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Wypukłość, ortogonalność a iteracje operatorów różnicowych

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Difference operators

It is well-known that the functional equation

$$\Delta_h^{n+1}\varphi(x) = 0\,, \tag{1}$$

where Δ_h^p stands for the ***p*—th iterate** of the difference operator

$$\boldsymbol{\Delta_h\varphi(x) := \varphi(x+h) - \varphi(x),}$$

of *polynomial functions* characterizes the usual polynomials of at most $n-$th degree in the class of continuous functions $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$. Continuous solutions $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ of the functional inequality

$$\Delta_h^{n+1}\varphi(x) \geqslant 0\,, \tag{2}$$

where $x \in \mathbb{R}, h \in (0,\infty)$, are just $C^{n-1}-$functions whose derivatives $\varphi^{(n-1)}$ are convex (see e.g. M. Kuczma [8, Chapter XV]). Therefore, the solutions to (2) are used to be called $n-$*convex functions.* For $n = 1$ inequality (2) states that

$$\varphi\left(\frac{x+y}{2}\right) \leqslant \frac{\varphi(x)+\varphi(y)}{2}\,, \quad x, y \in \mathbb{R},$$

which is the functional inequality defining *Jensen-convex* functions. Motivated by this fact, in what follows, we shall be using the operator

$$\delta_y^n\varphi(x) := \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\varphi\left((1-\frac{j}{n+1})x + \frac{j}{n+1}y\right)\,,$$

instead of Δ_h^{n+1}. We have

$$\delta_y^n \varphi(x) = \Delta^{n+1}_{\frac{y-x}{n+1}} \varphi(x) \ ;$$

thus φ is $n-$convex (resp. $n-$concave) if and only if

$$x \leqslant y \implies \delta_y^n \varphi(x) \geqslant 0 \,, \tag{3}$$

(resp.

$$x \leqslant y \implies \delta_y^n \varphi(x) \leqslant 0 \ ). \tag{3'}$$

It is not hard to check that, for odd n's, condition (3) is equivalent to the following inequality

$$\delta_y^n \varphi(x) \geqslant 0 \,. \tag{4}$$

An interesting and exhaustive study of the class of *delta-convex* mappings (yielding a generalization of functions which are representable as a difference of two convex functions) has been given by L. Veselý and L. Zajiček [14]. Their definition of delta-convexity reads as follows:

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two real normed linear spaces and let D be a nonempty open and convex subset of X. A map $F : D \longrightarrow Y$ is termed delta-convex provided that there exists a continuous convex functional $f : D \longrightarrow \mathbb{R}$ such that $f + y^* \circ F$ is continuous and convex for any member y^* of the space Y^* dual to Y with $\| y^* \| = 1$. If this is the case then F is called to be *controlled* by f or F is a delta-convex mapping with a *control function* f.

It turns out that a *continuous* function $F : D \longrightarrow Y$ is a delta-convex mapping controlled by a *continuous* function $f : D \longrightarrow \mathbb{R}$ if and only if the functional inequality

$$\| F\left(\frac{x+y}{2}\right) - \frac{F(x)+F(y)}{2} \| \ \leqslant \ \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \tag{5}$$

is satisfied for all $x, y \in D$ (see Corollary 1.18 in [14]).

In a natural way, this leads to the following

Definition. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two real normed linear spaces and let $n \in \mathbb{N}$. Assume that we are given a proper cone $C \subset X$ and a nonempty open and convex set $D \subset X$. Write $x \leqslant y$ whenever $y - x \in C$. A mapping $F : D \longrightarrow Y$ is termed *delta-convex of* $n-$*th order* if and only if there exists a (control) functional $f : D \longrightarrow \mathbb{R}$ such that for all $x, y \in D$ one has

$$x \leqslant y \Longrightarrow \| \delta_y^n F(x) \| \leqslant \delta_y^n f(x) \; . \tag{6}$$

In the case where n is odd and the order relation $\leqslant$ is linear (or, what amounts the same, $C \cup (-C) = X$) relation (6) is equivalent to

$$\| \delta_y^n F(x) \| \leqslant \delta_y^n f(x) \; , \tag{7}$$

and the order structure in X is not needed any more; in particular, for $n = 1$ inequality (7) reduces to (5). In the case where n is even, the restriction $x \leqslant y$ in (6) turns out to be essential. Indeed, having just (7) for every $x, y \in D$ and for an even $n \in \mathbb{N}$ we obviously get (4) (with $\varphi = f$) for all $x, y \in D$ whence, by interchanging x and y, we obtain

$$\delta_y^n f(x) \leqslant 0 \, .$$

Consequently, f and a fortiori F would have to be polynomial mappings which are defined in much simpler way (see (1)).

Examples

Now, we are going to present some examples of delta-convex mappings of $n-$th order. We begin with

Proposition 1. *In the case where* $Y = \mathbb{R}$ *a function* $F : D \longrightarrow \mathbb{R}$ *is delta-convex of* $n-$*th order if and only if* F *is a difference of two* $n-$*convex functions.*

Proof. Assume $f : D \longrightarrow \mathbb{R}$ to be a control function for F. Then, for all $x, y \in D$ we have

$$x \leqslant y \Longrightarrow |\,\delta_y^n F(x)| \leqslant \delta_y^n f(x)\ .$$

Put $\varphi_1 := \frac{1}{2}(F + f)$ and $\varphi_2 := \frac{1}{2}(f - F)$. In view of the linearity of the operator δ_y^n, the latter inequality says that both φ_1 and φ_2 are solutions to (3) on D, i.e. both are $n-$convex functions. It remains to observe that $F = \varphi_1 - \varphi_2$.

Conversely, let $F = \varphi_1 - \varphi_2$, where φ_1 and φ_2 are solutions to (3) on D. Then, setting $f := \varphi_1 + \varphi_2$ we infer that both $f - F$ and $F + f$ satisfy condition (3) as well, whence, for every $x, y \in D$,

$$x \leqslant y \Longrightarrow |\,\delta_y^n F(x)| \leqslant \delta_y^n f(x)\ ,$$

which completes the proof.

Proposition 2. *Every real $C^{n+1}-$function on an open interval in $\mathbb{R}$ is delta-convex of $n-$th order.*

Proof. Let $F : (a,b) \longrightarrow \mathbb{R}$ be a $C^{n+1}-$function. Then $\varphi := F^{(n-1)}$ is a C^2-function; thus, $\varphi = \varphi_1 - \varphi_2$, where $\varphi_1, \varphi_2 : (a,b) \longrightarrow \mathbb{R}$ are both convex (see Ch. O. Kiselman [7, Proposition 3.1]). Consequently, taking any functions $\psi_1, \psi_2 : (a,b) \longrightarrow \mathbb{R}$ such that $\psi_1^{(n-1)} = \varphi_1$ and $\psi_2^{(n-1)} = \varphi_2$ we have $F = \psi_1 - \psi_2 + p_{n-2}$ where p_{n-2} is a polynomial of at most $(n-2)-$th degree restricted to (a,b). Obviously, the functions $\psi_3 := \psi_1 + p_{n-2}$ and ψ_2 are both of class C^{n-1} on (a,b) with convex $(n-1)-$derivatives. Hence ψ_3 and ψ_2 are both $n-$convex (see e.g. M. Kuczma [8, Theorem 15.8.4]) and $F = \psi_3 - \psi_2$, which was to be proved.

Proposition 3 ($n-$th order delta-convexity of the Nemyckii operator). *Let $\Omega \subset \mathbb{R}^k$ be a Lebesgue measurable set of positive Lebesgue measure ℓ_k, $1 \leqslant p < \infty$, and let $\varphi, \psi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be such that*

a) *there exist nonnegative constants c_1, c_2 and functions $w_1, w_2 \in L^1(\Omega)$ such that*

$$|\varphi(t, \cdot)| \leqslant w_1(t) + c_1 |\cdot|^p$$

and

$$|\psi(t, \cdot)| \leqslant w_2(t) + c_2 |\cdot|^p$$

for ℓ_k-almost all $t \in \Omega$;

b) *for ℓ_k-almost all $t \in \Omega$ the function $\varphi(t, \cdot)$ is delta-convex of $n-$th order controlled by $\psi(t, \cdot)$;*

c) *for every $s \in \mathbb{R}$ the sections $\varphi(\cdot, s)$ and $\psi(\cdot, s)$ are Lebesgue measurable.*

Then the Nemyckii operator F given by the formula $F(x)(t) := \varphi(t, x(t))$, $t \in \Omega$, $x \in L^p(\Omega)$, acts from $L^p(\Omega)$ (equiped with the cone of all nonnegative functions) into $L^1(\Omega)$ and is delta-convex of $n-$th order with the control functional $f : L^p(\Omega) \longrightarrow \mathbb{R}$ given by the formula

$$f(x) := \int_\Omega \psi(\cdot, x(\cdot))\, d\ell_k\ , \quad x \in L^p(\Omega)\,.$$

Proof. First we observe that the Nemyckii operators: F and $G(x)(t) := \psi(t, x(t))$, $t \in \Omega$, $x \in L^p(\Omega)$, act (continuously) from $L^p(\Omega)$ into $L^1(\Omega)$ (see M. M. Vajnberg [13] and L. Veselý & L. Zajiček [14]). Now, to check (6), fix arbitrarily $x, y \in L^p(\Omega)$, $x \leqslant y$, and put

$$z_j := \left(1 - \frac{j}{n+1}\right) x + \frac{j}{n+1} y \quad \text{for} \quad j \in \{0, 1, ..., n+1\}.$$

Then

$$\begin{aligned}
\| \delta_y^n F(x) \| &= \int_\Omega \left| \left(\delta_y^n F(x) \right)(t) \right| \, d\ell_k(t) \\
&= \int_\Omega \left| \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} F(z_j)(t) \right| \, d\ell_k(t) \\
&= \int_\Omega \left| \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \varphi\left(t, z_j(t)\right) \right| \, d\ell_k(t) \\
&= \int_\Omega \left| \delta_{y(t)}^n \varphi\left(t, x(t)\right) \right| \, d\ell_k(t) \leqslant \int_\Omega \delta_{y(t)}^n \psi\left(t, x(t)\right) \, d\ell_k(t) \\
&= \int_\Omega \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \psi\left(t, z_j(t)\right) \, d\ell_k(t) \\
&= \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \int_\Omega \psi\left(t, z_j(t)\right) \, d\ell_k(t) \\
&= \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(z_j) = \delta_y^n f(x),
\end{aligned}$$

and the proof is completed.

Proposition 4 ($n-$th order delta-convexity of the Hammerstein operator). *Under the assumptions of* Proposition 3 *if, additionally, $K : \mathbb{R}\times\Omega \longrightarrow \mathbb{R}$ is a Lebesgue measurable function such that for some $c \geqslant 0$*

$$\int_{\mathbb{R}} |K(s,t)| d\ell_1(s) \ \leqslant \ c$$

for ℓ_k-almost all $t \in \Omega$, then the Hammerstein operator

$$H(x) := \int_{\Omega} K(\cdot,t)\varphi(t,x(t))\, d\ell_k(t)$$

is well defined on $L^p(\Omega)$ and yields a delta-convex mapping with the control functional $g : L^p(\Omega) \longrightarrow \mathbb{R}$ given by the formula

$$g(x) := c \cdot \int_{\Omega} \psi(\cdot, x(\cdot))\, d\ell_k \ , \quad x \in L^p(\Omega)\,.$$

Proof. We argue like in [14, Proposition 6.9]. It is not hard to check that the linear operator

$$T(z)(s) := \int_{\Omega} K(s,t)\, z(t)\, d\ell_k(t)\,, \quad z \in L^1(\Omega),\ s \in \mathbb{R}\,,$$

acts continuously from $L^1(\Omega)$ into $L^1(\mathbb{R})$ and $\|\,T\| \leqslant c$. Moreover, $H = T \circ F$, where F is the Nemyckii operator spoken of in Proposition 3. In view of the (just established) $n-$th order delta-convexity of F, for arbitrarily fixed $x, y \in L^p(\Omega)$, $x \leqslant y$, we get

$$\begin{aligned} \|\,\delta_y^n H(x)\| \ &= \ \|\,\delta_y^n\,(T\circ F)\,(x)\| = \|\,T\left(\delta_y^n F(x)\right)\| \\ &\leqslant \ \|\,T\|\,\|\,\delta_y^n F(x)\| \ \leqslant \ c\,\delta_y^n f(x) \ = \ \delta_y^n (c\,f)(x)\ , \end{aligned}$$

which was to be proved.

Equivalent conditions

The following result establishes necessary and sufficient conditions for a given map to be delta-convex of $n-$th order.

Theorem 1. *Under the assumptions of the* Definition *the following conditions are pairwise equivalent:*

(i) *F is a delta-convex mapping controlled by f ;*

(ii) *for every $y^* \in Y^*$ the function $y^* \circ F - \|y^*\| \cdot f$ is $n-$concave;*

(iii) *for every $y^* \in Y^*$ the function $y^* \circ F + \|y^*\| \cdot f$ is $n-$convex;*

(iv) *for every $y^* \in Y^*$, $\|y^*\| = 1$, the function $y^* \circ F + f$ is $n-$convex ;*

(v) *for every $y^* \in Y^*$, $\|y^*\| = 1$, the function $y^* \circ F - f$ is $n-$concave;*

(vi) *for every choice of rationals $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1} = 1$ and for every pair $x, y \in D$, $x \leqslant y$, one has*

$$(8) \qquad \|\sum_{j=0}^{n+1}(-1)^{n+1-j}V(\lambda_0, \lambda_1, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots \lambda_n, \lambda_{n+1}) \times F((1-\lambda_j)x + \lambda_j y)\| \leqslant \sum_{j=0}^{n+1}(-1)^{n+1-j}V(\lambda_0, \lambda_1, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots \lambda_n, \lambda_{n+1}) \times f((1-\lambda_j)x + \lambda_j y) \ ,$$

where V stands for the Vandermonde's determinant of the variables considered.

If, moreover, the function $D \ni x \longmapsto \|F(x)\| + |f(x)| \in \mathbb{R}$ *is upper bounded on a second category Baire subset of* D*, then each of these conditions is equivalent to*

(vii) *for every choice of real numbers* $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1} = 1$ *and for every pair* $x, y \in D, \ \ x \leqslant y,$ *one has* (8) *.*

Proof. (i) implies (ii). Let $F : D \longrightarrow Y$ be an $n-$th order delta-convex mapping with a control functional $f : D \longrightarrow \mathbb{R}$. This means that relation (6) holds true for all $x, y \in D$. Fix arbitrarily a nontrivial continuous linear functional $y^* : Y \longrightarrow \mathbb{R}$. Obviously, it follows from (6) that

$$\frac{y^*}{\|y^*\|}\left(\delta_y^n F(x)\right) \leqslant \delta_y^n f(x)\,,$$

whenever $x, y \in D, x \leqslant y,$ whence, in view of the linearity of the operator δ_y^n, we infer that

$$\delta_y^n \left(y^* \circ F - \|y^*\|\, f\right)(x) \leqslant 0$$

provided that $x, y \in D, x \leqslant y$.

(ii) implies (iii). Replace y^* by $-y^*$ in (ii).

(iii) implies (iv). Trivial.

(iv) implies (v). Replace y^* by $-y^*$ in (iv).

(v) implies (vi). Fix arbitrarily points $x, y \in D, x \leqslant y,$ rational numbers $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1} = 1$ and a continuous real functional $y^* \in Y^*, \|y^*\| = 1$. On account of (v), the function $\varphi := y^* \circ F - f$ is $n-$concave , i.e.

$$\delta_y^n \varphi(x) \leqslant 0\,.$$

Since the points

(9) $x_j := x + \lambda_j(y - x) = (1 - \lambda_j)x + \lambda_j y\,, \quad j \in \{0, 1, ..., n+1\}\,,$

divide rationally the segment $[x, y]$, in virtue of T. Popoviciu's result from [9] (see also: M. Kuczma [8] and R. Ger [3], [4]) we get

$$\sum_{j=0}^{n+1} (-1)^{n+1-j} V(\lambda_0, \lambda_1, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots \lambda_n, \lambda_{n+1})\, \varphi(x_j) \leqslant 0\,,$$

i.e.

$$y^* \left(\sum_{j=0}^{n+1} (-1)^{n+1-j} V(\lambda_0, \lambda_1, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots \lambda_n, \lambda_{n+1})\, F(x_j) \right)$$
$$\leqslant \sum_{j=0}^{n+1} (-1)^{n+1-j} V(\lambda_0, \lambda_1, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots \lambda_n, \lambda_{n+1})\, f(x_j)\,,$$

whence, in view of the arbitrarness of y^*, we get (vi).

(vi) implies (i). An elementary calculation shows that the number

$$\alpha_n := \frac{1}{\binom{n+1}{j}} V\left(0, \frac{1}{n+1}, ..., \frac{j-1}{n+1}, \frac{j+1}{n+1}, ..., \frac{n}{n+1}, 1\right)$$

is positive and does not depend upon $j \in \{0, 1, ..., n+1\}$. Therefore, having arbitrarily fixed $x, y \in D, x \leqslant y$, and putting $\lambda_j := \frac{j}{n+1}\,, j \in \{0, 1, ..., n+1\}$, in (vi), we get

$$\| \sum_{j=0}^{n+1} (-1)^{n+1-j} \alpha_n \binom{n+1}{j} F\left((1 - \frac{j}{n+1})x + \frac{j}{n+1} y\right) \|$$
$$\leqslant \sum_{j=0}^{n+1} (-1)^{n+1-j} \alpha_n \binom{n+1}{j} f\left((1 - \frac{j}{n+1})x + \frac{j}{n+1} y\right)\,,$$

which gives (i).

To prove the last part of the theorem assume (i) and take an arbitrary functional $y^* \in Y^*, \|y^*\| = 1$. By means of (iv), the function $\varphi := y^* \circ F + f$ is $n-$convex. Since

$$|\,\varphi(x)| \leqslant \|\,F(x)\| + |\,f(x)|\,, \quad x \in D\,,$$

we infer that both f and φ are $n-$convex functions bounded on a second category Baire subset of D and hence continuous (see R. Ger [5]). Consequently, F is weakly continuous. Since (i) implies (vi), we have (8) for every choice of rational numbers $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1} = 1$ and for every pair $x, y \in D, \ x \leqslant y$. Thus

$$\tag{10} \begin{aligned} &|\sum_{j=0}^{n+1}(-1)^{n+1-j}V(\lambda_0, \lambda_1, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots \lambda_n, \lambda_{n+1})\,(y^* \circ F)(x_j)\,| \\ &\qquad \leqslant \sum_{j=0}^{n+1}(-1)^{n+1-j}V(\lambda_0, \lambda_1, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots \lambda_n, \lambda_{n+1})\,f(x_j)\ , \end{aligned}$$

where the x_j's are defined by (9). In view of the continuity of f, $y^* \circ F$ and V inequality (10) holds true for all real numbers $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1} = 1$, and condition (vii) is proved. Since the converse implication is trivial, the proof has been completed.

Stability results

The following result was obtained in [5]: under some mild regularity condition upon the control function f, for every solution F of inequality (5) there exists an affine mapping A (i.e. a polynomial function of the first order) and a point x_o such that $\| F(x) - A(x)\| \leqslant f(x) - f(x_o)$ for all x's from the domain of F. In what follows we are going to extend this result to the case of polynomial mappings of higher orders.

Theorem 2. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two real normed linear spaces and let n be a fixed odd positive integer. Assume that we are given a nonempty open and convex set $D \subset X$. If $F : D \longrightarrow Y$ and $f : D \longrightarrow \mathbb{R}$ are two $C^{n+1}-$mappings such that inequality*

$$\| \delta_y^n F(x)\| \leqslant \delta_y^n f(x) \; , \tag{7}$$

holds true for all $x, y \in D$, then for every $x_o \in D$ there exist $C^\infty-$polynomial functions $Q : D \longrightarrow Y$ and $q : D \longrightarrow \mathbb{R}$ of at most $n-$th order such that $F(x_o) = Q(x_o)$, $\;f(x_o) = q(x_o)$, and

$$\| F(x) - Q(x)\| \leqslant f(x) - q(x)$$

for all $x \in D$.

Proof. Let us recall first, that for every $p-$additive and symmetric mapping
$M : X^p \longrightarrow Y$ its diagonalization $m : X \longrightarrow Y$ given by the formula

$$m(x) := M(\underbrace{x, x, ..., x}_{p \text{ times}}),\ x \in X,$$

(*a monomial function of* $p-$*th order*) has the following property (see e.g. L. Székelyhidi [7] or L. M. Kuczma [8]): for all $x, h \in X$ one has

$$\Delta_h^k m(x) = \begin{cases} k!\ m(h) & \text{for } k = p \\ 0 & \text{for } k > p\ . \end{cases}$$

In what follows, $D^k g(x)$ will stand for the $k-$th Fréchet differential of a map g; plainly, $D^k g(x)$ is a $k-$additive (actually, $k-$linear) and symmetric mapping. The monomial generated by $D^k g(x)$ will be denoted by $d^k g(x)$.

Fix arbitrarily an $x_o \in D$. For each $y^* \in Y^*$, $\| y^* \| = 1$, the $C^{n+1}-$function $\varphi := y^* \circ F - f$ is (unconditionally) $n-$concave, whence for any $x, y \in D$, we get

$$\begin{aligned} 0 \geqslant \delta_y^n \varphi(x) &= \Delta_{\frac{y-x}{n+1}}^{n+1} \varphi(x) = \Delta_{\frac{y-x}{n+1}}^{n+1} \left(\sum_{k=0}^{n} \frac{1}{k!} d^k \varphi(x_o)(x - x_o) \right. \\ &\left. + \frac{1}{(n+1)!} d^{n+1} \varphi\left(x_o + \theta(x - x_o)\right)(x - x_o) \right) \\ &= \frac{1}{(n+1)!} \Delta_{\frac{y-x}{n+1}}^{n+1} \left(d^{n+1} \varphi\left(x_o + \theta(x - x_o)\right)(x - x_o) \right) \\ &= d^{n+1} \varphi\left(x_o + \theta(x - x_o)\right) \left(\frac{y - x}{n+1} \right), \end{aligned}$$

with some $\theta \in (0, 1)$. In particular, taking here $y := x_o$ we infer that

$$d^{n+1} \varphi\left(x_o + \theta(x - x_o)\right)(x - x_o) \leqslant 0\,.$$

Consequently, for every $x \in D$ we have

$$
\begin{aligned}
y^* (F(x)) - f(x) = \varphi(x) = \sum_{k=0}^{n} \frac{1}{k!} d^k \varphi(x_o)(x - x_o) \\
+\frac{1}{(n+1)!}\, d^{n+1}\varphi\left(x_o + \theta(x - x_o)\right)(x - x_o) \\
\leqslant \sum_{k=0}^{n} \frac{1}{k!} d^k \varphi(x_o)(x - x_o) \\
= y^* \left(\sum_{k=0}^{n} \frac{1}{k!} d^k F(x_o)(x - x_o) \right) - \sum_{k=0}^{n} \frac{1}{k!} d^k f(x_o)(x - x_o)\ .
\end{aligned}
$$

Thus, setting

$$
Q(x) := \sum_{k=0}^{n} \frac{1}{k!} d^k F(x_o)(x - x_o) \quad \text{and} \quad q(x) := \sum_{k=0}^{n} \frac{1}{k!} d^k f(x_o)(x - x_o)
$$

for $x \in X$, we get two $C^\infty-$polynomial functions such that $F(x_o) = Q(x_o)$, $f(x_o) = q(x_o)$, and

$$
y^* (F(x) - Q(x)) \leqslant f(x) - q(x)
$$

for all $x \in D$ and all $y^* \in Y^*$ with $\| y^* \| = 1$. This implies that

$$
\| F(x) - Q(x) \| \leqslant f(x) - q(x)
$$

for all $x \in D$, which was to be proved.

Corollary 1. *Under the assumptions of* Theorem 2, *with D being a ball $B(x_o, \varepsilon)$ centered at x_o and having radius $\varepsilon > 0$, if we have*

$$\| D^{n+1} f(z)\| \leqslant c \qquad for \;\; z \in B(x_o, \varepsilon)\,,$$

then

$$\| F(x) - Q(x)\| \leqslant \frac{c}{(n+1)!} \| x - x_o\|^{n+1}$$

for all $x \in B(x_o, \varepsilon)$.

Proof. As a matter of fact, we have proved that

$$\| F(x) - Q(x)\| \leqslant \frac{c}{(n+1)!} d^{n+1} f\left(x_o + \theta(x - x_o)\right)(x - x_o)$$

whence

$$\| F(x) - Q(x)\| \leqslant \frac{c}{(n+1)!} \| D^{n+1} f\left(x_o + \theta(x - x_o)\right) \| \, \| x - x_o\|^{n+1}$$

for all $x \in B(x_o, \varepsilon)$.

Corollary 2 (on supporting polynomial functionals). *Let $(X, \| \cdot \|)$ a real normed linear space and let n be a fixed odd positive integer. Assume that we are given a nonempty open and convex set $D \subset X$ and a $C^{n+1}-$functional $f : D \longrightarrow \mathbb{R}$ such that inequality*

$$\delta_y^n f(x) \geqslant 0$$

holds true for all $x, y \in D$. Then, for every $x_o \in D$, there exists a $C^{\infty}-$polynomial functional $q : D \longrightarrow \mathbb{R}$ of at most $n-th$ order such that $f(x_o) = q(x_o)$ and

$$f(x) \geqslant q(x)$$

for all $x \in D$.

Proof. Take $F := 0$ in Theorem 2.

Remark. To avoid a reduction to polynomial functions in the case of even n's, the $n-$convexity is defined by (3). However, in such a case, for even n's, even the Corollary is no longer valid. To see this, consider the cubic function on $\mathbb{R}$: $f(x) = x^3$, $x \in \mathbb{R}$. We have $\delta_y^2 f(x) = \frac{2}{9}(y-x)^3 \geqslant 0$ whenever $x \leqslant y$, $x, y \in \mathbb{R}$, but, obviously, there exists no quadratic polynomial supporting f.

T-orthogonality

The so called Suzuki's property of isosceles trapezoids (see F. Suzuki [11]) on the real plane π reduced to the case of an (anticlockwise oriented) rectangle $ABCD \subset \pi$ states that for any point $S \in \pi$ the distances between S and the vertices of the rectangle satisfy the relationship: $AS^2 - BS^2 = DS^2 - CS^2$. This observation expressed in terms of vectors from a given real normed linear space $(X, \|\cdot\|)$, $\dim X \geqslant 2$, has led C. Alsina, P. Cruells and M. S. Tomás [1] to the following very interesting orthogonality relation $\perp^T \subset X \times X$: we say that two vectors $x, y \in X$ are $T-$*orthogonal* and write $x \perp^T y$ if and only if for every vector $z \in X$ one has

$$\|z - x\|^2 + \|z - y\|^2 = \|z\|^2 + \|z - (x + y)\|^2.$$

It turns out that, among others, any two $T-$orthogonal vectors $x, y \in X$ are also

- orthogonal in the classical sense: $(x|y) = 0$ provided that the norm $\|\cdot\|$ comes from an inner product $(\cdot|\cdot)$
- orthogonal in the sense of Pythagoras: $\|x + y\|^2 = \|x\|^2 + \|y\|^2$
- orthogonal in the sense of James: $\|x + y\| = \|x - y\|$
- orthogonal in the sense of Birkhoff: $\|x + \lambda y\| \geqslant \|x\|\,,\ \lambda \in \mathbb{R}.$

If so, one might conjecture that $T-$orthogonality must simply coincide with the classical orthogonality coming from an inner product structure. That is really the case in two-dimensional spaces; however, such a conjecture fails to be true in normed linear spaces of higher dimensions. Is there any deeper explanation of that phenomenon?

To proceed, observe first that the $T-$orthogonality relation may equivalently be expressed in terms of difference operators.

Indeed, setting

$$\Delta_{h,k} := \Delta_h \circ \Delta_k \quad \text{for} \quad h, k \in X$$

and writing our equation (with fixed $x, y \in X$) in an equivalent form

$$\|z + x + y\|^2 - \|z + x\|^2 - \|z + y\|^2 + \|z\|^2 = 0 \,,$$

valid for every $z \in X$, we see that

$$x \perp^T y \quad \text{if and only if} \quad \Delta_{x,y}\| \cdot \|^2 = 0 \,.$$

This gives rise to study a more general orthogonality relation determined by a fixed real functional φ defined on an Abelian group $(G, +)$. Namely, we shall say that two elements x, y from G are $\varphi-$*orthogonal* and write $x \perp_\varphi y$ if and only if for every element $z \in G$ one has

$$\Delta_{x,y}\varphi(z) = 0 \,.$$

In the case where $(G, +)$ stands for the additive group of a normed linear space $(X, \| \cdot \|)$ and $\varphi := \| \cdot \|^2$, the $\varphi-$orthogonality just defined coincides with $T-$orthogonality introduced and examined by C. Alsina, P. Cruells and M. S. Tomás [1]. In some cases we will admit another Abelian group $(H, +)$ in place of the additive group of all real numbers as the target space of the map φ in question, preserving the name $\varphi-$orthogonality for the corresponding orthogonality relation.

In 1985 J. Rätz [10], slightly modifying an idea of S. Gudder & D. Strawther [6], introduced the notion of an *orthogonality space* in an axiomatic way.

Let X be a real linear space of dimension greater or equal 2 and let $\perp \subset X \times X$ be a binary relation with the following properties:

(a) $x \perp 0$ and $0 \perp x$ for every $x \in X$;

(b) if $x, y \in X \setminus \{0\}$ and $x \perp y$ then x and y are linearly independent;

(c) if $x, y \in X$ and $x \perp y$ then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(d) if P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in (0, \infty)$, then there exists a $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

Then the pair $(X, \perp)$ is termed an orthogonality space. While the first three axioms seem to be unquestionable (observe the lack of symmetry) the last axiom (d) is rather strong. Nevertheless, beside the usual orthogonality in inner product spaces, Birkhoff orthogonality stands for an interesting example to produce an orthogonality space in the sense of Rätz. Even slightly exotic orthogonality relation: $0 \neq x \perp y \neq 0$ if and only if x and y are linearly independent also satisfies this axiomatic system proving that the axioms are not too restrictive.

In what follows we shall answer a natural question to determine functions φ on a given real linear space X such that the pair $(X, \perp_\varphi)$ happens to be a Rätz orthogonality space.

Theorem 3. *Let X be a real linear space with* $\dim X \geqslant 2$ *and let* $\varphi : X \longrightarrow \mathbb{R}$ *be a functional enjoying the property*

$$\sup\{|\varphi(x) + \varphi(-x)| : x \in S\} < \infty,$$

for every segment $S \subset X$. Then the pair $(X, \perp_\varphi)$ *is an orthogonality space if and only if there exists an inner product* $(\cdot|\cdot) : X \times X \longrightarrow \mathbb{R}$ *such that*

$$\perp_\varphi = \{(x, y) \in X \times X : (x|y) = 0\}.$$

Another characterization of inner product space (complementary to this theorem) involving the notion of $\varphi-$orthogonality may be obtained by the requirement that $\perp_\varphi$ *admits diagonals*, i.e. for every two nonzero vectors x, y there exists an $\alpha \in \mathbb{R}$ such that

$$x + \alpha y \perp_\varphi x - \alpha y.$$

In the case of $T-$orthogonality that requirement reduces it to the usual orthogonality in inner product spaces. What about $\varphi-$orthogonality ?

Theorem 4. *Let X be a real linear topological space with* $\dim X \geqslant 2$ *and let* $\varphi : X \longrightarrow \mathbb{R}$ *be a nonzero continuous and even functional with the property:*

$$\text{there exists an } x_0 \in X\backslash\{0\} \quad \text{such that} \quad \varphi(\lambda x_0) = \lambda^2 \varphi(x_0) \text{ for all } \lambda \in (0,$$

Then the corresponding $\varphi-$orthogonality admits diagonals if and only if there exists an inner product $(\cdot|\cdot) : X \times X \longrightarrow \mathbb{R}$ *such that*

$$\perp_\varphi = \{(x, y) \in X \times X : (x|y) = 0\}.$$

Another interesting orthogonality relation has been introduced recently by C. Alsina, J. Sikorska and M. S. Tomás in [2]. Two vectors x and y in a real normed linear space with dim $X \geqslant 2$ are said to be w-orthogonal if and only if

$$\Big\| \|x\|y + \|y\|x \Big\| = \sqrt{2}\, \|x\| \cdot \|y\|,$$

or, alternatively,

$$\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^2 = 2\,,$$

for nonzero vectors x and y. This gives rise to study another more general orthogonality relation determined by a fixed real functional φ defined on a normed real linear space X, $\varphi(x) > 0$ for $x \in X \setminus \{0\}$, $\varphi(0) = 0$, . Namely, we shall say that nonzero vectors $x, y \in X$ are φ-orthogonal if and only if

$$\varphi(\frac{x}{\sqrt{\varphi(x)}} + \frac{y}{\sqrt{\varphi(y)}}) = 2\,.$$

This leads to interesting conditional functional equations; for instance, the equation

$$\varphi(x+y) = \varphi(x) + \varphi(y) \Longrightarrow \varphi(\frac{x}{\sqrt{\varphi(x)}} + \frac{y}{\sqrt{\varphi(y)}}) = 2,$$

expresses the fact that the generalized Pythagorean orthogonality implies the orthogonality just introduced.

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