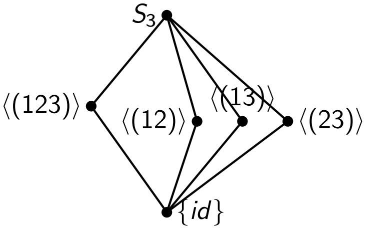

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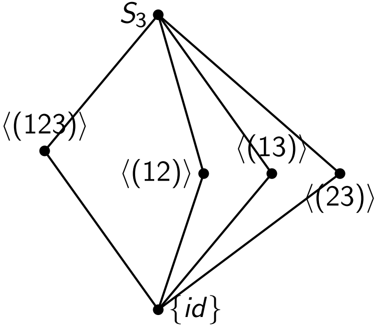

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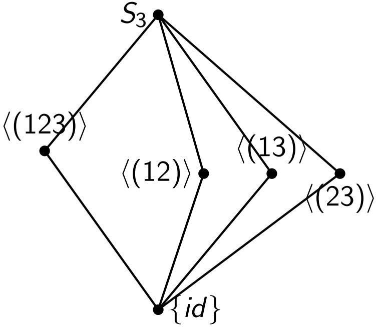

The picture is called the Hasse Diagram.

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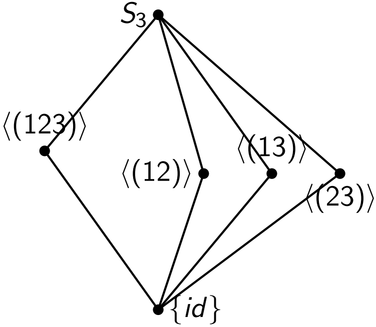

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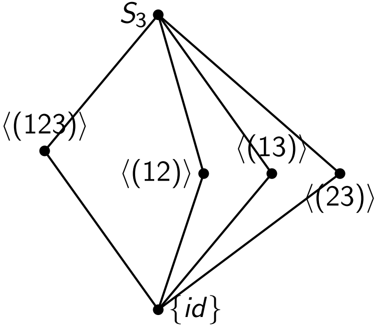

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- $L(G)$ forms together with $\subseteq$ an ordered set.

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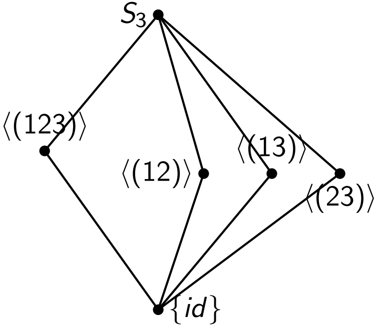

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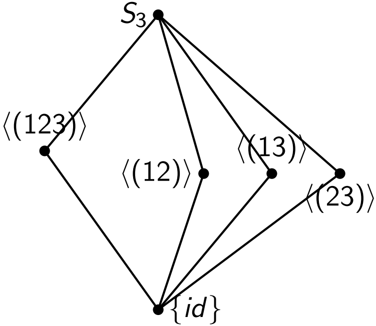

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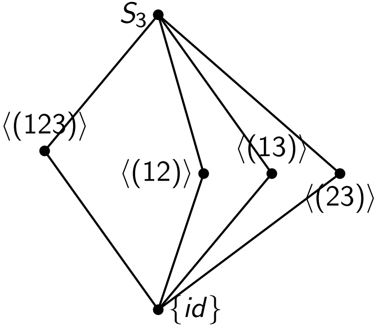

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- $\forall A, B \in L(G)$: $A \cap B \in L(G)$ infimum.
- $\forall A, B \in L(G)$: $\langle A, B\rangle \in L(G)$ supremum.

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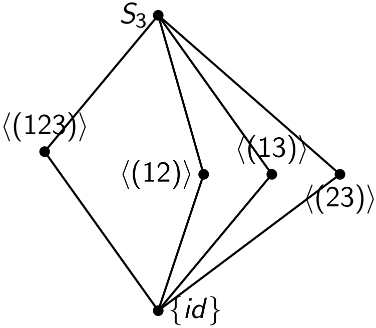

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Altogether $L(G)$ is a lattice.

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- Lattice of natural divisors of a natural number.

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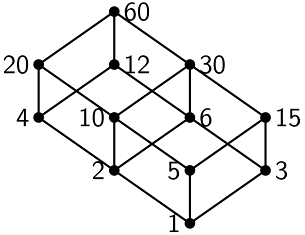

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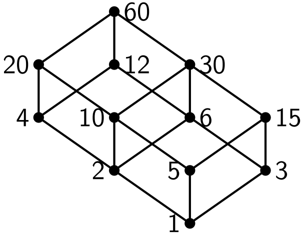

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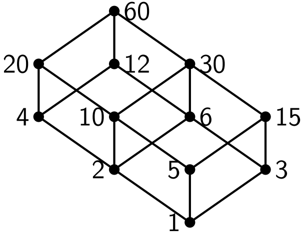

- The Boolean algebra.
  $x \leq y \Leftrightarrow x = x \wedge y$
  $x \wedge y$ is the infimum and
  $x \vee y$ is the supremum.

Examples of subgroup lattices

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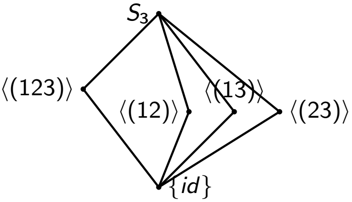
- 

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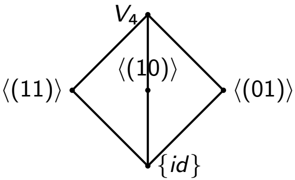
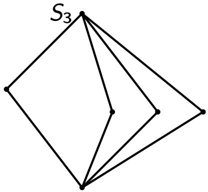
- 

- 

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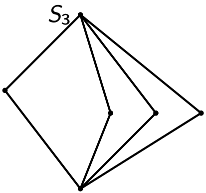
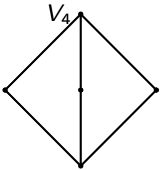
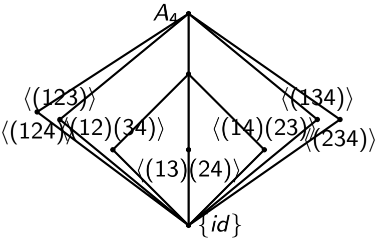
- 

- 

- 

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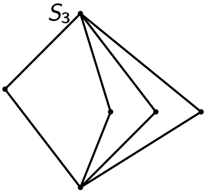
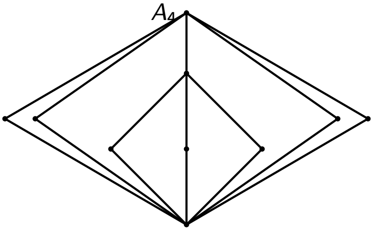
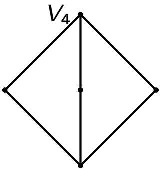
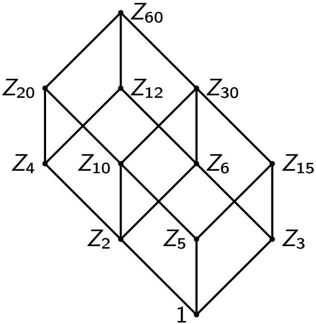
- 

- 

- 

- 

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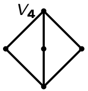

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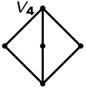

- Recent research. Under what conditions is it possible to describe the index of a subgroup in a subgroup lattice? (subgroup lattice index problem)
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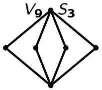

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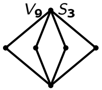

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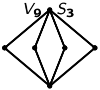

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    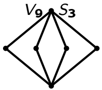

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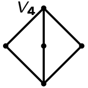
    The finite group G is cyclic if and only if $L(G)$ does not contain a sublattice isomorphic to $L(V_4)$.



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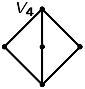

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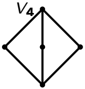

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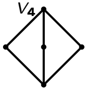

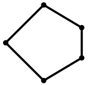

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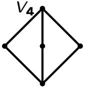
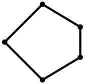


  - If G is finite a group, then $L(G)$ is modular if and only if $L(G)$ has no sublattice isomorphic to
    $\Leftrightarrow$ G is a direct product of P^*-groups and modular p-groups with relatively prime orders. (Kenkichi Iwasawa, 1941)



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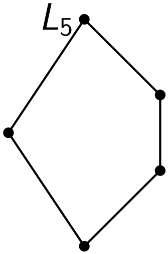

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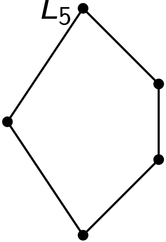
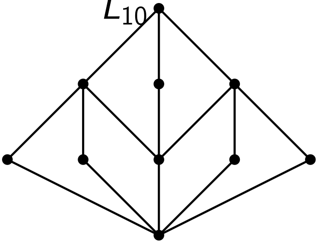
L_5



or

L_{10}



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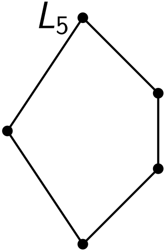

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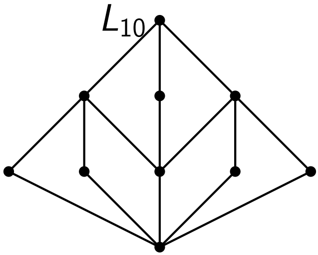

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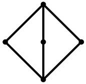
Let L be defined via 

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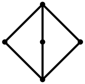
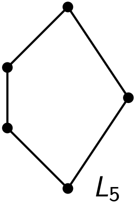
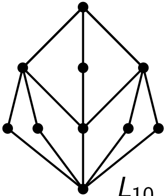
Let L be definied via 

Then a finite group is L-free if and only if it is cyclic.

We are interested in finite L-free groups for L one of:



L_5



L_{10}

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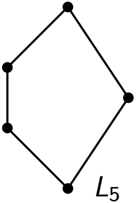

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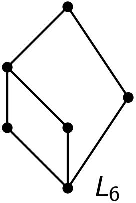
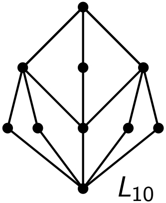


L_6



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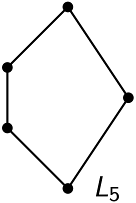

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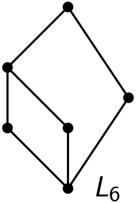

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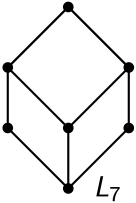
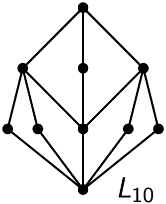


L_7



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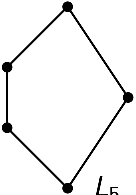

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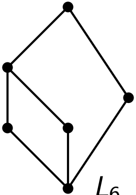

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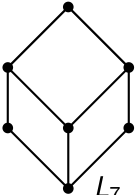

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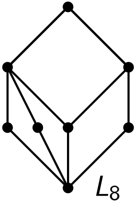
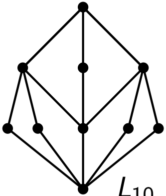


L_8



L_{10}

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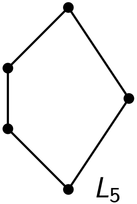

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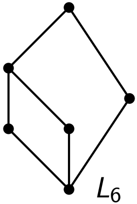
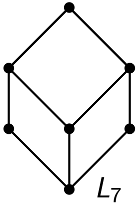
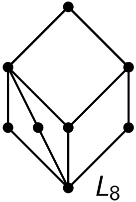


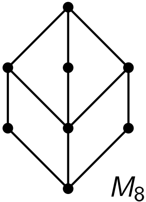
L_6



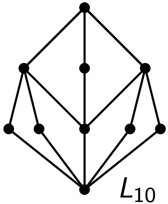
L_7



L_8

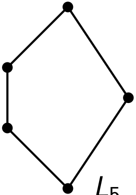


M_8

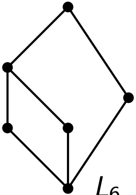


L_{10}

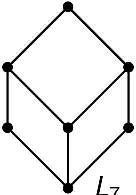
We are interested in finite L-free groups for L one of:



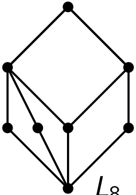
L_5



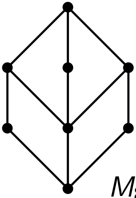
L_6



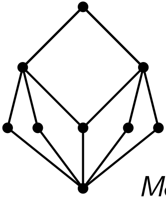
L_7



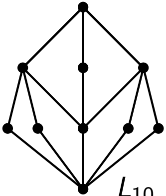
L_8



M_8

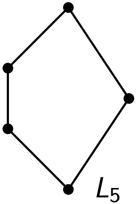


M_9

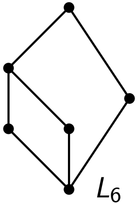


L_{10}

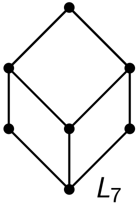
We are interested in finite L-free groups for L one of:



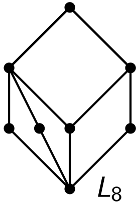
L_5



L_6



L_7



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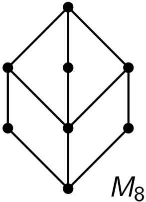
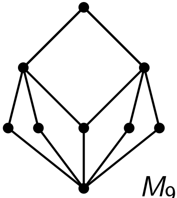
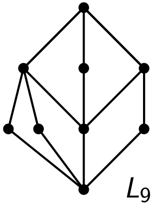


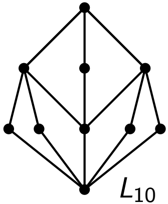
M_8



M_9

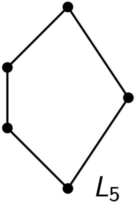


L_9

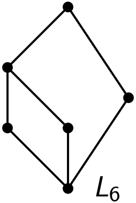


L_{10}

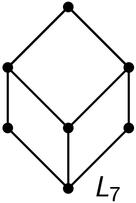
We are interested in finite L-free groups for L one of:



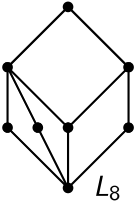
L_5



L_6

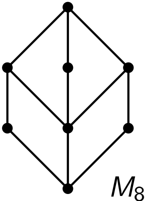


L_7

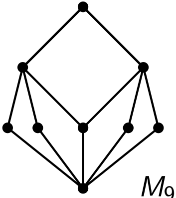


L_8

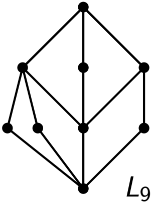
K. Iwasawa
1941



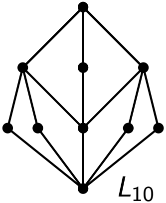
M_8



M_9

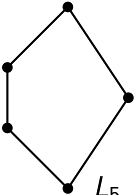


L_9



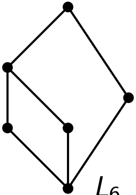
L_{10}

We are interested in finite L-free groups for L one of:



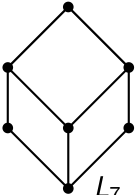
L_5

K. Iwasawa
1941



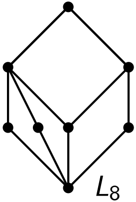
L_6

R. Schmidt
2003

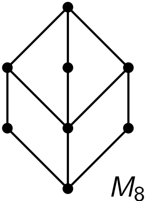


L_7

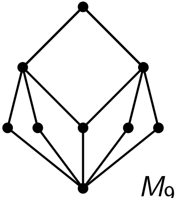
R. Schmidt
2003



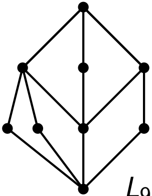
L_8



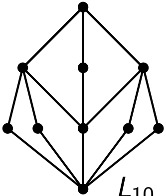
M_8



M_9

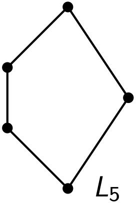


L_9



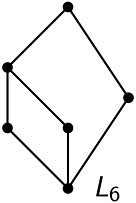
L_{10}

We are interested in finite L-free groups for L one of:



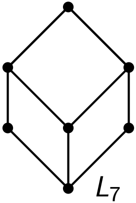
L_5

K. Iwasawa
1941



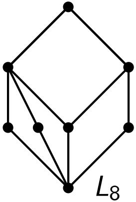
L_6

R. Schmidt
2003



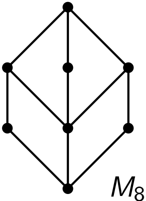
L_7

R. Schmidt
2003

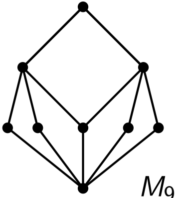


L_8

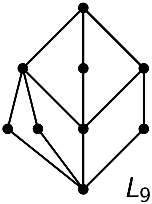
S. Andreeva
and R. Schmidt
2008



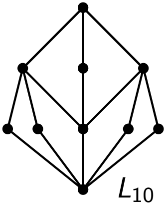
M_8



M_9

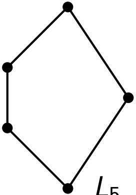


L_9

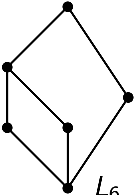


L_{10}

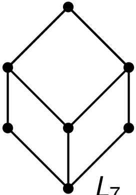
We are interested in finite L-free groups for L one of:


L_5

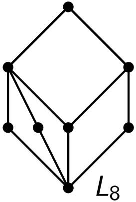
K. Iwasawa
1941


L_6

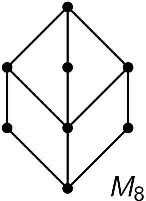
R. Schmidt
2003


L_7

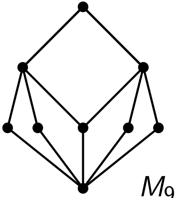
R. Schmidt
2003

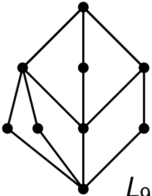

L_8

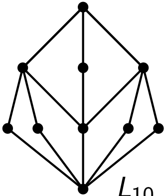
S. Andreeva
and R. Schmidt
2008


M_8

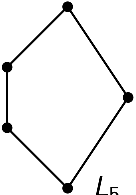
I. Toborg 2010


M_9


L_9

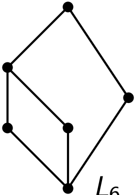

L_{10}

We are interested in finite L-free groups for L one of:



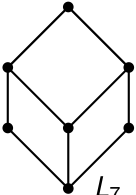
L_5

K. Iwasawa 1941



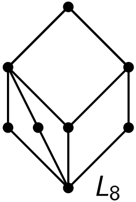
L_6

R. Schmidt 2003



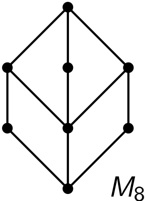
L_7

R. Schmidt 2003



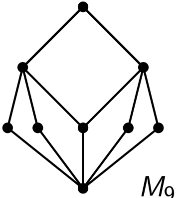
L_8

S. Andreeva and R. Schmidt 2008



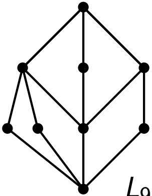
M_8

I. Toborg 2010

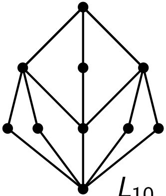


M_9

J. Pölzing and R. Waldecker 2013

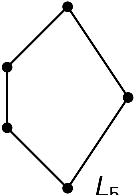


L_9



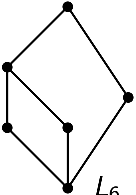
L_{10}

We are interested in finite L-free groups for L one of:



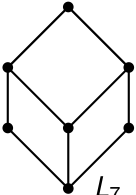
L_5

K. Iwasawa 1941



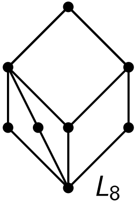
L_6

R. Schmidt 2003



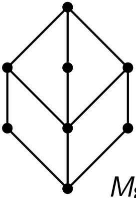
L_7

R. Schmidt 2003



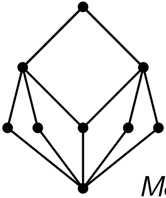
L_8

S. Andreeva and R. Schmidt 2008



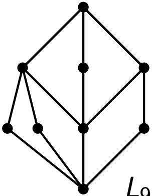
M_8

I. Toborg 2010



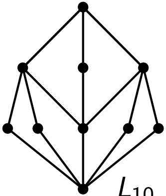
M_9

J. Pölzing and R. Waldecker 2013



L_9

work in progress

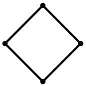


L_{10}

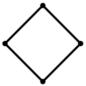
work in progress

Babyexample

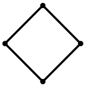
Babyexample

Let G be a finite group and L be defined via 

Babyexample

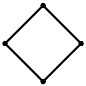
Let G be a finite group and L be defined via 
Then G is L-free if and only if G is cyclic from prime power order.

Babyexample

Let G be a finite group and L be defined via 
Then G is L-free if and only if G is cyclic from prime power order.

Proof

Babyexample

Let G be a finite group and L be defined via 
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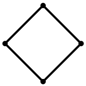
Let G be a finite group and L be defined via 
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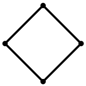
Let G be a finite group and L be defined via 
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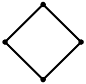
Let G be a finite group and L be defined via 
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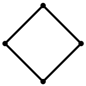
Let G be a finite group and L be defined via 
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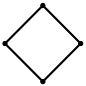
Let G be a finite group and L be defined via 
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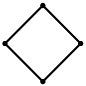
Let G be a finite group and L be defined via 
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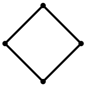
Let G be a finite group and L be defined via 
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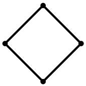
Let G be a finite group and L be defined via 
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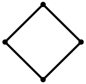
Let G be a finite group and L be defined via 
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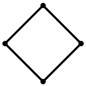
Let G be a finite group and L be defined via 
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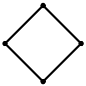
Let G be a finite group and L be defined via 
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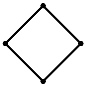
Let G be a finite group and L be defined via 
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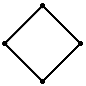
Let G be a finite group and L be defined via 
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Dziękuję bardzo!

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If you have any questions, please feel free to talk to me.