# ALGEBRAICITY OF SPECIAL $L$-VALUES ATTACHED TO SIEGEL-JACOBI MODULAR FORMS

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In this work we obtain algebraicity results on special $L$-values attached to Siegel-Jacobi modular forms. Our method relies on a generalization of the doubling method to the Jacobi group obtained in our previous work, and on introducing a notion of nearly holomorphicity for Siegel-Jacobi modular forms.

## 1. INTRODUCTION

This paper should be seen as a continuation of our earlier paper [3] on properties of the standard $L$-function attached to a Siegel-Jacobi modular form. Indeed, in [3] we have established various analytic properties (Euler product decomposition, analytic continuation and detection of poles) of the standard $L$-function attached to Siegel-Jacobi modular forms, and in this paper we turn our attention to algebraicity properties of some special $L$-values.

Shintani was the first person who attached an $L$-function to a Siegel-Jacobi modular form which is an eigenfunction of a properly defined Hecke algebra. He initiated the study of its analytic properties by finding an integral representation. His work was left unpublished, but then was took over by Murase [8, 9] and Arakawa [1] who obtained results on the analytic properties of this $L$-function using variants of the doubling method. In our previous work [3] we extended their results to a very general setting: non-trivial level, character and a totally real algebraic number field. For this purpose we applied the doubling method to the Jacobi group, and consequently related Siegel-type Jacobi Eisenstein series to the standard $L$-function. This identity has a further application in the current paper.

Here the starting point of our investigation is a result of Shimura in [11] on the arithmeticity of Siegel-Jacobi modular forms. Namely, if we let $S$ be a positive definite half-integral $l$ by $l$ symmetric matrix, write $M_{k,S}^n$ for the space of Siegel-Jacobi modular forms of weight $k$ and index $S$ (see next section for a definition), and of any congruence subgroup, and denote by $M_{k,S}^n(K)$ the subspace of $M_{k,S}^n$ consisting of those functions whose Fourier expansion at infinity has Fourier coefficients in a subfield $K$ of $\mathbb{C}$, then it is shown in (loc. cit.) that $M_{k,S}^n(K) = M_{k,S}^n(\mathbb{Q}) \otimes_{\mathbb{Q}} K$. In particular, for a

given $f \in M_{k,S}^n$ and a $\sigma \in Aut(\mathbb{C}/\mathbb{Q})$ one can define the element $f^\sigma \in M_{k,S}^n$ by letting $\sigma$ act on the Fourier coefficients of $f$.

The main result of this paper is Theorem 4.6. Without going into too much details, it may be vaguely stated as follows. Denote by $L(s, f, \chi)$ the standard $L$-function attached to a Siegel-Jacobi cuspidal eigenform $f \in M_{k,S}^n(\overline{\mathbb{Q}})$, which is twisted by a Dirichlet character $\chi$, and let

$$\mathbf{\Lambda}(s, f, \chi) := L(2s - n - l/2, f, \chi) \begin{cases} L_{\mathfrak{c}}(2s - l/2, \chi\psi_S) & \text{if } l \in 2\mathbb{Z}, \\ 1 & \text{if } l \notin 2\mathbb{Z}, \end{cases}$$

where $\psi_S$ is the non-trivial quadratic character attached to the extension $K_S := \mathbb{Q}(\sqrt{(-1)^{l/2} \det(2S)})$ if $K_S \neq \mathbb{Q}$, and otherwise $\psi_S = 1$. Then for certain integers $\sigma$ and for $k > 2n + l + 1$,

$$\frac{\mathbf{\Lambda}(\sigma/2, f, \chi)}{\pi^{e_\sigma} <f, f>} \in \overline{\mathbb{Q}},$$

for an explicit power $e_\sigma \in \mathbb{N}$, and where $<f, f>$ is a Petersson inner product on the space of cuspidal Siegel-Jacobi modular forms.

Let us now try to put the main result of this paper in some broader context. Results of the above form for the standard $L$-functions of automorphic forms associated to Shimura varieties, such as Siegel and Hermitian modular forms, were obtained by many researchers, most profoundly by Shimura (see for example [13]). These results can also be understood in the general framework of Deligne's Period Conjectures for critical values of motives [4]. Indeed, according to the general Langlands conjectures, the standard $L$-functions of automorphic forms related to Shimura varieties can be identified with motivic $L$-functions, and hence the algebraicity results for the special values of the automorphic $L$-functions can also be seen as a confirmation of Deligne's Period Conjecture, albeit is usually hard to actually show that the conjectural motivic period agrees with the automorphic one.

Siegel-Jacobi modular forms and - in particular - the algebraicity results obtained in this paper do not fit in this framework: since the Jacobi group is not reductive, it does not satisfy the necessary properties to be associated with a Shimura variety. Nevertheless, the Jacobi group can be actually associated with a geometric object, namely with a mixed Shimura variety, as it is explained for example in [6, 7]. It is thus very tempting to speculate - although we do not address this question in this paper - that the standard $L$-function studied here might be identified with an $L$-function of a mixed motive, and hence the theorem above could be seen as a confirmation of the generalization of Deligne's Period Conjecture to the mixed setting as for example stated by Scholl in [10].

We should mention here that even though in some cases one can identify the standard $L$-function associated to a Siegel-Jacobi form with the standard $L$-function associated to a Siegel modular form (see for example the remark on page 252 in [9]), this is possible under some quite restrictive conditions on both index and level of the Siegel-Jacobi form. Actually, even in the situation of classical Jacobi forms this correspondence becomes quite complicated when one considers an index different than 1 and/or non-trivial level, which is very clear for example in the work of [14].

**Remark:** In an earlier version of [3], which one can find on the arXiv ([2]), we had also included the results of this paper. However this had resulted in a rather long exposition, and for this reason we decided to keep the two main results of our investigations separately. Namely, [3] contains now our results towards the analytic properties of the standard $L$ function, whereas this paper focuses on the algebraic properties.

## 2. Preliminaries

In this section we recall basic facts regarding Siegel-Jacobi modular forms of higher index and set up the notation. We follow closely our previous work [3].

Let $F$ be a totally real algebraic number field of degree $d$, $\mathfrak{d}$ the different of $F$, and $\mathfrak{o}$ its ring of integers. For two natural numbers $l, n$, we consider the Jacobi group $\boldsymbol{G} := \boldsymbol{G}^{n,l} := H^{n,l} \rtimes \mathrm{Sp}_n$ of degree $n$ and index $l$ over $F$:

$$\boldsymbol{G}^{n,l}(F) := \{\boldsymbol{g} = (\lambda, \mu, \kappa)g : \lambda, \mu \in M_{l,n}(F), \kappa \in Sym_l(F), g \in G^n(F)\},$$

where $H(F) := H^{n,l}(F) := \{(\lambda, \mu, \kappa)1_{2n} \in \boldsymbol{G}^{n,l}(F)\}$ is the Heisenberg group, and

$$G^n(F) := \mathrm{Sp}_n(F) := \left\{ g \in \mathrm{SL}_{2n}(F) \colon {}^t g \left( {}_{1_n} {}^{-1_n} \right) g = \left( {}_{1_n} {}^{-1_n} \right) \right\}.$$

The group law is given by

$$(\lambda, \mu, \kappa)g(\lambda', \mu', \kappa')g' := (\lambda + \tilde{\lambda}, \mu + \tilde{\mu}, \kappa + \kappa' + \lambda\,{}^t\tilde{\mu} + \tilde{\mu}\,{}^t\lambda + \tilde{\lambda}\,{}^t\tilde{\mu} - \lambda'\,{}^t\mu')gg',$$

where $(\tilde{\lambda}\,\tilde{\mu}) := (\lambda'\,\mu')g^{-1} = (\lambda'\,{}^t d - \mu'\,{}^t c \quad \mu'\,{}^t a - \lambda'\,{}^t b)$, and the identity element of $\boldsymbol{G}^{n,l}(F)$ is $1_H 1_{2n}$, where $1_H := (0,0,0)$ is the identity element of $H^{n,l}(F)$ (whenever it does not lead to any confusion we suppress the indices $n, l$). For an element $g \in \mathrm{Sp}_n$ we write $g = \left( \begin{smallmatrix} a_g & b_g \\ c_g & d_g \end{smallmatrix} \right)$, where $a_g, b_g, c_g, d_g \in M_n$.

We write $\{\sigma_v : F \hookrightarrow \mathbb{R}, \ v \in \mathbf{a}\}$ for the set of real embeddings of $F$, $\mathbf{a}$ denoting the set of archimedean places of $F$. Each $\sigma_v$ induces an embedding $\boldsymbol{G}(F) \hookrightarrow \boldsymbol{G}(\mathbb{R})$; we will write $(\lambda_v, \mu_v, \kappa_v)g_v$ for $\sigma_v(\boldsymbol{g})$. The group $\boldsymbol{G}(\mathbb{R})^{\mathbf{a}}$ acts on $\mathcal{H}_{n,l} := (\mathbb{H}_n \times M_{l,n}(\mathbb{C}))^{\mathbf{a}}$ component wise via

$$\boldsymbol{g}z = \boldsymbol{g}(\tau, w) = (\lambda, \mu, \kappa)g(\tau, w) = \prod_{v \in \mathbf{a}}(g_v\tau_v, w_v\lambda(g_v, \tau_v)^{-1} + \lambda_v g_v\tau_v + \mu_v),$$

where $g_v\tau_v = (a_v\tau_v + b_v)(c_v\tau_v + d_v)^{-1}$ and $\lambda(g_v, \tau_v) := (c_v\tau_v + d_v)$ for $g_v = \left( \begin{smallmatrix} a_v & b_v \\ c_v & d_v \end{smallmatrix} \right)$.

For $k \in \mathbb{Z}^{\mathbf{a}}$ and a matrix $S \in Sym_l(\mathfrak{d}^{-1})$ we define the factor of automorphy of weight $k$ and index $S$ by

$$J_{k,S} \colon \boldsymbol{G}^{n,l}(F) \times \mathcal{H}_{n,l} \to \mathbb{C}$$

$$J_{k,S}(\boldsymbol{g}, z) = J_{k,S}(\boldsymbol{g}, (\tau, w)) := \prod_{v \in \mathbf{a}} j(g_v, \tau_v)^{k_v} \mathcal{J}_{S_v}(\boldsymbol{g_v}, \tau_v, w_v),$$

where $\boldsymbol{g} = (\lambda, \mu, \kappa)g$, $j(g_v, \tau_v) = \det(c_v\tau_v + d_v) = \det(\lambda(g_v, \tau_v))$ and

$$\mathcal{J}_{S_v}(\boldsymbol{g_v}, \tau_v, w_v) = e(-\mathrm{tr}\,(S_v\kappa_v) + \mathrm{tr}\,(S_v[w_v]\lambda(g_v, \tau_v)^{-1}c_v)$$
$$- 2\mathrm{tr}\,({}^t\lambda_v S_v w_v \lambda(g_v, \tau_v)^{-1}) - \mathrm{tr}\,(S_v[\lambda_v]g_v\tau_v))$$

with $e(x) := e^{2\pi i x}$, and we set $S[x] := {}^t x S x$; $J_{k,S}$ satisfies the usual cocycle relation:

$$J_{k,S}(\boldsymbol{gg'}, z) = J_{k,S}(\boldsymbol{g}, \boldsymbol{g'}\,z)J_{k,S}(\boldsymbol{g'}, z).$$

For a function $f \colon \mathcal{H}_{n,l} \to \mathbb{C}$ we define the action of $\boldsymbol{g} \in \boldsymbol{G}^{n,l}$ by

$$(f|_{k,S}\,\boldsymbol{g})(z) := J_{k,S}(\boldsymbol{g}, z)^{-1} f(\boldsymbol{g}\,z).$$

One can define a very general notion of a congruence subgroup of $\boldsymbol{G}(F)$ by considering any congruence subgroup $\Gamma$ of $Sp_n(F)$ and a lattice in $H^{n,l}(F)$ that is stable under the action of $\Gamma$. However in this paper we will mainly focus on functions $f$ which are invariant under the action of a particular congruence subgroup of $\boldsymbol{G}(F)$, namely,

$$\boldsymbol{\Gamma}[\mathfrak{b}, \mathfrak{c}] := C[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}] \rtimes \Gamma[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] \subset \boldsymbol{G}(F),$$

where $\mathfrak{b}$ and $\mathfrak{c}$ are respectively a fractional and an integral ideal of $F$, and

$$C[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}] := \{(\lambda, \mu, \kappa) \in H^{n,l}(F) : \lambda \in M_{l,n}(\mathfrak{o}), \mu \in M_{l,n}(\mathfrak{b}^{-1}), \kappa \in Sym_l(\mathfrak{b}^{-1})\},$$

$$\Gamma[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] := \left\{ g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix} \in G(F) : \begin{matrix} a_g \in M_n(\mathfrak{o}),\ b_g \in M_n(\mathfrak{b}^{-1}), \\ c_g \in M_n(\mathfrak{b}\mathfrak{c}),\ \ d_g \in M_n(\mathfrak{o}) \end{matrix} \right\}.$$

Now, let $S \in \mathfrak{b}\mathfrak{d}^{-1}\mathcal{T}_l$ where

$$\mathcal{T}_l := \{x \in Sym_l(F) : \mathrm{tr}\,(xy) \in \mathfrak{o} \text{ for all } y \in Sym_l(\mathfrak{o})\},$$

and assume additionally that $S$ is positive definite in the sense that if we write $S_v := \sigma_v(S) \in Sym_l(\mathbb{R})$ for $v \in \mathbf{a}$, then all $S_v$ are positive definite.

**Definition 2.1.** Let $k$ and $S$ be as above, and $\boldsymbol{\Gamma}$ a subgroup of $\boldsymbol{\Gamma}[\mathfrak{b}, \mathfrak{c}]$ (for some ideals $\mathfrak{b}, \mathfrak{c}$) equipped with a homorphism $\chi$. A Siegel-Jacobi modular form of weight $k \in \mathbb{Z}^{\mathbf{a}}$, index $S$, level $\boldsymbol{\Gamma}$ and Nebentypus $\chi$ is a holomorphic function $f \colon \mathcal{H}_{n,l} \to \mathbb{C}$ such that

(1) $f|_{k,S}\,\boldsymbol{g} = \chi(\boldsymbol{g})f$ for every $\boldsymbol{g} \in \boldsymbol{\Gamma}$,
(2) for each $g \in G^n(F)$, $f|_{k,S}\,g$ admits a Fourier expansion of the form

$$f|_{k,S}\,g(\tau, w) = \sum_{\substack{t \in L \\ t \geq 0}} \sum_{r \in M} c(g; t, r)\mathbf{e_a}(\mathrm{tr}\,(t\tau))\mathbf{e_a}(\mathrm{tr}\,({}^t rw))\quad (*)$$

for some appropriate lattices $L \subset Sym_n(F)$ and $M \subset M_{l,n}(F)$, where $t \geq 0$ means that $t_v$ is semi-positive definite for each $v \in \mathbf{a}$.
(Thanks to the Köcher principle for Siegel-Jacobi forms [15, Lemma 1.6], this condition is only necessary if $n = 1$ and $F = \mathbb{Q}$.)

We will denote the space of such functions by $M_{k,S}^n(\boldsymbol{\Gamma}, \chi)$.

We say that $f$ is a cusp form if in the expansion $(*)$ above for every $g \in G^n(F)$, we have $c(g; t, r) = 0$ unless $\begin{pmatrix} S_v & r_v \\ {}^t r_v & t_v \end{pmatrix}$ is positive definite for every $v \in \mathbf{a}$. The space of cusp forms will be denoted by $S_{k,S}^n(\boldsymbol{\Gamma}, \chi)$.

We define Petersson inner product of Siegel-Jacobi forms $f$ and $g$ of weight $k$ and level $\boldsymbol{\Gamma}$ under assumption that one of them is a cusp form as:

$$< f, g > := vol(A)^{-1} \int_A f(z)\overline{g(z)}\Delta_{S,k}(z)dz, \quad A := \boldsymbol{\Gamma} \backslash \mathcal{H}_{n,l},$$

where for $z = (\tau, w) \in \mathcal{H}_{n,l}$, $\tau = x + iy$ with $x, y \in Sym_n(F_{\mathbf{a}})$ and $w = u + iv$ with $u, v \in M_{l,n}(F_{\mathbf{a}})$, we set

$$dz := d(\tau, w) := \det(y)^{-(l+n+1)}dxdydudv\,, \qquad \Delta_{S,k}(z) := \det(y)^k \mathbf{e_a}(-4\pi\mathrm{tr}\,({}^t vSvy^{-1})).$$

In this way the inner product is independent of the group $\mathbf{\Gamma}$.

## 3. The standard $L$-function and the doubling method identity

In this section we recall some results and notation from [3] which will be necessary to establish results in the next section.

3.1. **The $L$-function.** We start by fixing some notation. For a fractional ideal $\mathfrak{b}$, and an integral ideal $\mathfrak{c}$ we let

$$\mathbf{\Gamma} := \mathbf{\Gamma}_1(\mathfrak{c}) := \{(\lambda, \mu, \kappa)g \in C[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}] \rtimes \Gamma[\mathfrak{b}^{-1}\mathfrak{c}, \mathfrak{b}\mathfrak{c}] : a_g - 1_n \in M_n(\mathfrak{c})\},$$

$$Q(\mathfrak{c}) := \{r \in \mathrm{GL}_n(F) \cap M_n(\mathfrak{o}) : r - 1_n \in M_n(\mathfrak{c})\},$$

$$R(\mathfrak{c}) := \{\mathrm{diag}[{}^t r^{-1}, r] : r \in Q(\mathfrak{c})\},$$

where $\mathrm{diag}[x_1, \ldots, x_n]$ stands for a block diagonal matrix with consecutive blocks on the diagonal $x_1, \ldots, x_n$.

For $r \in Q(\mathfrak{c})$ and $f \in M^n_{k,S}(\mathbf{\Gamma})$ we define a linear operator $T_r : M^n_{k,S}(\mathbf{\Gamma}) \to M^n_{k,S}(\mathbf{\Gamma})$ by

$$f|T_r := \sum_{\boldsymbol{\alpha} \in \boldsymbol{A}} f|_{k,S}\boldsymbol{\alpha},$$

where $\boldsymbol{A} \subset \boldsymbol{G}^n(F)$ is such that $\mathbf{\Gamma}\mathrm{diag}[{}^t r^{-1}, r]\mathbf{\Gamma} = \coprod_{\boldsymbol{\alpha} \in \boldsymbol{A}} \mathbf{\Gamma}\boldsymbol{\alpha}$. Further, for an integral ideal $\mathfrak{a}$ of $F$ we put

$$f|T(\mathfrak{a}) := \sum_{\substack{r \in Q(\mathfrak{c}) \\ \det(r)\mathfrak{o}=\mathfrak{a}}} f|T_r,$$

where we sum over all those $r$ for which the cosets $\mathrm{GL}_n(\mathfrak{o})\, r\mathrm{GL}_n(\mathfrak{o})$ are distinct.

We now consider a nonzero $f \in S^n_{k,S}(\mathbf{\Gamma})$ such that $f|T(\mathfrak{a}) = \lambda(\mathfrak{a})f$ for all integral ideals $\mathfrak{a}$ of $F$. For a Hecke character $\chi$ of $F$, and denoting by $\chi^*$ the corresponding ideal character, we define an absolutely convergent series

$$D(s, f, \chi) := \sum_{\mathfrak{a}} \lambda(\mathfrak{a})\chi^*(\mathfrak{a})N(\mathfrak{a})^{-s}, \qquad \mathrm{Re}(s) > 2n + l + 1.$$

In [3] we proved the following theorem regarding the Euler product representation of this Dirichlet series. For the condition $M_{\mathfrak{p}}^+$ for primes away from $\mathfrak{c}$ we refer to [3].

**Theorem 3.1** (Theorem 7.1, [3]). *Let $0 \neq f \in S^n_{k,S}(\mathbf{\Gamma})$ be such that $f|T(\mathfrak{a}) = \lambda(\mathfrak{a})f$ for all integral ideals $\mathfrak{a}$ of $F$. Assume that the matrix $S$ satisfies the condition $M_{\mathfrak{p}}^+$ for every prime ideal $\mathfrak{p}$ with $(\mathfrak{p}, \mathfrak{c}) = 1$. Then*

$$\mathfrak{L}(\chi, s)D(s + n + l/2, f, \chi) = L(s, f, \chi) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\chi^*(\mathfrak{p})N(\mathfrak{p})^{-s})^{-1},$$

*where for every prime ideal $\mathfrak{p}$ of $F$*

$$L_{\mathfrak{p}}(X) = \begin{cases} \prod_{i=1}^n \left((1 - \mu_{\mathfrak{p},i}X)(1 - \mu_{\mathfrak{p},i}^{-1}X)\right), \ \mu_{\mathfrak{p},i} \in \mathbb{C}^\times & \textit{if } (\mathfrak{p}, \mathfrak{c}) = 1, \\ 1 & \textit{if } (\mathfrak{p}, \mathfrak{c}) \neq 1. \end{cases}$$

*Moreover,* $\mathfrak{L}(\chi, s) = \prod_{(\mathfrak{p},\mathfrak{c})=1} \mathfrak{L}_\mathfrak{p}(\chi, s)$, *where*

$$\mathfrak{L}_\mathfrak{p}(\chi, s) := G_\mathfrak{p}(\chi, s) \cdot \begin{cases} \prod_{i=1}^n L_\mathfrak{p}(2s + 2n - 2i, \chi^2) & \text{if } l \in 2\mathbb{Z} \\ \prod_{i=1}^n L_\mathfrak{p}(2s + 2n - 2i + 1, \chi^2) & \text{if } l \notin 2\mathbb{Z} \end{cases}$$

*and* $G_\mathfrak{p}(\chi, s)$ *is a ratio of Euler factors which for almost all* $\mathfrak{p}$ *is equal to one. In particular, the function* $L(s, f, \chi)$ *is absolutely convergent for* $\mathrm{Re}(s) > n + l/2 + 1$.

We note here that the Euler product expression implies that

(1)                      $L(s, f, \chi) \neq 0, \qquad \mathrm{Re}(s) > n + l/2 + 1.$

For the rest of the paper, whenever the $L$ function attached to a Jacobi form $f$ is considered, **we will always assume** (without further notification) that the index matrix $S$ of $f$ satisfies the condition $M_\mathfrak{p}^+$ for every prime ideal $\mathfrak{p}$ away from the level $\mathfrak{c}$ of $f$.

Let us now also remark that if we replace $f$ with $f^c(z) := \overline{f(-\bar{z})}$, the $L$-function remains the same:

**Proposition 3.2** (Proposition 7.9, [3]). *Let* $f \in S_{k,S}^n(\Gamma)$ *be an eigenform with* $f|T(\mathfrak{a}) = \lambda(\mathfrak{a})f$ *for all fractional ideals* $\mathfrak{a}$ *prime to* $\mathfrak{c}$. *Then so is* $f^c$. *In particular,* $f^c|T(\mathfrak{a}) = \lambda(\mathfrak{a})f^c$ *and* $L(s, f, \chi) = L(s, f^c, \chi)$.

3.2. **Doubling method.** The $L$-function introduced above may be also obtained via a doubling method. We chose to take Arakawa's approach [1] and considered a homomorphism

$$\iota_A : \boldsymbol{G}^{m,l} \times \boldsymbol{G}^{n,l} \to \boldsymbol{G}^{m+n,l},$$

$$\iota_A((\lambda, \mu, \kappa)g) \times (\lambda', \mu', \kappa')g')) := ((\lambda \, \lambda'), (\mu \, \mu'), \kappa + \kappa')\iota_S(g \times g'),$$

where

$$\iota_S : G^m \times G^n \hookrightarrow G^{m+n}, \quad \iota_S\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \times \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)\right) := \begin{pmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{pmatrix}.$$

The map $\iota_A$ induces an embedding

$$\mathcal{H}_{m,l} \times \mathcal{H}_{n,l} \hookrightarrow \mathcal{H}_{n+m,l}, \quad z_1 \times z_2 \mapsto \mathrm{diag}[z_1, z_2],$$

defined by

$$(\tau_1, w_1) \times (\tau_2, w_2) \mapsto (\mathrm{diag}[\tau_1, \tau_2], (w_1 \, w_2)).$$

The doubling method suggests that computation of the Petersson inner product of a cuspidal Siegel-Jacobi modular form $f$ on $\mathcal{H}_{n,l}$ against a Siegel-type Jacobi Eisenstein series pull-backed from $\mathcal{H}_{n+m,l}$ leads to an $L$-function associated with $f$. Before we state the result, we define the Jacobi Eisenstein series which was used. It will appear again in the last section (Theorem 5.5), where the question of its nearly holomorphicity will be addressed. In order to avoid excessive and unnecessary notation, we state here an adelic definition of the Eisenstein series; the interested reader is referred to [3] for a derivation of a classical definition.

Fix a weight $k \in \mathbb{Z}^\mathbf{a}$ and consider a Hecke character $\chi$ such that for a fixed integral ideal $\mathfrak{c}$ of $F$ we have

   (1) $\chi_v(x) = 1$ for all $x \in \mathfrak{o}_v^\times$ with $x - 1 \in \mathfrak{c}_v$, $v \in \mathbf{h}$ (finite places of $F$),

(2) $\chi_{\mathbf{a}}(x_{\mathbf{a}}) = \mathrm{sgn}(x_{\mathbf{a}})^k := \prod_{v \in \mathbf{a}} \left(\frac{x_v}{|x_v|}\right)^{k_v}$, for $x_{\mathbf{a}} \in \mathbb{A}_{\mathbf{a}}$;

we will also write $\chi_{\mathfrak{c}} := \prod_{v|\mathfrak{c}} \chi_v$. Further, let

$$K^n := K_{\mathbf{h}}[\mathfrak{b}, \mathfrak{c}](\prod_{v \in \mathbf{a}} H^{n,l}(F_v) \rtimes D_\infty^{\mathbf{a}}),$$

where $K_{\mathbf{h}}[\mathfrak{b}, \mathfrak{c}] \subset \prod_{v \in \mathbf{h}}' \boldsymbol{G}(F_v)$ is defined so that $\boldsymbol{\Gamma}[\mathfrak{b}, \mathfrak{c}] = \boldsymbol{G}(F) \cap \prod_{v \in \mathbf{a}} \boldsymbol{G}(F_v) K_{\mathbf{h}}[\mathfrak{b}, \mathfrak{c}]$ and $D_\infty$ is the maximal compact subgroup of $\mathrm{Sp}_n(\mathbb{R})$.

We define an absolutely convergent adelic Eisenstein series of Siegel type on a Jacobi group with a parabolic subgroup

$$\boldsymbol{P}^n(F) := \{(0, \mu, \kappa)g : \mu \in M_{l,n}(F), \kappa \in Sym_l(F), g \in P^n(F)\},$$

where $P^n(F)$ is a Siegel subgroup of $G^n(F)$ as follows:

$$E(x, s; \chi) := \sum_{\gamma \in \boldsymbol{P}^n(F) \backslash \boldsymbol{G}^n(F)} \phi(\gamma x, s; \chi), \quad \mathrm{Re}(s) > \frac{1}{2}(n + l + 1),$$

where $\phi(x, s; \chi) := 0$ if $x \notin \boldsymbol{P}^n(\mathbb{A})K^n$ and otherwise, if $x = \boldsymbol{p}\boldsymbol{w}$ with $\boldsymbol{p} \in \boldsymbol{P}^n(\mathbb{A})$ and $\boldsymbol{w} \in K^n$, we set

$$\phi(x, s; \chi) := \chi(\det(d_p))^{-1}\chi_{\mathfrak{c}}(\det(d_w))^{-1}J_{k,S}(\boldsymbol{w}, \boldsymbol{i}_0)^{-1}|\det(d_p)|_{\mathbb{A}}^{-2s},$$

where $p, w \in \mathrm{Sp}_n(\mathbb{A})$ denote symplectic parts of $\boldsymbol{p}, \boldsymbol{w}$, respectively. The classical Jacobi Eisenstein series which corresponds to $E(x, s; \chi)$ is given by

$$E(z, s; \chi) := J_{k,S}(x, \boldsymbol{i}_0)E(x, s; \chi),$$

where $\boldsymbol{i}_0 = (i1_n, 0)^{\mathbf{a}} \in \mathcal{H}_{n,l}$, and $x \in \prod_{v \in \mathbf{a}} \boldsymbol{G}(F_v)$ is such that $x\boldsymbol{i}_0 = z$; for a formula which does not involve $x$ see [3, equation (11)].

**Theorem 3.3** ([3]). *Let $f \in S_{k,S}^n(\boldsymbol{\Gamma})$ be a Hecke eigenform and $E(z, s; \chi)$ an Eisenstein series defined above. Then:*

$$G(\chi, 2s - n - l/2)N(\mathfrak{b})^{2ns}\chi_{\mathbf{h}}(\theta)^{-n}(-1)^{n(s-k/2)}vol(A)\Lambda_{k-l/2,\mathfrak{c}}^{2n}(s - l/4, \chi\psi_S)$$

$$\cdot < (E|_{k,S}\boldsymbol{\rho})(\mathrm{diag}[z_1, z_2], s; \chi), (f_{k,S}|\boldsymbol{\eta}_n)^c(z_2) >$$

(2) $$= \nu_{\mathfrak{c}}c_{S,k}(s - k/2)\boldsymbol{\Lambda}(s, f, \chi)f(z_1),$$

*where*

$$\Lambda_{k-l/2,\mathfrak{c}}^{2n}(s, \chi\psi_S) := \begin{cases} L_{\mathfrak{c}}(2s - l/2, \chi\psi_S) \prod_{i=1}^n L_{\mathfrak{c}}(4s - l - 2i, \chi^2) & \text{if } l \in 2\mathbb{Z}, \\ \prod_{i=1}^{[(2n+1)/2]} L_{\mathfrak{c}}(4s - l - 2i + 1, \chi^2) & \text{if } l \notin 2\mathbb{Z}, \end{cases}$$

$$\boldsymbol{\Lambda}(s, f, \chi) := L(2s - n - l/2, f, \chi) \begin{cases} L_{\mathfrak{c}}(2s - l/2, \chi\psi_S) \prod_{i=n+1}^n L_{\mathfrak{c}}(4s - l - 2i, \chi^2), & l \in 2\mathbb{Z}, \\ \prod_{i=n+1}^{[(2n+1)/2]} L_{\mathfrak{c}}(4s - l - 2i + 1, \chi^2), & l \notin 2\mathbb{Z}, \end{cases}$$

(3) $$G(\chi, 2s - n - l/2) := \prod_{(\mathfrak{p}, \mathfrak{c}) = 1} G_{\mathfrak{p}}(\chi, 2s - n - l/2),$$

*and the rest of notation is as in [3, Section 6]; in particular:* $\boldsymbol{\eta}_n = 1_H \left(\begin{smallmatrix} & -1_n \\ 1_n & \end{smallmatrix}\right)$,

$$c_{S,k}(s) = \prod_{\nu \in \mathbf{a}} \left(\pm \det(2S_\nu)^{-n}2^{n(n+3)/2 - 4s_\nu - nk_\nu}\pi^{n(n+1)/2}\frac{\Gamma_n(s_\nu + k_\nu - \frac{l}{2} - \frac{n+1}{2})}{\Gamma_n(s_\nu + k_\nu - \frac{l}{2})}\right)$$

*and* $\Gamma_n(s) := \pi^{n(n-1)/4} \prod_{i=0}^{n-1} \Gamma(s - \frac{i}{2})$.

Statement of the above theorem expresses a combination of equations (30) and (31) from [3, Section 9] before multiplying them by the factor $\mathcal{G}_{k-l/2,2n}(s - l/4)$.

*Remark* 3.4. In fact, the results proved in [3] are more general than the ones presented above. Indeed, we worked with congruence subgroups of the form

$$\tilde{\mathbf{\Gamma}} := \{(\lambda, \mu, \kappa)g \in C[\mathfrak{o}, \mathfrak{b}^{-1}, \mathfrak{b}^{-1}] \rtimes \Gamma[\mathfrak{b}^{-1}\mathfrak{e}, \mathfrak{b}\mathfrak{c}] : a_g - 1_n \in M_n(\mathfrak{e})\},$$

where $\mathfrak{e}$ is an integral ideal such that $\mathfrak{c} \subset \mathfrak{e}$ and $\mathfrak{e}$ is prime to $\mathfrak{e}^{-1}\mathfrak{c}$; then

$$\mathbf{\Gamma}_1(\mathfrak{c}) \subseteq \tilde{\mathbf{\Gamma}} \subseteq \mathbf{\Gamma}_0(\mathfrak{c}),$$

where the last group is obtained by setting $\mathfrak{e} = \mathfrak{o}$. However, in this paper we decided to work with $\mathfrak{e} = \mathfrak{c}$, because for simplicity reasons we restricted the proof of our main theorem to this case.

## 4. Arithmetic properties of Siegel-Jacobi modular forms

In this section we will write $\boldsymbol{f}$ (instead of $f$) for Siegel-Jacobi modular forms and use $f$ for other types of forms.

For a congruence subgroup $\mathbf{\Gamma}$ of $\boldsymbol{G}(F)$ and a subfield $K$ of $\mathbb{C}$ we define the set

$$M_{k,S}^n(\mathbf{\Gamma}, K) := \{\boldsymbol{f} \in M_{k,S}^n(\mathbf{\Gamma}) : \boldsymbol{f}(\tau, w) = \sum_{t,r} c(t,r) \mathbf{e}_{\mathbf{a}}(\mathrm{tr}\,(t\tau + {}^t rw)),\ c(t,r) \in K\};$$

the subspace $S_{k,S}^n(\mathbf{\Gamma}, K)$ consisting of cusp forms is defined in a similar way. Moreover, we write $M_{k,S}^n(K)$ for the union of all spaces $M_{k,S}^n(\mathbf{\Gamma}, K)$ for all congruence subgroups $\mathbf{\Gamma}$.

For an element $\sigma \in Aut(\mathbb{C})$ and an element $k = (k_v) \in \mathbb{Z}^{\mathbf{a}}$ we define $k^\sigma := (k_{v\sigma}) \in \mathbb{Z}^{\mathbf{a}}$, where $v\sigma$ is the archimedean place corresponding to the embedding $K \overset{\tau_v}{\hookrightarrow} \mathbb{C} \overset{\sigma}{\to} \mathbb{C}$, if $\tau_v$ is the embedding in $\mathbb{C}$ corresponding to the archimedean place $v$.

**Proposition 4.1.** *Let $k \in \mathbb{Z}^{\mathbf{a}}$, and let $\Phi$ be the Galois closure of $F$ in $\overline{\mathbb{Q}}$, and $\Phi_k$ the subfield of $\Phi$ such that*

$$Gal(\Phi/\Phi_k) := \{\sigma \in Gal(\Phi/F) : \ k^\sigma = k\}.$$

*Then $M_{k,S}^n(\mathbb{C}) = M_{k,S}^n(\Phi_k) \otimes_{\Phi_k} \mathbb{C}$.*

*Proof.* If $F = \mathbb{Q}$, this is [11, Proposition 3.8]. A careful examination of the proof [11, page 60] shows that the proof is eventually reduced to the corresponding statement for Siegel modular forms of integral (if $l$ is even) or half-integral (if $l$ is odd) weight. However, in both cases the needed statement does generalize to the case of totally real fields, as it was established in [13, Theorems 10.4 and 10.7]. □

Given an $\boldsymbol{f} \in M_{k,S}^n(\mathbb{C})$, we define

$$\boldsymbol{f}_*(\tau, w) := \mathbf{e}_{\mathbf{a}}(Sw(\tau - \overline{\tau})^{-1}{}^t w)\boldsymbol{f}(\tau, w)$$

and write $\mathbb{Q}^{ab}$ for the maximal abelian extension of $\mathbb{Q}$. Moreover, for $k \in \frac{1}{2}\mathbb{Z}^{\mathbf{a}}$ such that $k_v - \frac{1}{2} \in \mathbb{Z}$ for all $v \in \mathbf{a}$ we write $M_k^n$ for the space of Siegel modular forms of weight $k$,

and of any congruence subgroup, and $M_k^n(K)$ for those with the property that all their Fourier coefficients at infinity lie in $K$ (see for example [13, Chapter 2] for a detailed study of these sets).

**Proposition 4.2.** *Let $K$ be a field that contains $\mathbb{Q}^{ab}$ and $\Phi$ as above. Then*

(1) $\boldsymbol{f} \in M_{k,S}^n(K)$ *if and only if* $\boldsymbol{f}_*(\tau, v\Omega_\tau) \in M_k^n(K)$, *where* $\Omega_\tau := {}^t(\tau \ 1_n)$, *and* $v \in M_{l,2n}(F)$.

(2) *For any element* $\gamma \in \mathrm{Sp}_n(F) \hookrightarrow \boldsymbol{G}^n(F)$ *and* $\boldsymbol{f} \in M_{k,S}^n(K)$, *we have*

$$\boldsymbol{f}|_{k,S}\gamma \in M_{k,S}^n(K).$$

*Moreover if* $\boldsymbol{f} \in M_{k,S}^n(\boldsymbol{\Gamma}, K)$, *it follows that* $\boldsymbol{f}|T_r \in M_{k,S}^n(\boldsymbol{\Gamma}, K)$ *for any* $r \in Q(\mathfrak{c})$.

*Proof.* If $F = \mathbb{Q}$, this is [11, Proposition 3.2]. It is easy to see that the proof generalizes to the case of any totally real field. Indeed, the first part of the proof is a direct generalization of the argument used by Shimura. The second part requires the fact that the space $M_k^n(K)$ is stable under the action of elements in $\mathrm{Sp}_n(F)$, which is true for any totally real field, as it is proved in [13, Theorem 10.7 (6)]. The last statement follows from the definition of the Hecke operator $T_r$. $\qquad\square$

For a symmetric positive definite matrix $S \in Sym_l(F)$, $h \in M_{l,n}(F)$ and a lattice $L \subset M_{l,n}(F)$ we define the Jacobi theta series of characteristic $h$ by

$$\Theta_{S,L,h}(\tau, w) = \sum_{x \in L} \mathbf{e_a}(\mathrm{tr}\,(S(\frac{1}{2}(x+h)\tau\,{}^t(x+h) + w\,{}^t(x+h)))).$$

**Theorem 4.3.** *Assume that $n > 1$ or $F \neq \mathbb{Q}$, and let $K$ be any subfield of $\mathbb{C}$. Let $A \in \mathrm{GL}_l(F)$ be such that $AS\,{}^tA = \mathrm{diag}[s_1, \ldots, s_l]$, and define the lattices $\Lambda_1 := AM_{l,n}(\mathfrak{o}) \subset M_{l,n}(F)$ and $\Lambda_2 := 2\mathrm{diag}[s_1^{-1}, \ldots, s_l^{-1}]M_{l,n}(\mathfrak{o}) \subset M_{l,n}(F)$. Then there is an isomorphism*

$$\Phi : M_{k,S}^n(K) \cong \bigoplus_{h \in \Lambda_1/\Lambda_2} M_{k-l/2}^n(K)$$

*given by $\boldsymbol{f} \mapsto (f_h)_h$, where the $f_h \in M_{k-l/2}^n(K)$ are defined by the expression*

$$\boldsymbol{f}(\tau, w) = \sum_{h \in \Lambda_1/\Lambda_2} f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w).$$

*Moreover, under the above isomorphism,*

$$\Phi^{-1}\left(\bigoplus_{h \in \Lambda_1/\Lambda_2} S_{k-l/2}^n(K)\right) \subset S_{k,S}^n(K).$$

*Remark* 4.4. We remark here that the assumption of $n > 1$ or $F \neq \mathbb{Q}$ is needed to guarantee that the $f_h$'s are holomorphic at the cusps, which follows from the Köcher principle. However, even in the case of $F = \mathbb{Q}$ and $n = 1$, if we take $\boldsymbol{f}$ to be of trivial level, then the $f_h$'s are holomorphic at infinity (see for example [5, page 59]).

*Proof of Theorem 4.3.* The first statement is [11, Proposition 3.5] for $F = \mathbb{Q}$ and it easily generalizes to the case of any totally real field. We explain the statement about cusp forms.

Consider first expansions around the cusp at infinity. Fix $h \in \Lambda_1/\Lambda_2$ and let $f_h(\tau) = \sum_{t_2>0} c(t_2)\mathbf{e_a}(\operatorname{tr}(t_2\tau))$. It is known that Fourier coefficients $c(t_1, r)$ of a Jacobi theta series

$$\Theta_{2S,\Lambda_2,h}(\tau, w) = \sum_{t_1,r} c(t_1, r)\mathbf{e_a}(\operatorname{tr}(t_1\tau))\mathbf{e_a}(\operatorname{tr}({}^t rw))$$

are nonzero only if $4t_1 = {}^t r S^{-1} r$ (see [15, p. 210]). Hence, the coefficients of

$$f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w) = \sum_{t,r}\left(\sum_{t_1+t_2=t} c(t_1, r)c(t_2)\right)\mathbf{e_a}(\operatorname{tr}(t\tau))\mathbf{e_a}(\operatorname{tr}({}^t rw))$$

are nonzero only if $4t = 4(t_1 + t_2) = {}^t r S^{-1} r + 4t_2 > {}^t r S^{-1} r$. This means that the function $f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w)$ satisfies cuspidality condition at infinity.

Now let $\gamma$ be any element in $\operatorname{Sp}_n(F)$. The first statement in the Theorem states that for every $h_1 \in \Lambda_1/\Lambda_2$ there exist $f_{h_1,h_2} \in M^n_{k-l/2}(K), h_2 \in \Lambda_1/\Lambda_2$, such that

$$\Theta_{2S,\Lambda_2,h_1}|_{k,S}\gamma(\tau, w) = \sum_{h_2} f_{h_1,h_2}(\tau)\Theta_{2S,\Lambda_2,h_2}(\tau, w).$$

Hence, for some cusp forms $f_{h_1} \in S^n_{k-l/2}(K)$,

$$\boldsymbol{f}|_{k,S}\gamma(\tau, w) := \sum_{h_1} f_{h_1}|_k\gamma(\tau)\left(\sum_{h_2} f_{h_1,h_2}(\tau)\Theta_{2S,\Lambda_2,h_2}(\tau, w)\right)$$

$$= \sum_{h_2}\left(\sum_{h_1} f_{h_1}|_k\gamma(\tau)f_{h_1,h_2}(\tau)\right)\Theta_{2S,\Lambda_2,h_2}(\tau, w).$$

The same argument as used for the cusp at infinity implies that the functions $\boldsymbol{f}|_{k,S}\gamma(\tau, w)$ and $\sum_{h_1} f_{h_1}|_k\gamma(\tau)f_{h_1,h_2}(\tau)$ are cuspidal. This finishes the proof. $\square$

Note that the above theorem does not state that $\Phi^{-1}\left(\bigoplus_{h\in\Lambda_1/\Lambda_2} S^n_{k-l/2}(K)\right) = S^n_{k,S}(K)$. For this reason we make the following definition.

**Property A.** We say that a cusp form $\boldsymbol{f} \in S^n_{k,S}(K)$ has the Property A if

$$\Phi(\boldsymbol{f}) \in \bigoplus_{h\in\Lambda_1/\Lambda_2} S^n_{k-l/2}(K).$$

**Examples of Siegel-Jacobi forms that satisfy the Property A:**

(1) Siegel-Jacobi forms over a field $F$ of class number one, and with trivial level, i.e. with $\mathfrak{c} = \mathfrak{o}$. Note that in this situation there is only one cusp. Then, keeping the notation as in the proof of the theorem above we need to verify that if $\boldsymbol{f}(\tau, w) = \sum_{t,r} c_{\mathbf{f}}(t, r)\mathbf{e_a}(\operatorname{tr}(t\tau))\mathbf{e_a}(\operatorname{tr}({}^t rw))$ with $4t > {}^t r S^{-1} r$ whenever $c(t, r) \neq 0$, then all the $f_h$ have to be cuspidal. Observe first that if $h_1, h_2 \in \Lambda_1/\Lambda_2$ are different, $\Theta_{2S,\Lambda_2,h_1}(\tau, w) = \sum_{t,r} c_1(t, r)\mathbf{e_a}(\operatorname{tr}(t\tau))\mathbf{e_a}(\operatorname{tr}({}^t rw))$, and $\Theta_{2S,\Lambda_2,h_2}(\tau, w) = \sum_{t,r} c_2(t, r)\mathbf{e_a}(\operatorname{tr}(t\tau))\mathbf{e_a}(\operatorname{tr}({}^t rw))$, then there is no $r$ such that at the same time

$c_1(t, r) \neq 0$ and $c_2(t, r) \neq 0$. Indeed, if it was not the case then there would be $\lambda_1, \lambda_2 \in \Lambda_2$ such that $r = 2S(\lambda_1 + h_1)$ and $r = 2S(\lambda_2 + h_2)$, that is, $\lambda_1 + h_1 = \lambda_2 + h_2$ or, equivalently, $h_1 - h_2 \in \Lambda_2$; contradiction. Hence, for any given $r$ there is a unique $h \in \Lambda_1/\Lambda_2$ such that $\Theta_{2S,\Lambda_2,h}$ has a nonzero coefficient $c(t, r)$. This means that there exists a unique $h$ such that $c_{\mathbf{f}}(t, r)$ is the Fourier coefficient of $f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w) = \sum_{t,r} \sum_{t_1+t_2=t} c(t_1, r)c(t_2)\mathbf{e_a}(\mathrm{tr}\,(t\tau))\mathbf{e_a}(\mathrm{tr}\,(^t rw))$. But then $^t r S^{-1} r < 4t = 4(t_1 + t_2) = {}^t r S^{-1} r + 4t_2$ and so $t_2 > 0$, which proves that $f_h$ is cuspidal.

(2) Siegel-Jacobi forms of index $S$ such that $\det(2S) \in \mathfrak{o}^\times$, as in this case the lattices $\Lambda_1$ and $\Lambda_2$ from Theorem 4.3 are equal.

(3) Siegel-Jacobi forms of non-parallel weight, that is, if there exist distinct $v, v' \in \mathbf{a}$ such that $k_v \neq k_{v'}$. Indeed, in this case $M^n_{k-l/2}(K) = S^n_{k-l/2}(K)$ for all $h \in \Lambda_1/\Lambda_2$ (see [12, Proposition 10.6]).

**Remark:** It is tempting to claim that Property A always holds. Actually the first example above suggests a possible way of establishing it for all Jacobi forms. However in the presence of more than one cusps, and of non-trivial index $[\Lambda_1 : \Lambda_2]$ one needs to understand the behavior of the theta series $\Theta_{2S,\Lambda_2,h}(\tau, w)$ at all cusps, which seems to be quite hard to understand in a general situation.

Let us now explain the significance of the Property A. Recall first that we have defined a Petersson inner product $< \boldsymbol{f}, \boldsymbol{g} >$ when $\boldsymbol{f}, \boldsymbol{g} \in M^n_{k,S}(K)$ and one of them, say, $\boldsymbol{f}$ is cuspidal. If $\boldsymbol{f}$ satisfies the Property A, then we claim that

$$< \boldsymbol{f}, \boldsymbol{g} >= N(\det(4S))^{-n/2} \sum_{h \in \Lambda_1/\Lambda_2} < f_h, g_h > .$$

Indeed, as in [15, Lemma 3.4],

$$< \boldsymbol{f}, \boldsymbol{g} >= N(\det(4S))^{-n/2} vol(A)^{-1} \int_A \sum_{h \in \Lambda_1/\Lambda_2} f_h(\tau)\overline{g_h(\tau)} \det(\mathrm{Im}(\tau))^{k-l/2-(n+1)} d\tau,$$

where $A = \Gamma \backslash \mathbb{H}^{\mathbf{a}}_n$ and a congruence subgroup $\Gamma$ is deep enough. We obtain the claimed equality after exchanging the order of integration and summation. This can be done exactly because each $f_h$ is cuspidal, which makes each individual integral well defined.

**Lemma 4.5.** *Assume that $n > 1$ or $F \neq \mathbb{Q}$ and that $\boldsymbol{f} \in S^n_{k,S}(\overline{\mathbb{Q}})$ satisfies the Property A and one of the following two conditions holds:*

*(i) there exist $v, v' \in \mathbf{a}$ such that $k_v \neq k_{v'}$;*
*(ii) $k = \mu\mathbf{a} = (\mu, \dots, \mu) \in \mathbb{Z}^{\mathbf{a}}$, with $\mu \in \mathbb{Z}$ depending on $n$ and $F$ in the following way:*

| $n > 2$ | $n = 2, F = \mathbb{Q}$ | $n = 2, F \neq \mathbb{Q}$ | $n = 1$ |
|---|---|---|---|
| $\mu > 3n/2 + l/2$ | $\mu > 3$ | $\mu > 2$ | $\mu \geq 1/2$ |

*Then for any $\boldsymbol{g} \in M^n_{k,S}(\overline{\mathbb{Q}})$ there exists $\widetilde{\boldsymbol{g}} := \mathfrak{q}(\boldsymbol{g}) \in S^n_{k,S}(\overline{\mathbb{Q}})$ such that*

$$< \boldsymbol{f}, \boldsymbol{g} >=< \boldsymbol{f}, \widetilde{\boldsymbol{g}} > .$$

*Proof.* There is nothing to show in the case of non-parallel weight, since as it was mentioned above there is no (holomorphic) Eisenstein part in this case. In the parallel

weight case, as $\boldsymbol{f}$ has the Property A, $< \boldsymbol{f}, \boldsymbol{g} >= N(\det(4S))^{-n/2} \sum_{h \in \Lambda_1/\Lambda_2} < f_h, g_h >$. It is known (see [13, Theorem 27.14]) that under the assumptions stated in (ii), there exists a projection $\widetilde{\mathfrak{q}} : M_{k-l/2}^n(\overline{\mathbb{Q}}) \to S_{k-l/2}^n(\overline{\mathbb{Q}})$ such that $< f, g >=< f, \widetilde{g} >$ for all $f \in S_{k-l/2}^n(\overline{\mathbb{Q}})$, $g \in M_{k-l/2}^n(\overline{\mathbb{Q}})$. Then, if we put $\widetilde{g}_h := \widetilde{\mathfrak{q}}(g_h)$ for all $h \in \Lambda_1/\Lambda_2$, we get

$$< \boldsymbol{f}, \boldsymbol{g} >= N(\det(4S))^{-n/2} \sum_{h \in \Lambda_1/\Lambda_2} < f_h, g_h >= N(\det(4S))^{-n/2} \sum_{h \in \Lambda_1/\Lambda_2} < f_h, \widetilde{g}_h > .$$

In particular, if we set $\widetilde{\boldsymbol{g}} := \Phi^{-1}((\widetilde{g}_h)_h)$, we obtain the statement of the lemma.    $\square$

Now, fix a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$ of $F$, set $\boldsymbol{\Gamma} := \boldsymbol{\Gamma}_1(\mathfrak{c})$ and let $\boldsymbol{f} \in S_{k,S}^n(\boldsymbol{\Gamma}, \overline{\mathbb{Q}})$ be a non-zero Siegel-Jacobi form. Furthermore, assume that $\boldsymbol{f}$ is an eigenfunction of the operators $T(\mathfrak{a})$ for all integral ideals $\mathfrak{a}$, and write $\boldsymbol{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\boldsymbol{f}$. We define the space

$$V(\boldsymbol{f}) := \{\widetilde{\boldsymbol{f}} \in S_{k,S}^n(\boldsymbol{\Gamma}, \overline{\mathbb{Q}}) : \widetilde{\boldsymbol{f}}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\widetilde{\boldsymbol{f}} \text{ for all } \mathfrak{a}\}.$$

We are now ready to state the main theorem of this paper on algebraic properties of

$$\boldsymbol{\Lambda}(s, \boldsymbol{f}, \chi) = L(2s - n - l/2, \boldsymbol{f}, \chi) \begin{cases} L_{\mathfrak{c}}(2s - l/2, \chi\psi_S) & \text{if } l \in 2\mathbb{Z}, \\ 1 & \text{if } l \notin 2\mathbb{Z}. \end{cases}$$

Recall that the $L$-function $L(s, \boldsymbol{f}, \chi)$ is defined only when the index matrix $S$ of $\boldsymbol{f}$ satisfies the property $M_{\mathfrak{p}}^+$, as stated in Theorem 3.1.

**Theorem 4.6.** *Assume $n > 1$ or $F \neq \mathbb{Q}$. Let $\chi$ be a Hecke character of $F$ such that $\chi_{\mathbf{a}}(x) = \text{sgn}_a(x)^k$, and $0 \neq \boldsymbol{f} \in S_{k,S}^n(\boldsymbol{\Gamma}, \overline{\mathbb{Q}})$ an eigenfunction of all $T(\mathfrak{a})$. Set $\mu := \min_v k_v$ and assume that*

*(1) $\mu > 2n + l + 1$,*
*(2) Property A holds for all $\widetilde{\boldsymbol{f}} \in V(\boldsymbol{f})$,*
*(3) $k_v \equiv k_{v'} \mod 2$ for all $v, v' \in \mathbf{a}$.*

*Let $\sigma \in \mathbb{Z}$ be such that*

*(i) $2n + 1 - (k_v - l/2) \leq \sigma - l/2 \leq k_v - l/2$ for all $v \in \mathbf{a}$,*
*(ii) $|\sigma - \frac{l}{2} - \frac{2n+1}{2}| + \frac{2n+1}{2} - (k_v - l/2) \in 2\mathbb{Z}$ for all $v \in \mathbf{a}$,*
*(iii) $k_v > l/2 + n(1 + k_v - l/2 - |\sigma - l/2 - (2n+1)/2| - (2n+1)/2)$ for all $v \in \mathbf{a}$,*

*but exclude the cases*

*(a) $\sigma = n + 1 + l/2$, $F = \mathbb{Q}$ and $\chi^2\psi_i^2 = 1$ for some $\psi_i$,*
*(b) $\sigma = l/2$, $\mathfrak{c} = \mathfrak{o}$ and $\chi\psi_S\psi_i = 1$ for some $\psi_i$,*
*(c) $0 < \sigma - l/2 \leq n$, $\mathfrak{c} = \mathfrak{o}$ and $\chi^2\psi_i^2 = 1$ for some $\psi_i$.*
*(d) $\sigma \leq l + n$ in case $F$ has class number larger than one.*

*Under these conditions*

$$\frac{\boldsymbol{\Lambda}(\sigma/2, \boldsymbol{f}, \chi)}{\pi^{e_\sigma} < \boldsymbol{f}, \boldsymbol{f} >} \in \overline{\mathbb{Q}},$$

*where*

$$e_\sigma = n \sum_{v \in \mathbf{a}} (k_v - l + \sigma) - de, \quad e := \begin{cases} n^2 + n - \sigma + l/2, & \text{if } 2\sigma - l \in 2\mathbb{Z} \text{ and } \sigma \geq 2n + l/2, \\ n^2, & \text{otherwise.} \end{cases}$$

This theorem will be proved at the end of the next section. First we need to introduce the notion of nearly holomorphic Siegel-Jacobi modular forms $N_{k,S}^{n,r}(\mathbf{\Gamma})$ for $r \in \mathbb{Z}^{\mathbf{a}}$.

## 5. Nearly holomorphic Siegel-Jacobi modular forms and algebraicity of special L-values

**Definition 5.1.** A $C^\infty$ function $\boldsymbol{f}(\tau, w) : \mathcal{H}_{n,l} \to \mathbb{C}$ is said to be a nearly holomorphic Siegel-Jacobi modular form (of weight $k$ and index $S$) for the congruence subgroup $\mathbf{\Gamma}$ if

  (1) $\boldsymbol{f}$ is holomorphic with respect to the variable $w$ and nearly holomorphic with respect to the variable $\tau$, that is, $\boldsymbol{f}$ as a function on $\tau$ belongs to the space $N^r(\mathbb{H}_n^d)$, $r \in \mathbb{N}$, where the space $N^r(\mathbb{H}_n^d)$ is defined in [13, page 99];
  (2) $\boldsymbol{f}|_{k,S}\gamma = \boldsymbol{f}$ for all $\gamma \in \mathbf{\Gamma}$.

We denote this space by $N_{k,S}^{n,r}(\mathbf{\Gamma})$ and write $N_{k,S}^{n,r} := \bigcup_{\mathbf{\Gamma}} N_{k,S}^{n,r}(\mathbf{\Gamma})$ for the space of all nearly holomorphic Siegel-Jacobi modular forms of weight $k$ and index $S$.

We note that if $\boldsymbol{f} \in N_{k,S}^{n,r}$, then $\boldsymbol{f}_*(\tau, v\,\Omega_\tau) \in N_k^{n,r}$, the space of nearly holomorphic Siegel modular forms, where recall $\Omega_\tau := {}^t(\tau\ 1_n)$, and $v \in M_{l,2n}(F)$. Below we extend Theorem 4.3 to the nearly-holomorphic situation.

**Theorem 5.2.** *Assume that $n > 1$ or $F \neq \mathbb{Q}$. Let $A \in \mathrm{GL}_l(F)$ be such that $AS\,{}^tA = \mathrm{diag}[s_1, \ldots, s_l]$, and define the lattices $\Lambda_1 := AM_{l,n}(\mathfrak{o}) \subset M_{l,n}(F)$ and $\Lambda_2 := 2\mathrm{diag}[s_1^{-1}, \ldots, s_l^{-1}]M_{l,n}(\mathfrak{o}) \subset M_{l,n}(F)$. Then there is an isomorphism*

$$\Phi : N_{k,S}^{n,r} \cong \bigoplus_{h \in \Lambda_1/\Lambda_2} N_{k-l/2}^{n,r}$$

*given by $\boldsymbol{f} \mapsto (f_h)_h$, where the $f_h \in N_{k-l/2}^{n,r}$ are defined by the expression*

$$\boldsymbol{f}(\tau, w) = \sum_{h \in \Lambda_1/\Lambda_2} f_h(\tau) \Theta_{2S, \Lambda_2, h}(\tau, w).$$

*Proof.* Given an $\boldsymbol{f} \in N_{k,S}^{n,r}$, the modularity properties with respect to the variable $w$ show that (see for example [11, proof of Proposition 3.5]) we may write

$$\boldsymbol{f}(\tau, w) = \sum_{h \in \Lambda_1/\Lambda_2} f_h(\tau) \Theta_{2S, \Lambda_2, h}(\tau, w)$$

for some functions $f_h(\tau)$ with the needed modularity properties. In order to establish that they are actually nearly holomorphic one argues similarly to the holomorphic case. Indeed, a close look at the proof of [11, Lemma 3.4] shows that the functions $f_h$ have the same properties (real analytic, holomorphic, nearly holomorphic, meromorphic, etc.)

with respect to the variable $\tau$ as $\boldsymbol{f}(\tau, w)$, since everything is reduced to a linear system of the form

$$\boldsymbol{f}(\tau, w_i) = \sum_{h \in \Lambda_1/\Lambda_2} f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w_i), \ \ i = 1, \ldots, \sharp\Lambda_1/\Lambda_2,$$

for some $\{w_i\}$ such that $\det(\Theta_{2S,\Lambda_2,h}(\tau, w_i)) \neq 0$. In particular, after solving the linear system of equations we see that the nearly holomorphicity of $f_h$ follows from that of $\boldsymbol{f}$ since the $\Theta_{2S,\Lambda_2,h}(\tau, w_i)$ are holomorphic with respect to the variable $\tau$.    □

The above theorem immediately implies the following.

**Corollary 5.3.** *For a congruence subgroup* $\boldsymbol{\Gamma}$, $N_{k,S}^{n,r}(\boldsymbol{\Gamma})$ *is a finite dimensional* $\mathbb{C}$ *vector space.*

*Proof.* The theorem above states that $N_{k,S}^{n,r}(\boldsymbol{\Gamma}) \cong \bigoplus_h N_{k-l/2}^{n,r}(\Gamma_h)$ for some congruence subgroups $\Gamma_h$, which are known to be finite dimensional (see [13, Lemma 14.3]).    □

Given an automorphism $\sigma \in Aut(\mathbb{C})$ and $\boldsymbol{f} \in N_{k,S}^{n,r}$, we define

$$\boldsymbol{f}^\sigma(\tau, w) := \sum_{h \in \Lambda_1/\Lambda_2} f_h^\sigma(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w),$$

where $f_h \in N_{k-l/2}^{n,r}$, and $f_h^\sigma$ is defined as in [13, page 117]. Also, for a subfield $K$ of $\mathbb{C}$, define the space $N_{k,S}^{n,r}(K)$ to be the subspace of $N_{k,S}^{n,r}$ such that $\Phi(N_{k,S}^{n,r}(K)) = \bigoplus_{h \in \Lambda_1/\Lambda_2} N_{k-l/2}^n(K)$. In particular, $\boldsymbol{f} \in N_{k,S}^{n,r}$ belongs to $N_{k,S}^{n,r}(K)$ if and only if $\boldsymbol{f}^\sigma = \boldsymbol{f}$ for all $\sigma \in Aut(\mathbb{C}/K)$. Moreover, if $K$ contains the Galois closure of $F$ in $\overline{\mathbb{Q}}$ and $\mathbb{Q}^{ab}$, then $N_{k,S}^{n,r} = N_{k,S}^{n,r}(K) \otimes_K \mathbb{C}$ as the same statement holds for $N_{k-l/2}^{n,r}$. Similarly it follows that if $\boldsymbol{f} \in N_{k,S}^{n,r}(\overline{\mathbb{Q}})$, then $\boldsymbol{f}|_{k,S}\boldsymbol{\gamma} \in N_{k,S}^{n,r}(\overline{\mathbb{Q}})$ for all $\boldsymbol{\gamma} \in \boldsymbol{G}(F)$. At this point we also remark that for an $\boldsymbol{f} \in M_{k,S}^n$ the $\boldsymbol{f}^c$ defined in Section 3 is nothing else than $\boldsymbol{f}^\rho$ where $1 \neq \rho \in Gal(\mathbb{C}/\mathbb{R})$ i.e. a complex conjugation.

We now define a variant of the holomorphic projection in the Siegel-Jacobi case. We define a map $\mathfrak{p}: N_{k,S}^{n,r}(\overline{\mathbb{Q}}) \to M_{k,S}^n(\overline{\mathbb{Q}})$ whenever $k_v - l/2 > n + r_v$ for all $v \in \mathbf{a}$ by

$$\mathfrak{p}(\boldsymbol{f}) := \mathfrak{p}\left(\sum_{h \in \Lambda_1/\Lambda_2} f_h(\tau)\Theta_{2S,\Lambda_2,h}(\tau, w)\right) := \sum_{h \in \Lambda_1/\Lambda_2} \widetilde{\mathfrak{p}}(f_h(\tau))\Theta_{2S,\Lambda_2,h}(\tau, w),$$

where $\widetilde{\mathfrak{p}}: N_{k-l/2}^{n,r}(\overline{\mathbb{Q}}) \to M_{k-l/2}^n(\overline{\mathbb{Q}})$ is the holomorphic projection operator defined for example in [13, Chapter III, Section 15] and its algebraic properties are established in [13, Lemma 28.2].

**Lemma 5.4.** *Assume* $n > 1$ *or* $F \neq \mathbb{Q}$, *and that* $\boldsymbol{f} \in S_{k,S}^n$ *satisfies the Property A, and* $k_v - l/2 > n + r_v$ *for all* $v \in \mathbf{a}$. *Then for any* $\boldsymbol{g} \in N_{k,S}^{n,r}(\overline{\mathbb{Q}})$,

$$< \boldsymbol{f}, \boldsymbol{g} > = < \boldsymbol{f}, \mathfrak{p}(\boldsymbol{g}) > .$$

*Proof.* This follows from the fact that the above property holds for nearly holomorphic Siegel modular forms, and the fact that the Property A allows us to write the Petersson

inner product of Siegel-Jacobi forms as a sum of Petersson inner products of Siegel modular forms, in a similar way as we did in the proof of Lemma 4.5. □

Let us write $F_1$ for the Hilbert class field extension of $F$ and denote by $\{\psi_i\}$ the ideal characters corresponding to the characters of $Gal(F_1/F)$. We can now state a theorem regarding the nearly holomorphicity of Siegel-type Jacobi Eisenstein series.

**Theorem 5.5.** *Consider the normalized Siegel-type Jacobi-Eisenstein series*

$$D(s) := D(z, s; k, \chi) := \Lambda^n_{k-l/2,\mathfrak{c}}(s - l/4, \chi\psi_S)E(z, \chi, s).$$

*Let $\mu \in \mathbb{Z}$ be such that*

(i) $n + 1 - (k_v - l/2) \le \mu - l/2 \le k_v - l/2$ *for all $v \in \mathbf{a}$, and*
(ii) $|\mu - l/2 - \frac{n+1}{2}| + \frac{n+1}{2} - k_v + l/2 \in 2\mathbb{Z}$,

*but exclude the cases*

(a) $\mu = \frac{n+2}{2} + l/2$, $F = \mathbb{Q}$ *and* $\chi^2\psi_i^2 = 1$ *for some $\psi_i$,*
(b) $\mu = l/2$, $\mathfrak{c} = \mathfrak{o}$ *and* $\chi\psi_S\psi_i = 1$ *for some $\psi_i$,*
(c) $0 < \mu - l/2 \le n/2$, $\mathfrak{c} = \mathfrak{o}$ *and* $\chi^2\psi_i^2 = 1$ *for some $\psi_i$.*
(d) $\mu \le l + n$ *if $F$ has class number larger than one.*

*Then*

$$D(\mu/2) \in \pi^\beta N^{n,r}_{k,S}(\overline{\mathbb{Q}}),$$

*where*

$$
r = \begin{cases}
\frac{n(k-\mu+2)}{2} & \text{if } \mu = \frac{n+2}{2} + \frac{l}{2},\ F = \mathbb{Q}, \chi^2 = 1, \\
\frac{k}{2} - \frac{l}{4} & \text{if } n = 1, \mu = 2 + \frac{l}{2}, F = \mathbb{Q}, \chi\psi_S = 1, \\
\frac{n}{2}(k - \frac{l}{2} - |\mu - \frac{l}{2} - \frac{n+1}{2}|\mathbf{a} - \frac{n+1}{2}\mathbf{a}) & \text{otherwise.}
\end{cases}
$$

*Moreover, $\beta = \frac{n}{2}\sum_{v \in \mathbf{a}}(k_v - l + \mu) - de$, where*

$$
e := \begin{cases}
[\frac{(n+1)^2}{4}] - \mu + \frac{l}{2} & \text{if } 2\mu - l + n \in 2\mathbb{Z}, \mu \ge n + \frac{l}{2}, \\
[\frac{n^2}{4}] & \text{otherwise.}
\end{cases}
$$

*Proof.* The proof is similar to the proof of Theorem 8.3 in [3], where the analytic properties of the series $D(s)$ were established. Indeed, the relation between the Jacobi Eisenstein series and the classical Siegel Eisenstein series (for details see [3, Section 8]) allows us to read off the nearly holomorphicity of $D(s)$ from the nearly holomorphicity of the latter series, which was established in [13, Theorem 17.9].

To be more precise, in [3] (page 45 there, just before Theorem 8.2) we related the Jacobi Eisenstein series $E(z, s, \chi)$ to a sum involving the Siegel Eisenstein series $E(\tau, s - l/4; \chi\psi_S\psi_i)$ (with the notation as in [3]), where $\psi_i$'s vary over all the finite unramified characters of $F$, that is the characters corresponding to the Hilbert class group of $F$. However, the normalizing factor for the Jacobi Eisenstein series $D(s)$ is $\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2 - l/4, \chi\psi_S)$ whereas for the various Siegel Eisenstein series is $\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2 - l/4, \chi\psi_S\psi_i)$, which of course depends on the characters $\psi_i$.

Therefore we need to assure that the series

$$\frac{\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2 - l/4, \chi\psi_S)}{\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2 - l/4, \chi\psi_S\psi_i)}\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2 - l/4, \chi\psi_S\psi_i)E(\tau, s - l/4; \chi\psi_S\psi_i)$$

has the same algebraic properties, i.e. whether it is defined over $\overline{\mathbb{Q}}$, as the normalized series

$$\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2 - l/4, \chi\psi_S\psi_i)E(\tau, s - l/4; \chi\psi_S\psi_i).$$

Of course, we need to exclude the cases where the factor $\frac{\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2-l/4,\chi\psi_S)}{\Lambda^n_{k-l/2,\mathfrak{c}}(\mu/2-l/4,\chi\psi_S\psi_i)}$ has a pole. Then it remains to check whether

$$\frac{\Lambda^n_{k-l/2}(\mu/2 - l/4, \chi\psi_S)}{\Lambda^n_{k-l/2}(\mu/2 - l/4, \chi\psi_S\psi_i)} \in \overline{\mathbb{Q}}.$$

This should follow from the general Bellinson conjectures for motives associated to finite Hecke characters over totally real fields (see for example [10]). However this is not known in general, and hence we are forced to set the condition $\mu > n + l$ in case $F$ has class number larger than one, in which case we obtain values whose ratio is known to be algebraic, since we are then considering critical values. $\square$

**Lemma 5.6.** *Consider the embedding*

$$\Delta : \mathcal{H}_{n,l} \times \mathcal{H}_{m,l} \hookrightarrow \mathcal{H}_{n+m,l}, \quad (\tau_1, w_1) \times (\tau_2, w_2) \mapsto (\mathrm{diag}[\tau_1, \tau_2], (w_1 \; w_2)).$$

*Then the pullback*

$$\Delta^* \left( N^{n+m,r}_{k,S}(\overline{\mathbb{Q}}) \right) \subset N^{n,r}_{k,S}(\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} N^{m,r}_{k,S}(\overline{\mathbb{Q}}).$$

*Proof.* The proof of this lemma is identical to the Siegel modular form case (see [13, Lemma 24.11]). Let $\boldsymbol{f} \in N^{n+m,r}_{k,S}(\boldsymbol{\Gamma}^{n+m}, \overline{\mathbb{Q}})$ for a sufficiently deep congruence subgroup $\boldsymbol{\Gamma}^{n+m}$. Note that the function $\boldsymbol{g}(z_1, z_2) := \Delta^*\boldsymbol{f}(\mathrm{diag}[z_1, z_2])$ is in $N^{n,r}_{k,S}(\boldsymbol{\Gamma}^n)$ as a function in $z_1$ and in $N^{m,r}_{k,S}(\boldsymbol{\Gamma}^m)$ as a function in $z_2$ for appropriate congruence subgroups $\boldsymbol{\Gamma}^n$ and $\boldsymbol{\Gamma}^m$. Hence, by Corollary 5.3 and the fact that $N^{n,r}_{k,S} = N^{n,r}_{k,S}(\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$, for each fixed $z_1$ we may write

$$\boldsymbol{g}(z_1, z_2) = \sum_i \boldsymbol{g}_i(z_1)\boldsymbol{h}_i(z_2),$$

where $\boldsymbol{g}_i(z_1) \in \mathbb{C}$, and $\boldsymbol{h}_i \in N^{n,r}_{k,S}(\overline{\mathbb{Q}})$ form a basis of the space. The general argument used in [13, Lemma 24.11], which is based on the linear independence of the basis $\boldsymbol{h}_i$, shows that the functions $\boldsymbol{g}_i(z_1)$ have the same properties as the function $\boldsymbol{g}$ when viewed as a function of the variable $z_1$. Hence, $\boldsymbol{g}_i \in N^{n,r}_{k,S}$. Now, for any $\sigma \in Aut(\mathbb{C}/\overline{\mathbb{Q}})$,

$$\boldsymbol{g}(z_1, z_2) = \boldsymbol{g}^\sigma(z_1, z_2) = \sum_i \boldsymbol{g}_i^\sigma(z_1)\boldsymbol{h}_i^\sigma(z_2) = \sum_i \boldsymbol{g}_i^\sigma(z_1)\boldsymbol{h}_i(z_2).$$

Hence, $\boldsymbol{g}_i^\sigma(z_2) = \boldsymbol{g}_i(z_2)$ for all $\sigma \in Aut(\mathbb{C}/\overline{\mathbb{Q}})$, and thus $\boldsymbol{g}_i \in N^{n,r}_{k,S}(\overline{\mathbb{Q}})$. $\square$

We can now establish a theorem which is the key result towards Theorem 4.6.

**Theorem 5.7.** *Assume $n > 1$ or $F \neq \mathbb{Q}$. Let $0 \neq \boldsymbol{f} \in S^n_{k,S}(\boldsymbol{\Gamma}, \overline{\mathbb{Q}})$ be an eigenfunction of $T(\mathfrak{a})$ for all integral ideas $\mathfrak{a}$ with $(\mathfrak{a}, \mathfrak{c}) = 1$. Define $\mu := \min_{v \in \mathbf{a}}\{k_v\}$ and assume that*

(1) $\mu > 2n + l + 1$,
(2) Property A holds for all $\widetilde{\boldsymbol{f}} \in V(\boldsymbol{f})$,
(3) $k_v \equiv k_{v'} \mod 2$ for all $v, v' \in \mathbf{a}$.
(4) $k_v > l/2 + n(1 + k_v - \mu)$ for all $v \in \mathbf{a}$.

Then for any $\boldsymbol{g} \in M_{k,S}^n(\overline{\mathbb{Q}})$,

$$\frac{<\boldsymbol{f}, \boldsymbol{g}>}{<\boldsymbol{f}, \boldsymbol{f}>} \in \overline{\mathbb{Q}}.$$

*Proof.* By Lemma 4.5 it suffices to prove this theorem for $\boldsymbol{g} \in S_{k,S}^n(\overline{\mathbb{Q}})$. Furthermore, as it was shown in [3, Section 7.4], the Hecke operators are normal and Proposition 4.2 states that the Hecke operators $T(\mathfrak{a})$ preserve $S_{k,S}^n(\boldsymbol{\Gamma}, \overline{\mathbb{Q}})$. That is, we have a decomposition

$$S_{k,S}^n(\boldsymbol{\Gamma}, \overline{\mathbb{Q}}) = V(\boldsymbol{f}) \oplus \boldsymbol{U},$$

where $\boldsymbol{U}$ is a $\overline{\mathbb{Q}}$-vector space orthogonal to $V(\boldsymbol{f})$. Therefore, without loss of generality, we may assume that $\boldsymbol{g} \in V(\boldsymbol{f})$.

Now consider a character $\chi$ of conductor $\mathfrak{f}_\chi \neq \mathfrak{o}$ such that $\chi_\mathbf{a}(x) = \mathrm{sgn}_\mathbf{a}(x)^k$, $\chi^2 \neq 1$ and $G(\chi, \mu - n - l/2) \in \overline{\mathbb{Q}}^\times$, where $G(\chi, \mu - n - l/2)$ is as in equation (3). The existence of such a character follows from the fact that $G(\chi, 2s - n - l/2)$ is the ratio of products of finitely many Euler polynomials.

We recall that if $\tilde{\boldsymbol{f}} \in V(\boldsymbol{f})$, then so is $\tilde{\boldsymbol{f}}^c \in V(\boldsymbol{f})$ and their $L$-functions agree. In particular, up to some non-zero algebraic number, the identity (2) becomes:

$$\Lambda_{k-l/2,\mathfrak{c}}^{2n}(\mu/2 - l/4, \chi\psi_S)vol(A) < (E|_{k,S}\boldsymbol{\rho})(\mathrm{diag}[z_1, z_2], \mu/2; \chi), (\tilde{\boldsymbol{f}}^c|_{k,S}\boldsymbol{\eta}_n)^c(z_2) >$$
$$= \overline{\mathbb{Q}}^\times c_{S,k}(\mu/2 - k/2)\Lambda(\mu/2, \boldsymbol{f}, \chi)\tilde{\boldsymbol{f}}^c(z_1).$$

Since we take $\mu > 2n + l + 1$, and $\mu$ is equal to the smallest of the $k_v$'s, it follows that for all $v \in \mathbf{a}$:

(1) $2n + 1 - (k_v - l/2) \leq \mu - l/2 \leq k_v - l/2$,
(2) $\mu - k_v \in 2\mathbb{Z}$;

the last condition is possible because all $k_v$ have the same parity. Thanks to the above choice of $\mu$ we can apply Theorem 5.5 (note that the excluded cases there do not apply for $\mu$ larger than $2n + l + 1$) so that

$$\Lambda_{k-l/2,\mathfrak{c}}^{2n}(\mu/2 - l/4, \chi\psi_S)E(z, \mu/2; \chi) \in \pi^\beta N_{k,S}^{2n,r}(\overline{\mathbb{Q}})$$

for some $\beta \in \mathbb{N}$ and

$$r = n(k - \mu).$$

The same then holds for

$$\Lambda_{k-l/2,\mathfrak{c}}^{2n}(\mu/2 - l/4, \chi\psi_S)E(z, \mu/2; \chi)|_{k,S}\boldsymbol{\rho}.$$

In particular,

$$\pi^{-\beta}\Lambda_{k-l/2,\mathfrak{c}}^{2n}(\mu/2 - l/4, \chi\psi_S)(E|_{k,S}\boldsymbol{\rho})(\mathrm{diag}[z_1, z_2], \mu/2; \chi) = \sum_i \boldsymbol{f}_i(z_1)\boldsymbol{g}_i(z_2),$$

where $\boldsymbol{f}_i, \boldsymbol{g}_i \in N_{k,S}^{n,r}(\overline{\mathbb{Q}})$ by Lemma 5.6. Moreover, $vol(A) = \pi^{d_0}\mathbb{Q}^{\times}$, where $d_0$ is the dimension of $\mathbb{H}_n^d$ since the volume of the Heisenberg part is normalized to one. Furthermore,

$$c_{S,k}(\mu/2 - k/2) \in \pi^{\delta}\overline{\mathbb{Q}}^{\times}, \ \ \delta \in \frac{1}{2}\mathbb{Z}.$$

Altogether we obtain

$$\sum_i \boldsymbol{f}_i(z_1) < \boldsymbol{g}_i(z_2), \boldsymbol{g}(z_2) >= \overline{\mathbb{Q}}^{\times} \pi^{\delta - d_0 + \beta}\boldsymbol{\Lambda}(\mu/2, \boldsymbol{f}, \chi)\tilde{\boldsymbol{f}}^c(z_1),$$

where $\boldsymbol{g} := (\tilde{\boldsymbol{f}}^c|_{k,S}\boldsymbol{\eta}_n)^c = \tilde{\boldsymbol{f}}|_{k,S}\boldsymbol{\eta}_n^{-1} \in S_{k,S}^n(\overline{\mathbb{Q}})$. Considering the Fourier expansion of $\boldsymbol{f}_i$'s and $\boldsymbol{f}$, and comparing any $(r,t)$ coefficients for which $c(r,t; \tilde{\boldsymbol{f}}^c) \neq 0$, we find that

$$< \sum_i \alpha_{i,r,t}\boldsymbol{g}_i(z_2), \boldsymbol{g}(z_2) >= \overline{\mathbb{Q}}^{\times} \pi^{\delta - d_0 + \beta}\boldsymbol{\Lambda}(\mu/2, \boldsymbol{f}, \chi) \neq 0$$

for some $\alpha_{i,r,t} \in \overline{\mathbb{Q}}$, where the non-vanishing follows from (1), a corollary to Theorem 3.1. Setting $\boldsymbol{h}_{r,t}(z_2) := \sum_i \alpha_{i,r,t}\boldsymbol{g}_i(z_2) \in N_{k,S}^{n,r}(\overline{\mathbb{Q}})$, we obtain

$$< \boldsymbol{h}_{r,t}(z_2), \boldsymbol{g}(z_2) >= \overline{\mathbb{Q}}^{\times} \pi^{\delta - d_0 + \beta}\boldsymbol{\Lambda}(\mu/2, \boldsymbol{f}, \chi) \neq 0,$$

or,

$$< \mathfrak{p}^0(\boldsymbol{h}_{r,t}|_{k,S}\boldsymbol{\eta}_n)(z_2), \tilde{\boldsymbol{f}}(z_2) >= \overline{\mathbb{Q}}^{\times} \pi^{\delta - d_0 + \beta}\boldsymbol{\Lambda}(\mu/2, \boldsymbol{f}, \chi) \neq 0,$$

where

$$\mathfrak{p}^0 := \begin{cases} \mathfrak{p}, & k \text{ not parallel,} \\ \mathfrak{q} \circ \mathfrak{p}, & k \text{ parallel,} \end{cases}$$

and we have used the assumptions (2) and (4) in the statement of the theorem in order to be able to apply Lemma 5.4.

Since $\tilde{\boldsymbol{f}} \in V(\boldsymbol{f})$ was arbitrary, the forms $\widetilde{\boldsymbol{h}}_{r,t} := \mathfrak{p}^0(\boldsymbol{h}_{r,t}|_{k,S}\boldsymbol{\eta}_n) \in S_{k,S}^n(\overline{\mathbb{Q}})$ (or rather their projections to $V(\boldsymbol{f})$) for the various $(r,t)$ span the space $V(\boldsymbol{f})$ over $\overline{\mathbb{Q}}$ and

$$< \widetilde{\boldsymbol{h}}_{r,t}, \tilde{\boldsymbol{f}} >\in \pi^{\delta - d_0 + \beta}\boldsymbol{\Lambda}(\mu/2, \boldsymbol{f}, \chi)\overline{\mathbb{Q}}^{\times}.$$

That is, for any $\boldsymbol{g} \in V(\boldsymbol{f})$ we have $< \boldsymbol{g}, \boldsymbol{f} >\in \pi^{\delta - d_0 + \beta}\boldsymbol{\Lambda}(\mu/2, \boldsymbol{f}, \chi)\overline{\mathbb{Q}}^{\times}$. In particular, the same holds for $\boldsymbol{g} = \boldsymbol{f}$, and that concludes the proof.            $\square$

We are now ready to give the proof of Theorem 4.6.

*Proof of Theorem 4.6.* We follow the same steps as in the proof of Theorem 5.7 but this time we apply Theorem 5.5 with $\mu = \sigma/2$. Note that the restrictions on $\sigma$ in conditions $(i)$ and $(ii)$ are the ones that make the corresponding Eisenstein series nearly holomorphic (compare with conditions in Theorem 5.5). Condition $(iii)$ allows us to use the holomorphic projection operator. Finally, the restrictions on the minimal weight are set so that we can apply the above Theorem 5.7 on the algebraicity of the ratio of the Petersson inner products.

In exactly the same way as above we obtain

$$< \boldsymbol{h}_{r,t}(z_2), \boldsymbol{f}(z_2) >= \overline{\mathbb{Q}}^{\times} \pi^{\delta - d_0 + \beta} \boldsymbol{\Lambda}(\sigma/2, \boldsymbol{f}, \chi),$$

for some $\boldsymbol{h}_{r,t} \in \boldsymbol{N}_{k,S}^n(\overline{\mathbb{Q}})$. Thanks to Theorem 5.7 the proof will be finished after dividing the above equality by $< \boldsymbol{f}, \boldsymbol{f} >$ if we make the powers of $\pi$ precise. Recall that

$$c_{S,k}(\sigma/2 - k/2) = \overline{\mathbb{Q}}^{\times} \pi^{dn(n+1)/2} \prod_{v \in \mathbf{a}} \frac{\Gamma_n(\sigma/2 + k_v - l/2 - (n+1)/2)}{\Gamma_n(\sigma/2 + k_v - l/2)}$$

$$= \pi^{dn(n+1)/2} \prod_{v \in \mathbf{a}} \frac{\prod_{i=0}^{n-1} \Gamma(\sigma/2 + k_v - l/2 - (n+1)/2 - i/2)}{\prod_{i=0}^{n-1} \Gamma(\sigma/2 + k_v - l/2 - i/2)} = \overline{\mathbb{Q}}^{\times} \pi^{dn(n+1)/2}.$$

Hence, $\delta = dn(n+1)/2$. However, this is also equal to the dimension of the space $\mathbb{H}_n^d$, which we denoted by $d_0$. We are then left only with $\beta$, which is provided by Theorem 5.5; namely,

$$\beta = n \sum_{v \in \mathbf{a}} (k_v - l + \sigma) - de,$$

where $e := n^2 + n - \sigma + l/2$ if $2\sigma - l \in 2\mathbb{Z}$ and $\sigma \geq 2n + l/2$, and $e := n^2$ otherwise. This concludes the proof of the theorem. $\qquad \square$

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