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Summary of professional accomplishments

Zbigniew Leśniak

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- 1988 – Master of Sciences in Mathematics (Jagiellonian University in Kraków), the title of Master Thesis: The Martin axiom (in Polish), supervisor: dr B. Grell

- 1994 – Ph.D. in Mathematics (Faculty of Mathematics, Physics and Technical Science, Pedagogical University of Cracow), the title of Ph.D. Thesis: The Abel equation on the plane and its applications in the iteration theory (in Polish), supervisor: prof. dr hab. M. C. Zdun

Studies and employment in scientific institutions:

- 1983-1988 – Jagiellonian University in Kraków, Faculty of Mathematics and Physics, field of study: mathematics

- from 1 October 1988 till now – employed in the Department of Mathematics, Pedagogical University of Cracow - research and teaching position: 1988-1989 junior assistant, 1989-1994 assistant, from 1994 assistant professor

Contents

Chapter 1

The scientific achievement and remaining publications

The series of articles indicated as a scientific achievement consists of 6 articles and is entitled:

Flows of Brouwer homeomorphisms – the form, topological equivalence and conjugacy

1.1 List of articles constituting the scientific achievement

The indicated series consists of the following articles:

[A1] Z. Leśniak, *On boundaries of parallelizable regions of flows of free mappings*, Abstr. Appl. Anal., Vol. 2007 (2007), Article ID 31693, 8 pp.

[A2] Z. Leśniak, *On a decomposition of the plane for a flow free mappings*, Publ. Math. Debrecen 75 (2009), No. 1-2, 191–202.

[A3] Z. Leśniak, *On fractional iterates of a Brouwer homeomorphism embeddable in a flow*, J. Math. Anal. Appl. 366 (2010), No. 1, 310–318.

[A4] Z. Leśniak, *On the topological equivalence of flows of Brouwer homeomorphisms*, J. Difference Equ. Appl. 22 (2016), 853–864.

[A5] Z. Leśniak, *On properties of the set of invariant lines of a Brouwer homeomorphism*, J. Difference Equ. Appl. 24 (2018), 746–752.

[A6] Z. Leśniak, *On the topological conjugacy of Brouwer flows*, Bull. Malays. Math. Sci. Soc., DOI: 10.1007/s40840-017-0567-8.

1.2 List of remaining articles

The remaining articles listed in the chronological order are the following:

[B1] Z. Leśniak, *On homeomorphic and diffeomorphic solutions of the Abel equation on the plane*, Ann. Polon. Math. 58 (1993), No. 1, 7–18.

[B2] Z. Leśniak, *On simultaneous Abel inequalities*, Opuscula Math. 14 (1994), 107–115.

[B3] M.C. Zdun, Z. Leśniak, *On iteration groups of singularity-free homeomorphisms of the plane*, Ann. Math. Sil. 8 (1994), 203–210.

[B4] Z. Leśniak, *On the system of the Abel equations on the plane*, Ann. Math. Sil. 9 (1995), 105–122.

[B5] Z. Leśniak, *Constructions of fractional iterates of Sperner homeomorphisms of the plane*, Förg-Rob, W. (ed.) et al., Iteration theory. Proceedings of the European conference, ECIT '92, Batschuns, Austria, September 13–19, 1992, World Scientific, Singapore (1996), 182–192.

[B6] Z. Leśniak, *On continuous iteration groups of some homeomorphisms of the plane*, Grazer Math. Ber. 334 (1997), 193–198.

[B7] Z. Leśniak, *On fractional iterates of a homeomorphism of the plane*, Ann. Polon. Math. 79 (2002), No. 2, 129–137.

[B8] Z. Leśniak, *On an equivalence relation for free mappings embeddeable in a flow*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 17 (2003), No. 7, 1911–1915.

[B9] Z. Leśniak, *On parallelizability of flows of free mappings*, Aequationes Math. 71 (2006), No. 3, 280–287.

[B10] Z. Leśniak, *On parallelizable regions of flows of the plane*, Grazer Math. Ber. 350 (2006), 175–183.

[B11] Z. Leśniak, *On maximal parallelizable regions of flows of the plane*, Int. J. Pure Appl. Math. 30 (2006), No. 2, 151–156.

[B12] Z. Leśniak, *On boundary orbits of a flow of free mappings of the plane*, Int. J. Pure Appl. Math. 42 (2008), No. 1, 5–11.

[B13] Z. Leśniak, *On the first prolongational limit set of flows of free mappings*, Tamkang J. Math. 39 (2008), No. 3, 263–269.

[B14] Z. Leśniak, *On the existence of analytic solutions of the d'Alembert equation*, Int. J. Pure Appl. Math. 48 (2008), No. 3, 385–397.

[B15] Z. Leśniak, Yong-Guo Shi, *One class of planar rational involutions*, Nonlinear Anal. 74 (2011), No. 17, 6097–6104.

[B16] Z. Leśniak, *On the structure of Brouwer homeomorphisms embeddable in a flow*, Abstr. Appl. Anal., Vol. 2012 (2012), Article ID 248413, 8 pp.

[B17] Yong-Guo Shi, Lin Li, Z. Leśniak, *On conjugacy of r-modal interval maps with non-monotonicity height equal to 1*, J. Difference Equ. Appl. 19 (2013), 573–584.

[B18] K. Ciepliński, Z. Leśniak, *On conjugacy equation in dimension one*, Banach Center Publ. 99 (2013), 31–44.

[B19] Z. Leśniak, *On strongly irregular points of a Brouwer homeomorphism embeddable in a flow*, Abstr. Appl. Anal., Vol. 2014 (2014), Article ID 638784, 7 pp.

[B20] J. Brzdęk, K. Ciepliński, Z. Leśniak, *On Ulam's type stability of the linear equation and related issues*, Discrete Dyn. Nat. Soc., Vol. 2014 (2014), Art. ID 536791, 14 pp.

[B21] A. Bahyrycz, J. Brzdęk, Z. Leśniak, *On approximate solutions of the generalized Volterra integral equation*, Nonlinear Anal. Real World Appl. 20 (2014), 59–66.

[B22] Z. Leśniak, Yong-Guo Shi, *Topological conjugacy of piecewise monotonic functions of nonmonotonicity height* $\geq$ *1*, J. Math. Anal. Appl. 423 (2015), 1792–1803.

[B23] J. Brzdęk, L. Cădariu, K. Ciepliński, A. Fošner, Z. Leśniak, *Survey on recent Ulam stability results concerning derivations*, J. Funct. Spaces, Vol. 2016 (2016), Article ID 1235103, 9 pp.

[B24] J. Brzdęk, El-s. El-hady, W. Förg-Rob, Z. Leśniak, *A note on solutions of a functional equation arising in a queueing model for a LAN gateway*, Aequationes Math. 90 (2016), 671–681.

[B25] J. Brzdęk, Z. Leśniak, R. Malejki, *On the generalized Fréchet functional equation with constant coefficients and its stability*, Aequationes Math. 92 (2018), 355–373.

[B26] J. Brzdęk, El-s. El-hady, Z. Leśniak, *On fixed points of a linear operator of polynomial form of order 3*, J. Fixed Point Theory Appl. 20 (2018), No. 2, Article:85, 10 pp.

[B27] J. Brzdęk, El-s. El-hady, Z. Leśniak, *On Fixed-point theorem in classes of function with values in a dq-metric space*, J. Fixed Point Theory Appl. 20 (2018), No. 4, Article:143, 16 pp.

Chapter 2

Results of the scientific achievement

The present chapter is an essential part of this report. It covers the results constituting the indicated scientific achievement. It is divided into seven sections according to respective considered issues.

In the first section we present the definitions and theorems which are the starting point for studies on Brouwer homeomorphisms. After introducing the relevant definitions we present the Brower translation theorem and the Brouwer lemma. Then, there are definitions of regular and irregular points and the theorem about the structure of any Brouwer homeomorphism given by T. Homma and H. Terasaka. This theorem along with the Brower translation theorem was what guided the research programme. This section also contains the basic results describing properties of flows of Brouwer homeomorphisms.

In the second section we describe properties of the codivergency relation. The most of the results presented here hold for any Brouwer homeomorphism, without the assumption that it is embeddable in a flow. Moreover, in this section we show the application of the theorem saying that the set of all regular points of a Brouwer homeomorphism embeddable in a flow is equal to the first prolongational limit set of the flow which contains this homeomorphism. Using this theorem we study further properties of the codivergency relation defined for a Brouwer homeomorphism embeddable in a flow.

The third section contains theorems concerning parallelizable regions of a flow of Brouwer homeomorphisms. Trajectories contained in the boundaries such regions pay an important role in our considerations. Therefore, the most of the results presented in this section describe properties of the first prolongational limit set of the boundary of a parallelizable region.

The main outcome of the fourth section is the theorem about the form of a flow of Brouwer homeomorphisms. One can also find here the result which describes the relationship between parallelizing homeomorphisms of maximal parallelizable regions forming a family covering the plane which occurs in this result.

In the fifth section we apply the theorem about the form of a flow of Brouwer homeomorphisms to determine iterative roots of a Brouwer homeomorphism embeddable in a flow. To show the continuity of the constructed roots we use properties of trajectories which are contained in the boundary of maximal parallelizable regions of the family occurring in the main result of the previous section.

The sixth section is devoted to the problem of the topological equivalence of flows of Brouwer homeomorphisms. It includes, among others, the result which says that a homeomorphism which realizes the topological equivalence of such flows, maps the first prolongational limit set of one of these flows onto the first prolongational limit set of the second one.

The most important results are provided in the seventh section. They concern the topological conjugacy of flows of Brouwer homeomorphisms. In the proof of the theorem about the topological conjugacy of such flows we use the result describing the form of a flow of Brouwer homeomorphisms and theorems regarding the topological equivalence of such flows.

The bibliography contains a list of papers and books that had a significant impact on the results presented in this report (direct or indirect).

2.1 Brouwer homeomorphisms

In this section we review fundamental properties of plane mappings called *Brouwer homeomorphisms*, i.e. homeomorphisms of the plane onto itself which have no fixed points and preserves orientation. In particular we recall the Brouwer translation theorem and the theorem describing the structure of any Brouwer homeomorphism given by T. Homma and H. Terasaka.

Before we explain the notion of preserving orientation we fix the terminology which will be used throughout this report. By a *curve* we mean a continuous mapping $\gamma : [0,1] \to \mathbb{R}^2$. A curve is called an *arc*, if it is a one-to-one mapping. A curve γ is said to be *closed* if $\gamma(0) = \gamma(1)$. By *Jordan curve* we mean a closed curve such that $\gamma_{|\,[0,1)}$ is a one-to-one mapping. As the image of a curve will be sometimes called a curve, curves are denoted by Greek small letters and their images by Latin capital letters to avoid any misunderstanding. Similarly, the image of an arc will be also called an arc.

We define the *index* $\mathrm{Ind}_\gamma(p)$ of a point p with respect to a closed curve γ such that $p \in \mathbb{R}^2 \setminus \gamma([0,1])$ in a two-stage way. Firstly, we specify an index of a point $p = (x_0, y_0) \in \mathbb{R}^2$ with respect to the elements of the family $\{\gamma_k : k \in \mathbb{Z}\}$, where

$$\gamma_k(t) = (x_0 + \cos 2k\pi t,\ y_0 + \sin 2k\pi t),$$

putting $\mathrm{Ind}_{\gamma_k}(p) = k$ (dependence of the curve γ_k on the point p is not shown in the notation of this curve because there is no need to change the fixed point in our reasoning). The image of curve γ_k is equal to the circle with centre p and radius 1 for $k \neq 0$ and the one-element set containing $q = (x_0 + 1, y_0)$ for $k = 0$.

Next, applying the above mentioned theorem that states that for each closed curve γ such that $p \in \mathbb{R}^2 \setminus \gamma([0,1])$ there exists exactly one $k \in \mathbb{Z}$ such that curves γ

and γ_k are homotopic in $\mathbb{R}^2 \setminus \{p\}$ (cf. Newman [93], Theorem 8.6, p. 192) we define $\mathrm{Ind}_\gamma(p)$ as the index of the point p with respect to the closed curve γ_k homotopic with γ in $\mathbb{R}^2 \setminus \{p\}$.

In order to define the notion of orientation preserving homeomorphism of the plane onto itself, we use the following theorem.

Theorem 2.1. (Newman [93], Theorem 11.1, p. 197) *For each homeomorphism f of the plane onto itself there exists exactly one number $d_f \in \{-1, 1\}$ such that*

$$\mathrm{Ind}_\gamma(p) = d_f \cdot \mathrm{Ind}_{f\circ\gamma}(f(p))$$

for every $p \in \mathbb{R}^2$ and every closed curve $\gamma : [0,1] \to \mathbb{R}^2$ such that $p \notin \gamma([0,1])$.

If $d_f = 1$, then we say that the homeomorphism f *preserves orientation*, and if $d_f = -1$ we say that f *reverses orientation*. Since d_f does not depend on the choice of the point p and of the closed curve γ, to identify if a homeomorphism f of the plane onto itself preserves or reverses orientation it will suffice to take a point p and check the indices of points p and $f(p)$ with respect to γ and $f \circ \gamma$, respectively, for a Jordan curve γ such that $p \notin \gamma([0,1])$ and $\mathrm{Ind}_\gamma(p) \neq 0$. For a homeomorphism of the plane of class C^1 a necessary and sufficient condition for preserving orientation is the positivity of the Jacobian determinant of this homeomorphism in at least one point (cf. Newman [93], Theorem 11.2, p. 198).

The study of preserving orientation homeomorphisms of the plane onto itself without fixed points has been initiated by Luitzen E.J. Brouwer. In 1912 it has been published a theorem called the Brouwer plane translation theorem which can be formulated in the following way.

Theorem 2.2. (Brouwer [19], Translationssatz) *Let f be a Brouwer homeomorphism. Then for each $p \in \mathbb{R}^2$ there exists a simply connected region U_p such that $p \in U_p$, $f(U_p) = U_p$, and a homeomorphism $\varphi : U_p \to \mathbb{R}^2$ satisfying the Abel equation*

$$\varphi(f(x,y)) = \varphi(x,y) + (1,0), \qquad (x,y) \in U_p \tag{2.1}$$

such that for every $t \in \mathbb{R}$ the preimage $\varphi^{-1}(\{t\} \times \mathbb{R})$ is a closed set on the plane.

Condition (2.1) means that the restriction $f|_{U_p}$ of f to the region U_p is topologically conjugate with the translation T given by the formula $T(x_1, x_2) = (x_1 + 1, x_2)$ by the homeomorphism $\varphi : U_p \to \mathbb{R}^2$, i.e.

$$\varphi \circ f|_{U_p} = T \circ \varphi.$$

One can find another result called the Brouwer plane translation theorem. It has been stated by Stephen A. Andrea ([5], Proposition 1.1) and is a weaken version of the result given by Brouwer. It can be found in the book of S. Alpern i V.S. Prasad [4] in the following form.

Theorem 2.3. (Alpern, Prasad [4], Theorem 5.1, p. 32) *Let f be a Brouwer homeomorphism. Then if a continuum (i.e. a nonempty compact connected set) D satisfies the condition $f(D) \cap D = \emptyset$, then $f^n(D) \cap D = \emptyset$ for every $n \in \mathbb{Z} \setminus \{0\}$.*

In this report, by the Brouwer plane translation theorem we mean Theorem 2.2.

The Brouwer lemma presented below plays an important role in the proofs of results describing properties of Brouwer homeomorphisms.

Theorem 2.4. (Brouwer [19], Satz 1 & 2) *Let f be a Brouwer homeomorphism and let $p \in \mathbb{R}^2$. Assume that K is an arc with endpoints p and $f(p)$ such that*

$$f(K) \cap K = \{f(p)\}.$$

Then the set $\bigcup_{n\in\mathbb{Z}} f^n(K)$ is a homeomorphic image of the set of real numbers.

An arc K occurring in the Brouwer lemma is called a *translation arc*. Here, by an arc we mean the range of an one-to-one continuous function $\gamma : [0, 1] \to \mathbb{R}^2$, since in this case the essential thing is that $\gamma(0) = p$, $\gamma(1) = f(p)$, and the parametrization of the set K is not important. The set $\bigcup_{n\in\mathbb{Z}} f^n(K)$ will be said to be a *translation curve*.

Let us note the for a homeomorphism φ occurring in the Brouwer plane translation theorem the preimage $C_s := \varphi^{-1}(\mathbb{R} \times \{s\})$ is a translation curve for every $s \in \mathbb{R}$, but it does not have to be a closed set in the plane. Translation curves which are closed sets are essentially our concern. To shorten statements of the presented results, the homeomorphic image of a straight line which is a closed set will be called a *line*. Relations describing the mutual placement of triples of pairwise disjoint invariant lines play an important role for studying properties of Brouwer homeomorphisms.

Denote by $\mathcal{F}$ a family which consists of pairwise disjoint lines. According to the Jordan curve theorem for the two dimensional sphere, each element of the family $\mathcal{F}$ divides the plane into two simply connected regions. Thus any two different elements C_1, C_2 of the family $\mathcal{F}$ divide the plane into three simply connected regions in such a way that only one of them contains C_1 and C_2 in its boundary. This region will be called a *strip* between C_1 and C_2.

For any distinct elements C_1, C_2, C_3 of the family $\mathcal{F}$ one of the following two possibilities must be satisfied: exactly one of the elements C_1, C_2, C_3 is contained in the strip between the other two or each of the elements C_1, C_2, C_3 is contained in the strip between the other two. In the first case if C_j is the trajectory which lies in the strip between C_i and C_k we will write $C_i|C_j|C_k$ (i, j, $k \in \{1, 2, 3\}$ and i, j, k are different). In the second case we will write $|C_1, C_2, C_3|$. So we have, either exactly on the the elements C_i, C_j, C_k, say C_j, divides the plane in such a way that the other two are subsets of the different components of of its complement $\mathbb{R}^2 \setminus C_j$, or each the elements C_i, C_j, C_k divides the plane in such a way that the other two are subsets of the same component of its complement.

The mutual relations of triples of elements of a family of pairwise disjoint lines which covers the plane has been considered by Wilfred Kaplan [58]. The configurations $|C_1, C_2, C_3|^+$ and $|C_1, C_2, C_3|^-$ occurring in the Kaplan paper has been replaced here by the configuration $|C_1, C_2, C_3|$, since in our considerations it is not important whether a Jordan curve having exactly one common point with each of the sets C_1, C_2, C_3 and the orientation given by the order of these points is oriented consistently or inconsistently with the unit circle.

Now we discuss the definition and fundamental properties of the codivergency relation defined in a paper of Stephen Andrea [5]. In this definition the sequences of iterates of arcs are used. If f is a Brouwer homeomorphism, then for each point $p \in \mathbb{R}^2$ we have $f^n(p) \to \infty$ as $n \to \pm\infty$ (cf. Brouwer [19], Satz 8). However, in general this property does not hold if we replace a point by an arc.

The definition of the codivergency relation for a given Brouwer homeomorphism f can be formulated in the following way:

$$p \sim q, \quad \text{if } p = q \text{ or } p,\ q \text{ are endpoint of an arc } K \text{ for which } f^n(K) \to \infty \text{ as } n \to \pm\infty.$$

One can observe that the relation defined above is an equivalence relation. In order to avoid considering degenerated arcs, the reflexivity of the codivergency relation is guaranteed directly in the definition.

S. Andrea has proved that a Brouwer homeomorphism cannot have exactly two equivalence classes (cf. [5], Proposition 3.2). Moreover, he has noted that for each positive integer n different from 2, one can construct a Brouwer homeomorphism which has exactly n equivalence classes. In the survey paper of Morton Brown [21] we can find examples of Brouwer homeomorphisms with the countable family of equivalence classes as well as with the uncountable family of equivalence classes.

Now we proceed to the issue of invariance of equivalence classes of the codivergency relation. M. Brown, E.E. Slaminka, W. Transue [23] and E.W. Daw [29] have given examples of Brouwer homeomorphisms which have no invariant equivalence class of the codivergency relation.

M. Brown ([21], p. 56) has noted that a Brouwer homeomorphism has no invariant equivalence classes if and only if there are no invariant translation curves which are closed sets, i.e. every translation curve is not a closed set. If an equivalence class of the codivergency relation is invariant, then this class contains an invariant translation curve which is a closed set. A construction of such translation curve has been described in the proof of the above mentioned theorem which says that a Brouwer homeomorphism cannot have exactly two equivalence classes of the codivergency relation (cf. Andrea [5], Proposition 3.2).

Now we proceed to a result given by T. Homma and H. Terasaka [51] that describes the structure of any Brouwer homeomorphism. For any sequence of subsets $(A_n)_{n\in\mathbb{N}}$ of the plane we define *limit superior* $\limsup_{n\to\infty} A_n$ as the set of all points $p \in \mathbb{R}^2$ such that any neighbourhood of p has common points with infinitely many elements of the sequence $(A_n)_{n\in\mathbb{N}}$. We can write it in the following way

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \mathrm{cl}\,(\bigcup_{m=n}^{\infty} A_m).$$

Thus $\limsup_{n\to\infty} A_n$ is a closed set.

For a Brouwer homeomorphism f and a subset B of the plane we define the *positive limit set* $\omega_f(B)$ as the limit superior of the sequence of its iterates $(f^n(B))_{n\in\mathbb{N}}$ and the *negative limit set* $\alpha_f(B)$ as the limit superior of the sequence $(f^{-n}(B))_{n\in\mathbb{N}}$. Under the assumption that B is compact, the sets can be represented in the form (see Nakayama [91]):

$$\begin{aligned} \omega_f(B) = \{q \in \mathbb{R}^2 : \ & \text{there exist sequences } (p_j)_{j\in\mathbb{N}} \text{ and } (n_j)_{j\in\mathbb{N}} \text{ such} \\ & \text{that } p_j \in B,\ n_j \in \mathbb{N},\ n_j \to +\infty,\ f^{n_j}(p_j) \to q \text{ as} \\ & j \to +\infty\}, \\ \alpha_f(B) = \{q \in \mathbb{R}^2 : \ & \text{there exist sequences } (p_j)_{j\in\mathbb{N}} \text{ and } (n_j)_{j\in\mathbb{N}} \text{ such} \\ & \text{that } p_j \in B,\ n_j \in \mathbb{N},\ n_j \to +\infty,\ f^{-n_j}(p_j) \to q \\ & \text{as } j \to +\infty\}. \end{aligned}$$

T. Homma and H. Terasaka [51] have introduced the notions of positively irregular point and negatively irregular point for any Brouwer homeomorphism. A point p is called *positively irregular* if $\omega_f(B) \neq \emptyset$ for each Jordan domain B containing p in its interior, and *negatively irregular* if $\alpha_f(B) \neq \emptyset$ for each Jordan domain B containing p in its interior, where by a Jordan domain we mean the union of a Jordan curve J and the Jordan region determined by J (i.e. the bounded component of $\mathbb{R}^2 \setminus J$). A point which is positively or negatively irregular is called *irregular*, otherwise it is *regular*.

For an irregular point p of a Brouwer homeomorphism f the set $P^+(p)$ is defined as the intersection of all $\omega_f(B)$ and the set $P^-(p)$ as the intersection of all $\alpha_f(B)$, where B are Jordan domains containing p in its interior. Moreover, we put $P(p) := P^+(p)\cup P^-(p)$. A positively irregular point p is *strongly positively irregular* if $P^+(p) \neq \emptyset$. Similarly, a negatively irregular point p is *strongly negatively irregular* if $P^-(p) \neq \emptyset$. We say that p is *strongly irregular* if it is strongly positively irregular or strongly negatively irregular. Otherwise, an irregular point p is said to be *weakly irregular*.

The announced result describing the structure of any Brouwer homeomorphism can be stated as follows.

Theorem 2.5. (Homma, Terasaka [51], First structure theorem) *Let f be a Brouwer homeomorphism. Then the plane is divided into at most three kinds of disjoint sets: $\{O_i : i \in I\}$, where $I = \mathbb{N}$ or $I = \{1, \dots, n\}$ for a positive integer n, $\{O_i^{'} : i \in \mathbb{N}\}$ and F. The sets $\{O_i : i \in I\}$ and $\{O_i^{'} : i \in \mathbb{N}\}$ are the components of the set of all regular points such that each O_i is an unbounded invariant simply connected region and can be filled with a family of pairwise disjoint translation lines which are closed sets and each $O_i^{'}$ is a simply connected region satisfying the condition $O_i^{'} \cap f^n(O_i^{'}) = \emptyset$*

for $n \in \mathbb{Z} \setminus \{0\}$. The set F of all irregular points is equal to the closure of the set of all strongly irregular points.

Result presented in this report mainly concern Brouwer homeomorphisms embeddable in a flow. Now, we present notions used in the study of properties of such flows.

By a *flow* we mean a family $\{f^t : t \in \mathbb{R}\}$ of homeomorphisms of the plane onto itself with the composition operation which satisfies the conditions

(1) the function $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2, \phi(x,t) = f^t(x)$ is continuous,

(2) $f^t(f^s(x)) = f^{t+s}(x)$ for $x \in \mathbb{R}^2$, $t, s \in \mathbb{R}$.

We say that a Brouwer homeomorphism f is *embeddable in a flow*, if there exists a flow $\{f^t : t \in \mathbb{R}\}$ such that $f = f^1$.

One can show that every element of a flow $\{f^t : t \in \mathbb{R}\}$, where f^t is homeomorphism of the plane onto itself, has to preserve orientation. Moreover, if one of elements of a flow is a Brouwer homeomorphism, then each element of this flow except the identity mapping has no fixed point. This fact can be deduced from the following theorem.

Theorem 2.6. (Andrea [5], Proposition 2.1) *Let f be a Brouwer homeomorphism embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then for each $p \in \mathbb{R}^2$ we have $f^t(p) \to \infty$ as $t \to \pm\infty$.*

Thus if an element of a flow is a Brouwer homeomorphism, then each element of this flow except the identity mapping is a Brouwer homeomorphism. Then we will say that $\{f^t : t \in \mathbb{R}\}$ is a flow of Brouwer homeomorphisms.

From Theorem 2.6 we obtain that the trajectory of each point $p \in \mathbb{R}^2$, i.e. the set $C_p := \{f^t(p) : t \in \mathbb{R}\}$, is a translation curve and is a closed set. Therefore, the family of all trajectories of a flow of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$ will serve as an important example of the family $\mathcal{F}$ defined above and we can consider the two configurations of triples of pairwise disjoint invariant lines in the family of all trajectories of the flow.

Under the assumption that Brouwer homeomorphism f is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$, each equivalence class of the codivergency relation is invariant. More precisely, for each equivalence class G we have $f^t(G) = G$ for $t \in \mathbb{R}$ (cf. Andrea [5], Proposition 3.1). In particular, for each point $p \in \mathbb{R}^2$ the trajectory C_p is contained in the equivalence class G_p which contains p.

Let us recall the definition of a parallelizable region of a flow of Brouwer homeomorhisms. A region $U \subset \mathbb{R}^2$ is said to be a *parallelizable region* of a flow of Brouwer homeomorhisms $\{f^t : t \in \mathbb{R}\}$, if there exists a homeomorphism $\varphi : U \to \mathbb{R}^2$ such that

$$\varphi(f^t(x,y)) = \varphi(x,y) + (t,0), \qquad (x,y) \in U,\ t \in \mathbb{R}. \tag{2.2}$$

A parallelizable region U is called *maximal parallelizable region* of the flow $\{f^t : t \in \mathbb{R}\}$, if it is not contained in any other parallelizable region.

Condition (2.2) means that the flow $\{f^t|_U : t \in \mathbb{R}\}$ is topologically conjugate with the flow of translations $\{T^t : t \in \mathbb{R}\}$, where T^t is given by the formula $T^t(x, y) = (x + t, y)$ for $(x, y) \in \mathbb{R}^2$, $t \in \mathbb{R}$, i.e.

$$\varphi \circ f^t|_U = T^t \circ \varphi, \qquad t \in \mathbb{R}.$$

For every $t \in \mathbb{R}$ the preimage $\varphi^{-1}(\{t\} \times \mathbb{R})$ has exactly one common point with each trajectory of the flow $\{f^t : t \in \mathbb{R}\}$ contained in the region U. Any set $S \subset U$ having the property that for each $p \in U$ there exists exactly one number $\tau(p)$ such that $f^{\tau(p)}(p) \in S$ we call a section of the region U. The existence of a continuous section of a region U (i.e. a section for which the function $\tau : U \to \mathbb{R}$ is continuous) is equivalent to the parallelizability of this region (cf. Bhatia, Szegö [14], Theorem 2.4, p. 49).

The notion of the first prolongational limit set plays an important role for studying maximal parallelizable regions of a Brouwer homeomorphism embeddable in a flow. The definitions presented below can be found in a book of A. Pelczar [96] (cf. Bhatia, Szegö [14]).

For a flow $\{f^t : t \in \mathbb{R}\}$ we define

$$\begin{aligned} \mathrm{J}^+(p) := \{q \in X : \ & \text{there exist sequences } (p_n)_{n\in\mathbb{N}} \text{ and } (t_n)_{n\in\mathbb{N}} \\ & \text{such that } p_n \to p,\ t_n \to +\infty,\ f^{t_n}(p_n) \to q \\ & \text{as } n \to +\infty\}, \\ \mathrm{J}^-(p) := \{q \in X : \ & \text{there exist sequences } (p_n)_{n\in\mathbb{N}} \text{ and } (t_n)_{n\in\mathbb{N}} \\ & \text{such that } p_n \to p,\ t_n \to -\infty,\ f^{t_n}(p_n) \to q \\ & \text{as } n \to +\infty\ \}. \end{aligned}$$

The set $J(q) = J^+(q) \cup J^-(q)$ is said to be the *first prolongational limit set* of the point q. For a set $H \subset \mathbb{R}^2$ we define

$$J^+(H) = \bigcup_{q\in H} J^+(q), \quad J^-(H) = \bigcup_{q\in H} J^-(q), \quad J(H) = \bigcup_{q\in H} J(q).$$

The sets $J^+(q)$ i $J^-(q)$ are closed and invariant for every $q \in \mathbb{R}^2$ (cf. Bhatia, Szegö [14], Theorem 4.3, p. 26). If H is a compact set, then the sets $J^+(H)$ i $J^-(H)$ are closed (cf. Pelczar [96], Theorem 57.1, p. 135). Moreover, $J^+(q) = J^+(f^t(q))$ and $J^-(q) = J^-(f^t(q))$ for all $q \in \mathbb{R}^2$ and $t \in \mathbb{R}$ (cf. Pelczar [96], Theorem 57.2, p. 136). But the set $J(\mathbb{R}^2)$ may not be closed (cf. McCann [86], Example 3.10).

Directly from the definition of the first prolongational limit set, we obtain that $p \in J(q)^+$ if and only if $q \in J(p)^-$ for all $p, q \in \mathbb{R}^2$. If $\{f^t : t \in \mathbb{R}\}$ is a flow of Brouwer homeomorphisms, then for each $p \in \mathbb{R}^2$ we have $p \notin J(p)$ and $J^+(p) \cap J^-(p) = \emptyset$ (cf. McCann [86], Propositions 1.5 and 2.11).

2.2 Codivergency relation for a Brouwer homeomorphism

In this section we discuss properties of the codivergency relation introduced by Stephen A. Andrea [5]. The definition and basic properties of this relation has been presented in the previous section. The results contained in this section with the exception of the last one, has been obtained without the assumption that the considered Brouwer homeomorphism is embeddable in a flow.

The theorem ending this section applies to trajectories contained in different equivalence classes of the codivergency relation. In the proof of this theorem we use the result which characterizes the set of all strongly irregular points of a Brouwer homeomorphism embeddable in a flow in terms of continuous dynamical systems theory. The other results describing properties of the codivergency relation for Brouwer homeomorphisms embeddable in a flow are contained in the second chapter of this report which includes subsidiary results.

Let us remind that if f is a Brouwer homeomorphism, then each iterate f^n of f for $n \neq 0$, is also a Brouwer homeomorphism. Therefore, the codivergency relation can be defined for f and f^n.

Theorem 2.7. ([A3], Proposition 3.3) *Let f be a Brouwer homeomorphism and n be a nonzero integer. Then the Brouwer homeomorphisms f and f^n have the same equivalence classes of the codivergency relation.*

In the main step of the proof of this result we show that if for some points $p, q \in \mathbb{R}^2$ there exists an arc K with endpoints p and q such that $f^{nm}(K) \to \infty$ as $m \to \pm\infty$, then for this arc we have $f^k(K) \to \infty$ as $k \to \pm\infty$.

Now we proceed to the problem of invariance of equivalence classes of the codivergency relation. We start from a result which says that any Brouwer homeomorphism maps equivalence classes onto equivalence classes.

Theorem 2.8. ([A3], Proposition 3.4) *Let f be a Brouwer homeomorphism and $\{G_i\}_{i \in I}$ be the family of all equivalence classes of the codivergency relation. Then for every $i \in I$ there exists a $j \in I$ such that $f(G_i) = G_j$.*

Therefore, to show that an equivalence class G_i is invariant under a Brouwer homeomorphism f it suffices to show that for a point $p \in G_i$ we have $f(p) \in G_i$.

The next result presented here says that if an equivalence class is invariant under some iterate of a Brouwer homeomorphism f, then it is also invariant under f.

Theorem 2.9. ([A3], Proposition 3.6) *Let f be a Brouwer homeomorphism and n be a nonzero integer. Then for every equivalence class G_0 of the codivergency relation the equality $f^n(G_0) = G_0$ implies that $f(G_0) = G_0$.*

In the proof of this result, for a given equivalence class G_0, we consider the family $\{G_m : m \in \mathbb{Z}\}$, where $G_m := f^m(G_0)$ for all $m \in \mathbb{Z}$. Under the assumption that $f^n(G_0) = G_0$, we have that this family contains at most n distinct equivalence classes. Using a result of S. Andrea concerning a finite family of disjoint arcwise connected sets (cf. [5], Proposition 1.3), we obtain that each element of this family is equal to G_0.

Thus, in the case where the family of all equivalence classes of the codivergency relation defined for a Brouwer homeomorphism f is finite, each of the classes is invariant under f. It follows from Theorems 2.8 and 2.9, since in this case f permutes the elements of this finite family.

Now we present results concerning invariant lines of a given Brouwer homeomorphism f, i.e. homeomorphic images of a straight line which are closed sets and invariant under f. The following theorem says that each of such lines is a translation curve.

Theorem 2.10. ([A5], Proposition 2.1) *Let f be a Brouwer homeomorphism and C be a line. Assume that $f(C) = C$. Then for each $p_0 \in C$ we have*

$$\bigcup_{n\in\mathbb{Z}} f^n(K_{p_0 f(p_0)}) = C, \tag{2.3}$$

where $K_{p_0 f(p_0)}$ is the arc with endpoints p_0 and $f(p_0)$ contained in C. Moreover, $f^n(K_{pq}) \to \infty$ as $n \to \pm\infty$ for all $p, q \in C$, where K_{pq} is the arc with endpoints p and q contained in C.

Directly from Theorem 2.10 we obtain that each invariant line is a closed translation curve and is contained in an equivalence class of the codivergency relation. Hence, according to Theorem 2.8 this equivalence class is invariant.

Corollary 2.11. ([A5], Corollary 2.2) *Let f be a Brouwer homeomorphism and C be a line. Assume that $f(C) = C$. Then there exists an equivalence class G of the codivergency relation such that $C \subset G$. Moreover, $f(G) = G$.*

During studying properties of equivalence classes of the codivergency relation, there can arise the question whether the assumption that the Jordan curve being the boundary of a Jordan domain is contained in an equivalence class implies that this Jordan domain is also contained in this class. Using Corollary 2.11 we can show the following result concerning this question.

Theorem 2.12. ([A5], Proposition 2.3) *Let f be a Brouwer homeomorphism. Assume that for each $p \in \mathbb{R}^2$ there exists an invariant line C_p such that $p \in C_p$. Then each equivalence class G of the codivergency relation is simply connected.*

Let us note that in the above theorem we do not assume that the elements of the family $\{C_p : p \in \mathbb{R}^2\}$ are either disjoint or equal. In a paper of S. Andrea [5] one can find an example of a Brouwer homeomorphism with an equivalence class such that the intersection of all invariant lines contained in this class is a countable set.

Using Theorem 2.10 we obtain the following result concerning an arc which joins two invariant lines contained in the same equivalence class of the codivergency relation.

Theorem 2.13. ([A5], Theorem 3.1) *Let f be a Brouwer homeomorphism and C_1, C_2 be lines. Assume that $f(C_1) = C_1$, $f(C_2) = C_2$ and $C_1 \cap C_2 = \emptyset$. Let K_{pq} be an arc with endpoints p and q such that $p \in C_1$, $q \in C_2$ and $(K_{pq} \setminus \{p, q\}) \cap (C_1 \cup C_2) = \emptyset$. If C_1 and C_2 are contained in the same equivalence class of the codivergency relation, then $f^n(K_{pq}) \to \infty$ as $n \to \pm\infty$.*

In the proof of this theorem we starts from an arc K_0 with endpoints belonging to C_1 and C_2 such that $f^n(K_0) \to \infty$ as $n \to \pm\infty$. Under the assumption that C_1, C_2 are contained in the same equivalence class of the codivergency relation, the existence of such arc K_0 follows directly from the definition of this relation. To show that the sequence of iterates of the arc K_{pq} occurring in the assumptions of our theorem tends to infinity, we use the fact that the lines C_1, C_2 are closed translation curves. Therefore, using the arc K_0 we can construct a Jordan domain B_f which contains the given arc K_{pq} such that $f^n(B_f) \to \infty$ as $n \to \pm\infty$, where by a Jordan domain we mean the union of a Jordan curve J_f and the bounded component of $\mathbb{R}^2 \setminus J_f$.

The Jordan curve J_f being the boundary of the considered Jordan domain B_f is equal to the union of four arcs, one of which is contained in C_1, another is contained in C_2, while the other two have the property that the intersection of each of them with the lines C_1, C_2 consists of exactly one point being its endpoint. Hence we get that the Jordan domain B_f is contained in the same equivalence class as the lines C_1, C_2, since each point of B_f can be joined with C_1 and C_2 by an arc contained in the Jordan domain.

The following result says that for any two disjoint invariant lines contained in the same equivalence class of the codivergency relation, the strip between them is contained in the set of regular points.

Corollary 2.14. ([A5], Corollary 3.2) *Let f be a Brouwer homeomorphism and C_1, C_2 be lines such that $C_1 \cap C_2 = \emptyset$. Assume that $f(C_1) = C_1$, $f(C_2) = C_2$ and C_1, C_2 are contained in the same equivalence class G of the codivergency relation. Then each point of the strip between C_1, C_2 is a regular point and belongs to the class G.*

The above result is a corollary from the proof of Theorem 2.13. The only difference is that the Jordan domain B_f from the proof of Theorem 2.13, is modified to contain a neighbourhood of a given point p from the strip between C_1 and C_2. Since the sequence of iterates of the Jordan domain B_f tends to ∞, the point p is regular. Moreover, p belongs to the equivalence class G which contains the lines C_1, C_2, since we can join the point p with a point belonging to any point of the set $(C_1 \cup C_2) \cap B_f$ by an arc contained in B_f.

Now we proceed to discuss the relationship between configurations of triples of pairwise disjoint invariant lines and the codivergency relation. Using Theorems 2.10 and 2.13 we can show the following result.

Theorem 2.15. ([A5], Theorem 4.1) *Let f be a Brouwer homeomorphism and C_1, C_2, C_3 be pairwise disjoint lines. Assume that $f(C_i) = C_i$ for $i \in \{1, 2, 3\}$. If $|C_1, C_2, C_3|$, then each of the lines C_1, C_2, C_3 is contained in a different equivalence class of the codivergency relation.*

The main part of the proof of this theorem is to show that any arc K with endpoints belonging to two of the lines C_1, C_2, C_3 and having no common points with the third of them does not satisfy the condition $f^n(K) \to \infty$ as $n \to \pm\infty$. Thus, on account of Theorem 2.13, any two of these lines cannot be contained in the same equivalence class.

From the above theorem we get a result about configurations of triples of pairwise disjoint invariant lines in the case where two of them are contained in the same equivalence class.

Corollary 2.16. ([A5], Corollary 4.2) *Let f be a Brouwer homeomorphism and C_1, C_2, C_3 be pairwise disjoint lines. Assume that $f(C_i) = C_i$ for $i \in \{1, 2, 3\}$ and C_1, C_2 are contained in the same equivalence class G of the codivergency relation. If C_3 is a subset of the strip between C_1 and C_2, then $C_1|C_3|C_2$ and $C_3 \subset G$.*

The subsequent results presented in this section relate to a Brouwer homeomorphism f which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then the trajectories of the flow $\{f^t : t \in \mathbb{R}\}$ are pairwise disjoint invariant lines of f and the set of all regular points of f can be determined by using the codivergency relation. More precisely, the set of all regular points is equal to the union of the interiors of all equivalence classes of the codivergency relation (cf. [B16], Proposition 2.1). This result will be discuss in more details in the next chapter containing complementary results (see Theorem 3.21). There we will also give Theorem 3.24 which says that the set of all strongly irrregular points of a Brouwer homeomorphism f embeddable in a flow $\{f^t : t \in \mathbb{R}\}$ is equal to the first prolongational limit set of this flow (cf. [B19], Corollary 3).

Theorem 3.9 stated in the next chapter, says that each equivalence class of the codivergency relation is contained in a parallelizable region. Thus, for any points p, q belonging to the same equivalence class of this relation there exists a continuous section containing the points p, q. By Theorem 2.13 we have that the arc K_{pq} with endpoints p, q contained in this continuous section satisfies the condition $f^n(K_{pq}) \to \infty$ as $n \to \pm\infty$. In the same way as in the proof of Theorem 2.13 we get that $(f^t)^n(K_{pq}) \to \infty$ as $n \to \pm\infty$ for each $t \in \mathbb{R} \setminus \{0\}$. Thus the equivalence classes of the codivergency relation defined for a Brouwer homeomorphism f^t belonging to the flow $\{f^t : t \in \mathbb{R}\}$ do not depend on $t \in \mathbb{R} \setminus \{0\}$.

Therefore, from Theorem 3.21 mentioned above, we obtain that the set of all regular points is the same for each non-identity element f^t of a flow of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$, i.e. for each $t \in \mathbb{R} \setminus \{0\}$. Furthermore, Corollary 3.26 presented is the next chapter says that for each $t \in \mathbb{R} \setminus \{0\}$ the set of all strongly irregular points of the element f^t of a flow of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$ is the same. Hence, the sets of all weakly irregular points of the elements f^t of this flow are equal for all $t \in \mathbb{R} \setminus \{0\}$. Thus, we can say about the sets of regular, strongly irregular and weakly irregular points of a flow of Brouwer homeomorphisms.

Przejdziemy teraz do omówienia wyniku dotyczącego trajektorii potoku homeomorfizmów Brouwera zawartych w różnych klasach abstrakcji relacji współzbieżności do nieskończoności. Zasadniczą rolę w jego dowodzie odgrywa wspomniene wyżej Twierdzenie 3.24. W dowodzie tym skorzystamy również z następującego wyniku, który otrzymujemy z definicji pierwszego przedłużenia granicznego i trójargumentowych relacji zdefiniowanych w zbiorze trajektorii potoku homeomorfizmów Brouwera.

Now we proceed to a result concerning trajectories of a flow of Brouwer homeomorphisms contained in different equivalence classes of the codivergency relation. In the proof of this result, the crucial role is played by Theorem 3.24 mentioned above. In this proof we also use the following result which can be obtained from the definition of the first prolongational limit set and the definition of the 3-argument relations defined in the set of trajectories of the flow.

Theorem 2.17. ([A1], Proposition 3.1) *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. If $p \in J(q)$, then $|C_p, C_q, C_r|$ for every $r \in D_{pq}$, where D_{pq} denotes the strip between the trajectories of points p and q.*

The announced above result concerning mutual placement of trajectories a flow of Brouwer homeomorphisms can be formulated in the following way.

Theorem 2.18. ([A4], Theorem 3.6) *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Let $q_1 \in G_1$, $q_2 \in G_2$ and G_1, G_2 be different equivalence classes of the codivergency relation. Then there exists a point r such that $|C_{q_1}, C_r, C_{q_2}|$, where C_{q_1}, C_{q_2}, C_r denote the trajectories of the points q_1, q_2, r, respectively.*

The main part of the proof of this result concerns the case where $q_1 \in \operatorname{bd} G_1$ and the component of the set $\mathbb{R}^2 \setminus C_{q_1}$ containing C_{q_2} denoted by H is disjoint with G_1. Then, if q_1 belongs to the boundary of an equivalence class which is contained in H, then to show the existence of a point r such that $|C_{q_1}, C_r, C_{q_2}|$ we use properties of the codivergency relation presented in the next chapter (cf. Theorems 3.13 and 3.14).

The more difficult is the case where q_1 does not belong to the boundary of any equivalence class contained in H. Then, on account of the Whitney-Bebutov theorem (see Bhatia, Szegö [14], p. 52), we obtain the existence of a local section K containing q_1 which has no common points with the trajectory C_{q_2}. Next we fix a $q_0 \in K \cap H$. If $|C_{q_1}, C_{q_0}, C_{q_2}|$, then we can take $r = q_0$ to obtain the assertion of our theorem.

Now, let us consider the case where $C_{q_1}|C_{q_0}|C_{q_2}$. In this case the key step is to show that the strip $D_{q_1 q_0}$ contains a strongly irregular point q_3 such that $C_{q_1}|C_{q_3}|C_{q_0}$.

By Theorem 3.24 we obtain that $P(q_3) = J(q_3)$. Hence there exists a point p_3 such that $p_3 \in J(q_3)$. Then $p_3 \in D_{q_1 q_3}$ or $p_3 \in D_{q_0 q_3}$. If $p_3 \in D_{q_1 q_3}$, then by Theorem 2.17 we have $|C_{q_1}, C_{p_3}, C_{q_3}|$. If $p_3 \in D_{q_0 q_3}$, then by Theorem 2.17 we have $|C_{q_0}, C_{p_3}, C_{q_3}|$. Thus in any case we have $|C_{q_1}, C_{p_3}, C_{q_2}|$. Putting $r = p_3$ we obtain the assertion of our theorem.

Theorem 2.18 may be considered as an extension of Theorem 3.14 to the case where the boundaries of the equivalence classes G_1, G_2 are disjoint. But it is not generally true that each point $r \in D_{q_1 q_2} \setminus (G_1 \cup G_2)$ satisfies the condition $|C_{q_1}, C_r, C_{q_2}|$. Theorem 2.18 has been used in the proof of a property of a homeomorphism realizing the topological equivalence of flows of Brouwer homeomorhisms.

2.3 Parallelizable regions of a flow of Brouwer homeomorhisms

In this chapter we describe properties of parallelizable regions of a flow of Brouwer homeomorhisms, i.e. regions for which the restriction of the flow to them is topologically conjugate with the flow of translations. More precisely,

We start from results concerning the invariance of the boundary of a parallelizable region (we do not assume that this region is maximal region with respect to inclusion among all parallelizable regions).

Theorem 2.19. ([A1], Proposition 2.1) *Let U be a parallelizable region of a flow of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$. Then the boundary of the region U is invariant.*

In the proof of this result we use the invariance of any parallelizable region and the fact that the closure of a parallelizable region has no common point with one of the components of the complement of the trajectory of any point belonging to the boundary of this region (cf. Theorem 3.15).

From Theorem 2.19 we obtain that the boundary of a parallelizable region of a flow of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$ is a union of trajectories of the flow. Now we give a result which describes mutual relations between trajectories contained in the boundary of a parallelizable region.

Theorem 2.20. ([A1], Proposition 2.2) *Let U be a parallelizable region of a flow of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$. Then $|C_{p_1}, C_{p_2}, C_{p_3}|$ for all distinct trajectories C_{p_1}, C_{p_2}, C_{p_3} contained in $\operatorname{bd} U$.*

In the proof of this theorem we use Theorem 3.15 mentioned above. It implies that each of the three considered trajectories C_{p_1}, C_{p_2}, C_{p_3} divides the plane in such a way that the other two of them are contained in the same component of its complement.

From the proof of the latter theorem, by replacing one of the trajectories C_{p_1}, C_{p_2}, C_{p_3} by a trajectory contained in the parallelizable region U we get the following result.

Theorem 2.21. ([A1], Proposition 2.3) *Let U be a parallelizable region of a flow of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$. Let $r \in U$ and H be a component of $\mathbb{R}^2 \setminus C_r$. Then for all distinct trajectories C_{p_1}, C_{p_2} contained in* $\operatorname{bd} U \cap H$ *the relation $|C_{p_1}, C_{p_2}, C_r|$ holds.*

It is known that a region U is parallelizable if and only if $J(U) \cap U = \emptyset$ (cf. O. Bhatia, G. P. Szegö [14], Theorem 1.8, p. 46 and Theorem 2.4, p. 49). Hence for every parallelizable region U condition $J(U) \subset \operatorname{bd} U$ is satisfied, since from the definition of the first prolongational limit set we obtain that $J(U) \subset \operatorname{cl} U$.

Maximal parallelizable regions are of particular importance in describing flows of Brouwer homeomorphisms, where by a maximal parallelizable region we mean a parallelizable region for which there is no parallelizable region containing it as a proper subset. If U is a maximal parallelizable region, then $J(U) = \operatorname{bd} U$ (cf. R. McCann [86], Proposition 2.6).

In the description of maximal parallelizable regions we can also use the codivergency relation. A maximal parallelizable region U of a flow of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$ is equal to the union of a family of equivalence classes of this relation (cf. Theorem 3.16). Boundary trajectories of these equivalence classes can be contained either in this region or in its boundary.

Trajectories contained in a parallelizable region U which are boundary trajectories of equivalence classes are subsets of the set of all irregular points. It follows from Theorem 3.21 mentioned above which says that the set of all regular points is equal to the union of the interiors of all equivalence classes of the codivergency relation (cf. [B16], Proposition 2.1). Properties of boundary trajectories of equivalence classes of the codivergency relation are discussed in details in the next chapter containing supplementary results.

For every parallelizable region U trajectories contained in $\mathrm{J}(\operatorname{bd} U) \cap U$ consists of strongly irregular points (cf. Theorem 3.24). However, this does not mean that all other trajectories contained in U consists of regular points. In fact, a parallelizable region U can also contain trajectories which consists of weakly irregular points (see McCann [86], Example 3.10).

Subsequent results presented here mainly concern the set $\mathrm{J}(p) \cap U$ for $p \in \operatorname{bd} U$, where U is a parallelizable region. Among them will be included a result which says that about the uniqueness of trajectory contained in the first prolongational limit set of a trajectory contained in the boundary of a maximal parallelizable region which is a subset of this region. We start from a result which plays a crucial role in the proof of this fact.

Theorem 2.22. ([A1], Proposition 2.4) *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Let $q_1, q_2 \in J(p)$, $C_{q_1} \neq C_{q_2}$. Then $|C_{q_1}, C_{q_2}, C_r|$ for each $r \in D_{q_1,q_2} \setminus C_p$, where D_{q_1,q_2} is a strip between C_{q_1}, C_{q_2}.*

In the proof of this result we first show that $p \in D_{q_1,q_2}$. The main step is to exclude the case where $C_{q_1} \mid C_r \mid C_{q_2}$. Suppose, on the contrary, that this relation holds. Then the points q_1, q_2 belong to the different components of $\mathbb{R}^2 \setminus C_r$. Hence the point p belongs to one of the components of $\mathbb{R}^2 \setminus C_r$, since $p \notin C_r$. Thus p cannot be contained in the first prolongational limit set of this of the points q_1, q_2 which lies in the component of $\mathbb{R}^2 \setminus C_r$ not containing p, but this contradicts our assumption.

In the reasoning presented above we use the assumption that $p \notin C_r$. In the case where $p \in C_r$ the relation $C_{q_1} \mid C_r \mid C_{q_2}$ can occur.

From Theorem 2.22 we obtain a corollary describing properties of trajectories contained in the first prolongational limit set of a boundary point of a parallelizable region.

Corollary 2.23. ([A1], Corollary 2.5) *Let U be a parallelizable region of a flow of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$. Let $p \in \operatorname{bd} U$ and $q_1, q_2 \in U$. Assume that $q_1, q_2 \in J(p)$. Then $C_{q_1} = C_{q_2}$.*

To show this corollary we use proof by contradiction. Let us suppose that $C_{q_1} \neq C_{q_2}$. Then the strip D_{q_1,q_2} must contain a point $r \in U$. By the parallelizability of the region U we get $C_{q_1} \mid C_r \mid C_{q_2}$. On the other hand, by Theorem 2.22 we have $|C_{q_1}, C_{q_2}, C_r|$, which gives a contradiction.

If U is a maximal parallelizable region of a flow of Brouwer homeomorphisms and $p \in \operatorname{bd} U$, then the set $\mathrm{J}(p) \cap U$ is nonempty. Thus by Corollary 2.23 we get the following result announced above.

Corollary 2.24. ([A4], Corollary 1.4) *Let U be a maximal parallelizable region of a flow of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$. Let $p \in \operatorname{bd} U$. Then the set $\mathrm{J}(p) \cap U$ consists of exactly one trajectory.*

From the above corollary does not follow one-to-one correspondence between the set of trajectories contained in the boundary of a maximal parallelizable region U and the set of trajectories contained in $\mathrm{J}(\operatorname{bd} U) \cap U$. It can happen that for distinct trajectories C_{p_1}, C_{p_2} contained in the boundary of a maximal parallelizable region U corresponds the same trajectory contained in the set $\mathrm{J}(\operatorname{bd} U) \cap U$, i.e. $\mathrm{J}(p_1) \cap U = \mathrm{J}(p_2) \cap U$.

Next results presented here concerns an application of the codivergency relation to study properties of parallelizable regions of a flow of Brouwer homeomorphisms. Among such regions, maximal parallelizable regions are of particular importance, since they are elements of a cover of the plane which occurs in the theorem describing the general form of a flow of Brouwer homeomorphisms.

Now we present a sufficient condition for the property that the intersection of a parallelizable region with one of the components of the complement of a trajectory contained in this region contains only points belonging to one of the equivalence classes of the codivergency relation.

Theorem 2.25. ([A1], Proposition 4.1) *Let U be a parallelizable region of a flow of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$ and $r \in U$. Let H be a component of $\mathbb{R}^2 \setminus C_r$*

such that $H \cap J(\mathrm{bd}\, U) \cap U = \emptyset$. Then $H \cap U$ is contained in an equivalence class of the codivergency relation.

Now, we present a sketch of proof of this result. Using the assumption that $H \cap J(\mathrm{bd}\, U) \cap U = \emptyset$ we show that for all $p, q \in H \cap U$ the strip D_{pq} is contained in U. Next, from this fact we infer that for all $p_1, q_1 \in D_{pq}$ there exists an arc K with endpoints p_1, q_1 such that $f^n(K) \to \infty$ as $n \to \pm\infty$. Hence the strip D_{pq} is contained in an equivalence class of the codivergency relation. Thus, to prove that any two points $p_1, q_1 \in H \cap U$ belong to the same equivalence class we only have to choose $p, q \in H \cap U$ in such a way that $p_1, q_1 \in D_{pq}$.

Form the above theorem we obtain the following result.

Corollary 2.26. ([A1], Corollary 4.2) *Let U be a parallelizable region of a flow of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$ and $r \in U$. Let H be a component of $\mathbb{R}^2 \setminus C_r$ such that $H \cap \mathrm{bd}\, U = \emptyset$. Then $H \subset U$ and H is contained in an equivalence class of the codivergency relation.*

Under the assumptions of Theorem 2.25 it may exist a point $p \in H \cap U$ such that the strip D_{pr} contains points which do not belong to U. The assumptions of Corollary 2.26 exclude such a possibility.

Some results concerning maximal parallelizable region of a flow of Brouwer homeomorphisms which are not discussed here are presented in the next chapter of this report.

2.4 Form of a flow of Brouwer homeomorphisms

In this chapter we present the result describing the form of a flow of Brouwer homeomorphisms. For such a flow we can find a countable family of maximal parallelizable regions which covers the plane. In each of the regions we consider the coordinate systems given by parallelizing homeomorphisms of the region. We also describe the properties of the transitions functions between parallelizing homeomorphisms of non-disjoint regions of the cover.

In the construction of the family of maximal parallelizable regions we use the idea of Wilfred Kaplan [58], [59]. For any flow of Brouwer homeomorphisms, as the index set of these family we will take an admissible class of finite sequences described below.

Let X be a nonempty set. Denote by $X^{<\omega}$ the set of all finite sequences of elements of X. By a *tree* on X we mean a subset T of $X^{<\omega}$ which is closed under initial segments, i.e. for all positive integers m, n such that $n > m$ if $(x_1, \ldots, x_m, \ldots, x_n) \in T$, then $(x_1, \ldots, x_m) \in T$. Let $\alpha = (x_1, \ldots, x_n) \in X^{<\omega}$. Then, for any $x \in X$ by $\alpha\widehat{\ }x$ we denote the sequence $(x_1, \ldots, x_n, x)$.

A class A^+ of finite sequences of positive integers will be termed *admissible* if $A^+ \subset \mathbb{Z}_+^{<\omega}$ is a tree on $\mathbb{Z}_+$ and satisfies the conditions

(i) A^+ contains the sequence 1 and no other one-element sequence,

(ii) if $\alpha\widehat{\ }k$ is in A^+ and $k > 1$, then so also is $\alpha\widehat{\ }(k-1)$.

A class A^- of finite sequences of negative integers will be termed *admissible* if $A^- \subset \mathbb{Z}_-^{<\omega}$ is a tree on $\mathbb{Z}_-$ and satisfies the conditions

(i) A^+ contains the sequence 1 and no other one-element sequence,

(ii) if $\alpha\widehat{\ }k$ is in A^+ and $k > 1$, then so also is $\alpha\widehat{\ }(k-1)$,

(iii) A^- contains the sequence -1 and no other one-element sequence,

(iv) if $\alpha\widehat{\ }k$ is in A^- and $k < -1$, then so also is $\alpha\widehat{\ }(k+1)$.

The set $A := A^+ \cup A^-$, where A^+ and A^- are some admissible classes of finite sequences of positive and negative integers, respectively, will be said to be *admissible class of finite sequences.*

To obtain a cover of the plane consisting of maximal parallelizable regions for a given flow of Brouwer homeomorphisms we use the following result.

Lemma 2.27. ([A2], Lemma 2.1) *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Let $p \in \mathbb{R}^2$. Then there exists an at most countable family of maximal parallelizable regions $\{U_j : j \in J\}$, where J is the set of all positive integers or $J = \{1, \ldots, N\}$ for some positive integer N such that $p \in U_1$ and for each positive integer n the set $\operatorname{cl} B(p,n)$, where $B(p,n)$ is the ball centered at p with radius n, is covered by a finite subfamily $\{U_1, \ldots, U_{j_n}\}$ of $\{U_j : j \in J\}$. Moreover $j_n \le j_{n+1}$ for every n.*

Now we can recall the main result describing the structure of flows of Brouwer homeomorphisms. It says that for any such flow there exists a cover of the plane consisting of maximal parallelizable regions which can be indexed in a convenient way by an admissible class of finite sequences.

Theorem 2.28. (([A2], Theorem 2.2)) *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Then there exists a family of trajectories $\{C_\alpha : \alpha \in A\}$ and a family of maximal parallelizable regions $\{U_\alpha : \alpha \in A\}$, where $A = A^+ \cup A^-$ is an admissible class of finite sequences, such that $U_1 = U_{-1}$, $C_1 = C_{-1}$ and*

$$C_\alpha \subset U_\alpha \quad \text{for } \alpha \in A, \tag{2.4}$$

$$\bigcup_{\alpha \in A} U_\alpha = \mathbb{R}^2, \tag{2.5}$$

$$U_\alpha \cap U_{\alpha\widehat{\ }i} \neq \emptyset \quad \text{for } \alpha\widehat{\ }i \in A, \tag{2.6}$$

$$C_{\alpha\widehat{\ }i} \subset \operatorname{bd} U_\alpha \quad \text{for } \alpha\widehat{\ }i \in A, \tag{2.7}$$

$$|C_\alpha, C_{\alpha\widehat{\ }i_1}, C_{\alpha\widehat{\ }i_2}| \quad \text{for } \alpha\widehat{\ }i_1, \alpha\widehat{\ }i_2 \in A, \quad i_1 \neq i_2, \tag{2.8}$$

$$C_\alpha | C_{\alpha\widehat{\ }i} | C_{\alpha\widehat{\ }i\widehat{\ }j} \quad \text{for } \alpha\widehat{\ }i\widehat{\ }j \in A. \tag{2.9}$$

The construction of the families occurring in Theorem 2.28 starts from a trajectory denoted by C_1 of an arbitrary point p of the plane and a maximal parallelizable region U_1 occurring in Lemma 2.27. Next we take $C_{-1} := C_1$ and $U_{-1} := U_1$. In the case where the flow $\{f^t : t \in \mathbb{R}\}$ is not topological conjugate with a flow of translations (i.e. $J(\mathbb{R}^2) \neq \emptyset$), the point p can be taken to satisfy the condition $p \in J(\mathbb{R}^2)$.

Having constructed an index $\alpha \in A$ and the corresponding C_α and U_α, in case $\operatorname{bd} U_\alpha \cap H_\alpha \neq \emptyset$ we consider a bijective indexing of the set of all trajectories contained in

$$\operatorname{bd} U_\alpha \cap H_\alpha$$

by the sequences of the form $\alpha\widehat{\ }k$ starting from $k = 1$ and taking subsequent positive integers k if $\alpha \in A^+$ and starting from $k = -1$ and taking subsequent negative integers k if $\alpha \in A^-$, where H_1, H_{-1} are the components of $\mathbb{R}^2 \setminus C_1$, and for $\alpha = \beta\widehat{\ }l \in A$ the set H_α is the components of $\mathbb{R}^2 \setminus C_\alpha$ which has no common points with U_β. In case $\operatorname{bd} U_\alpha \cap H_\alpha = \emptyset$ the sequence $\alpha \in A$ will be a leaf of the tree A^+ or A^-, respectively.

Now we can extend A by all these sequences $\alpha\widehat{\ }k$ and for each $\alpha\widehat{\ }k$ we denote by $C_{\alpha\widehat{\ }k}$ the trajectory indexed by $\alpha\widehat{\ }k$. Then we take as $U_{\alpha\widehat{\ }k}$ the element of the subfamily $\{U_j : j = 1, \dots, j_{m_{\alpha\widehat{\ }k}}\}$ of the family occurring in Lemma 2.27 which contains $C_{\alpha\widehat{\ }k}$, where $m_{\alpha\widehat{\ }k}$ is the smallest positive integer greater then or equal to the distance of the trajectory $C_{\alpha\widehat{\ }k}$ from p. Lemma 2.27 guarantees that the constructed family of maximal parallelizable regions satisfies condition (2.5).

For the family $\{U_\alpha : \alpha \in A\}$ of maximal parallelizable regions occurring in Theorem 2.28 we consider the family of parallelizing homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{R}^2$ such that $\varphi_\alpha(C_\alpha) = \mathbb{R} \times \{0\}$ and

$$\varphi_\alpha(N_\alpha) = \mathbb{R} \times (0, +\infty) \quad \text{for } \alpha \in A^+,$$

$$\varphi_\alpha(N_\alpha) = \mathbb{R} \times (-\infty, 0) \quad \text{for } \alpha \in A^-,$$

where $N_\alpha := U_\alpha \cap H_\alpha$. Each of these homeomorphisms maps trajectories contained in U_α onto horizontal straight lines. The parallelizing homeomorphisms will be also called *charts*.

The result presented below describes relations between the parallelizing homeomorphisms of overlapping elements of the cover described in Theorem 2.28.

Theorem 2.29. ([A2], Proposition 3.1) *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms and $\{U_\alpha : \alpha \in A\}$ be a family of maximal parallelizable regions occurring in Theorem 2.28. Then there exists a family of the parallelizing homeomorphisms $\{\varphi_\alpha : \alpha \in A^+\}$, where $\varphi_\alpha : U_\alpha \to \mathbb{R}^2$, such that for each $\alpha\widehat{\ }i \in A^+$*

$$\varphi_{\alpha\widehat{\ }i}(U_\alpha \cap U_{\alpha\widehat{\ }i}) = \mathbb{R} \times (c_{\alpha\widehat{\ }i}, 0),$$

$$\varphi_\alpha(U_\alpha \cap U_{\alpha\widehat{\ }i}) = \mathbb{R} \times (c^\alpha_{\alpha\widehat{\ }i}, d^\alpha_{\alpha\widehat{\ }i}),$$

where $c^{\alpha}_{\alpha\frown i} \in \mathbb{R} \cup \{-\infty\}$, $d^{\alpha}_{\alpha\frown i} \in \mathbb{R} \cup \{+\infty\}$ *and* $c_{\alpha\frown i} \in [-\infty, 0)$ *are some constants such that* $c^{\alpha}_{\alpha\frown i} < d^{\alpha}_{\alpha\frown i}$ *and at least one of the constants* $c^{\alpha}_{\alpha\frown i}$, $d^{\alpha}_{\alpha\frown i}$ *is finite. Moreover, there exists a continuous function* $\mu_{\alpha\frown i} : (c^{\alpha}_{\alpha\frown i}, d^{\alpha}_{\alpha\frown i}) \to \mathbb{R}$ *and a homeomorphism* $\nu_{\alpha\frown i} : (c^{\alpha}_{\alpha\frown i}, d^{\alpha}_{\alpha\frown i}) \to (c_{\alpha\frown i}, 0)$ *such that the homeomorphism*

$$h_{\alpha\frown i} : \mathbb{R} \times (c^{\alpha}_{\alpha\frown i}, d^{\alpha}_{\alpha\frown i}) \to \mathbb{R} \times (c_{\alpha\frown i}, 0)$$

given by the relation

$$h_{\alpha\frown i} := \varphi_{\alpha\frown i} \circ (\varphi_{\alpha|_{U_\alpha \cap U_{\alpha\frown i}}})^{-1} \tag{2.10}$$

has the form

$$h_{\alpha\frown i}(t, s) = (\mu_{\alpha\frown i}(s) + t, \nu_{\alpha\frown i}(s)), \quad t \in \mathbb{R}, s \in (c^{\alpha}_{\alpha\frown i}, d^{\alpha}_{\alpha\frown i}). \tag{2.11}$$

Similarly, there exists a family of the parallelizing homeomorphisms $\{\varphi_\alpha : \alpha \in A^-\}$, *where* $\varphi_\alpha : U_\alpha \to \mathbb{R}^2$, U_α *are those occurring in Theorem 2.28,* $\varphi_{-1} = \varphi_1$, *such that for each* $\alpha\frown i \in A^-$

$$\varphi_{\alpha\frown i}(U_\alpha \cap U_{\alpha\frown i}) = \mathbb{R} \times (0, c_{\alpha\frown i}),$$
$$\varphi_\alpha(U_\alpha \cap U_{\alpha\frown i}) = \mathbb{R} \times (c^{\alpha}_{\alpha\frown i}, d^{\alpha}_{\alpha\frown i}),$$

where $c^{\alpha}_{\alpha\frown i} \in \mathbb{R} \cup \{-\infty\}$, $d^{\alpha}_{\alpha\frown i} \in \mathbb{R} \cup \{+\infty\}$ *and* $c_{\alpha\frown i} \in (0, +\infty)$ *are some constants such that* $c^{\alpha}_{\alpha\frown i} < d^{\alpha}_{\alpha\frown i}$ *and at least one of the constants* $c^{\alpha}_{\alpha\frown i}$, $d^{\alpha}_{\alpha\frown i}$ *is finite. Moreover, there exists a continuous function* $\mu_{\alpha\frown i} : (c^{\alpha}_{\alpha\frown i}, d^{\alpha}_{\alpha\frown i}) \to \mathbb{R}$ *and a homeomorphism* $\nu_{\alpha\frown i} : (c^{\alpha}_{\alpha\frown i}, d^{\alpha}_{\alpha\frown i}) \to (0, c_{\alpha\frown i})$ *such that the homeomorphism*

$$h_{\alpha\frown i} : \mathbb{R} \times (c^{\alpha}_{\alpha\frown i}, d^{\alpha}_{\alpha\frown i}) \to \mathbb{R} \times (0, c_{\alpha\frown i})$$

given by relation (2.10) satisfies (2.11).

The form of the homeomorphism $h_{\alpha\frown i}$ occurring in Theorem 2.29 is obtained by solving the following functional equation

$$h_{\alpha\frown i}(t_1 + t_2, s) = h_{\alpha\frown i}(t_1, s) + (t_2, 0), \qquad t_1, t_2, s \in \mathbb{R}.$$

Now we describe the properties of the homeomorphisms $\nu_{\alpha\frown i}$ occurring in Theorem 2.29. Let us recall that the parallelizing homeomorphisms $\varphi_{\alpha\frown i}$ are chosen in such a way that for each $\alpha\frown i \in A$ the trajectory $C_{\alpha\frown i}$ corresponds to the value 0 of the parameter s in the coordinate system given by $\varphi_{\alpha\frown i}$, since the second coordinate in the map $\varphi_{\alpha\frown i}$ of every point belonging to $C_{\alpha\frown i}$ is equal to 0, i.e. $\varphi_{\alpha\frown i}(C_{\alpha\frown i}) = \mathbb{R} \times \{0\}$.

The following considerations show that homeomorphism $\nu_{\alpha\frown i}$ can be extended into one of the endpoints of the interval $(c^{\alpha}_{\alpha\frown i}, d^{\alpha}_{\alpha\frown i})$ by giving the value 0 for this endpoint. To simplify the notation, we consider only the class A^+. The corresponding results for A^- can be stated analogously. The choice of the endpoint depends on whether homeomorphism $\nu_{\alpha\frown i}$ is increasing or decreasing. Then under the assumption that such extension exists we have $\nu_{\alpha\frown i}(c^{\alpha}_{\alpha\frown i}) = 0$ if $\nu_{\alpha\frown i}$ is decreasing, and $\nu_{\alpha\frown i}(d^{\alpha}_{\alpha\frown i}) = 0$ if $\nu_{\alpha\frown i}$ is increasing.

To define the considered extension we have to find the trajectory contained in U_α which will correspond to the value of $\nu_{\alpha\frown i}^{-1}(0)$ by the map φ_α, i.e. the trajectory $\varphi_\alpha^{-1}(\mathbb{R}\times\{\nu_{\alpha\frown i}^{-1}(0)\})$. By Corollary 2.24 the set $U_\alpha \cap \mathrm{J}(C_{\alpha\frown i})$ consists of just one trajectory and the unique trajectory contained in this set we denote by $C^\alpha_{\alpha\frown i}$. The trajectory $C^\alpha_{\alpha\frown i}$ corresponds either to $c^\alpha_{\alpha\frown i}$ or $d^\alpha_{\alpha\frown i}$. More precisely, the trajectory $C^\alpha_{\alpha\frown i}$ corresponds to $c^\alpha_{\alpha\frown i}$ in the case where $\nu_{\alpha\frown i}$ is decreasing, and to $d^\alpha_{\alpha\frown i}$ in the case where $\nu_{\alpha\frown i}$ is increasing.

The result presented below shows that the correspondence depends on how $C^\alpha_{\alpha\frown i}$ is situated in relation to C_α and $C_{\alpha\frown i}$, since the relation between the trajectories $C^\alpha_{\alpha\frown i}$, C_α and $C_{\alpha\frown i}$ determines the kind of monotonicity of $\nu_{\alpha\frown i}$.

Lemma 2.30. ([A2], Remark 3.2) *Let $\{f^t : t\in\mathbb{R}\}$ be a flow of Brouwer homeomorphisms and $\{U_\alpha : \alpha\in A\}$ be a family of maximal parallelizable regions occurring in Theorem 2.28. Assume that $\alpha\widehat{\ }i\in A^+$. If $C_\alpha|C^\alpha_{\alpha\frown i}|C_{\alpha\frown i}$ or $C_\alpha = C^\alpha_{\alpha\frown i}$, then homeomorphism $\nu_{\alpha\frown i} : (c^\alpha_{f,\alpha\frown i}, d^\alpha_{f,\alpha\frown i}) \to (c_{f,\alpha\frown i}, 0)$ occurring in Theorem 2.29 is decreasing and $c^\alpha_{\alpha\frown i} > 0$ or $c^\alpha_{\alpha\frown i} = 0$, respectively. If $|C_\alpha, C^\alpha_{\alpha\frown i}, C_{\alpha\frown i}|$, then homeomorphism $\nu_{\alpha\frown i}$ is increasing and $d^\alpha_{\alpha\frown i} > 0$.*

In the above result one of the numbers $c^\alpha_{\alpha\frown i}$, $d^\alpha_{\alpha\frown i}$ is equal to the value of the coordinate s corresponding to the trajectory $C^\alpha_{\alpha\frown i}$ in the coordinate system given by the homeomorphism φ_α. The fact that this value is nonnegative follows from the construction of the family $\{C_\alpha : \alpha\in A^+\}$ described in the sketch of the proof of Theorem 2.28.

Now we can state the result on the extension of homeomorphism $\nu_{\alpha\frown i}$ announced before.

Theorem 2.31. ([A6], Proposition 1.6) *Let $\{f^t : t\in\mathbb{R}\}$ be a flow of Brouwer homeomorphisms and $\{U_\alpha : \alpha\in A\}$ be a family of maximal parallelizable regions occurring in Theorem 2.28. Let $\alpha\widehat{\ }i\in A^+$. If $C_\alpha|C^\alpha_{\alpha\frown i}|C_{\alpha\frown i}$ or $C_\alpha = C^\alpha_{\alpha\frown i}$, then $\varphi_\alpha(C^\alpha_{\alpha\frown i}) = \mathbb{R}\times\{c^\alpha_{\alpha\frown i}\}$ and $\nu_{\alpha\frown i} : (c^\alpha_{\alpha\frown i}, d^\alpha_{\alpha\frown i}) \to (c_{\alpha\frown i}, 0)$ can be extended to a homeomorphism defined on $[c^\alpha_{\alpha\frown i}, d^\alpha_{\alpha\frown i})$ by putting $\nu_{\alpha\frown i}(c^\alpha_{\alpha\frown i}) = 0$. However, if $|C_\alpha, C^\alpha_{\alpha\frown i}, C_{\alpha\frown i}|$, then $\varphi_\alpha(C^\alpha_{\alpha\frown i}) = \mathbb{R}\times\{d^\alpha_{\alpha\frown i}\}$ and $\nu_{\alpha\frown i}$ can be extended to a homeomorphism defined on $(c^\alpha_{\alpha\frown i}, d^\alpha_{\alpha\frown i}]$ by putting $\nu_{\alpha\frown i}(d^\alpha_{\alpha\frown i}) = 0$.*

To present the idea of the proof of Proposition 2.31 let us consider the case where $\nu_{\alpha\frown i}$ is decreasing. To show that $\varphi_\alpha^{-1}(\mathbb{R}\times\{c^\alpha_{\alpha\frown i}\}) = C^\alpha_{\alpha\frown i}$ we observe that $\varphi_{\alpha\frown i}^{-1}(\mathbb{R}\times\{\nu_{\alpha\frown i}(s)\}) = \varphi_\alpha^{-1}(\mathbb{R}\times\{s\})$ for $s\in(c^\alpha_{\alpha\frown i}, d^\alpha_{\alpha\frown i})$, i.e. $\nu_{\alpha\frown i}(s)$ and s correspond to the same trajectory contained in $U_\alpha\cap U_{\alpha\frown i}$. Hence for any sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n\in(c^\alpha_{\alpha\frown i}, d^\alpha_{\alpha\frown i})$ for $n\in\mathbb{N}$, we have $\lim_{n\to\infty} s_n = c^\alpha_{\alpha\frown i}$ if and only if $\lim_{n\to\infty}\nu_{\alpha\frown i}(s_n) = 0$, since $C^\alpha_{\alpha\frown i}\subset \mathrm{J}(C_{\alpha\frown i})$.

The continuous functions $\mu_{\alpha\frown i}$ occurring in Theorem 2.29 describe the time needed for the flow $\{f^t : t\in\mathbb{R}\}$ to move from the point with coordinates $(0, \nu_{\alpha\frown i}(s))$ in the chart $\varphi_{\alpha\frown i}$ until it reaches the point with coordinates $(0, s)$ in the chart φ_α. In other words, $\mu_{\alpha\frown i}$ describe the time needed for the flow to move from a point from the section

$K_{\varphi_{\alpha\frown i}}$ in $U_{\alpha\frown i}$ to a point from the section K_{φ_α} in U_α, where $K_{\varphi_{\alpha\frown i}} := \varphi_{\alpha\frown i}^{-1}(\{0\} \times \mathbb{R})$ and $K_{\varphi_\alpha} := \varphi_\alpha^{-1}(\{0\} \times \mathbb{R})$.

The following result describes the limits of sequences $(\mu_{\alpha\frown i}(s_n))_{n\in\mathbb{N}}$ for the sequences $(s_n)_{n\in\mathbb{N}}$ considered in the proof of Proposition 2.31.

Theorem 2.32. ([B19],Proposition 8) *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms and $\{U_\alpha : \alpha \in A\}$ be a family of maximal parallelizable regions occurring in Theorem 2.28. The functions $\mu_{\alpha\frown i}$ occurring in Theorem 2.29 satisfy the condition*

$$\lim_{s\to c^\alpha_{\alpha\frown i}} \mu_{\alpha\frown i}(s) \;=\begin{cases} -\infty, & if \quad C_{\alpha\frown i} \subset \mathrm{J}^+_f(C^\alpha_{\alpha\frown i}), \\ +\infty, & if \quad C_{\alpha\frown i} \subset \mathrm{J}^-_f(C_{\alpha\frown i}) \end{cases}$$

in the case where $C_\alpha | C^\alpha_{\alpha\frown i} | C_{\alpha\frown i}$ or $C_\alpha = C^\alpha_{\alpha\frown i}$ or the condition

$$\lim_{s\to d^\alpha_{\alpha\frown i}} \mu_{\alpha\frown i}(s) \;=\begin{cases} -\infty, & if \quad C_{\alpha\frown i} \subset \mathrm{J}^+_f(C^\alpha_{\alpha\frown i}), \\ +\infty, & if \quad C_{\alpha\frown i} \subset \mathrm{J}^-_f(C_{\alpha\frown i}) \end{cases}$$

in the case where $|C_\alpha, C^\alpha_{\alpha\frown i}, C_{\alpha\frown i}|$.

Theorem 2.32 describes the property that the trajectories $C_{\alpha\frown i}$ and $C^\alpha_{\alpha\frown i}$ cannot be contained in the same parallelizable region, whereas Theorem 2.31 corresponds to the fact that the trajectories $C_{\alpha\frown i}$ and $C^\alpha_{\alpha\frown i}$ have no disjoint invariant neighbourhoods.

2.5 Iterative roots of a Brouwer homeomorphism embeddable in a flow

In this section we consider the problem of describing the form of orientation preserving iterative roots of a Brouwer homeomorphism embeddable in a flow, i.e. orientation preserving homeomorphic solutions g of the functional equation

$$g^n = f, \tag{2.12}$$

where n is a positive integer and f is a given Brouwer homeomorphism embeddable in a flow $\{f^t : t \in \mathbb{R}\}$.

The finding of iterative roots of a given function is one of the important issues arising in the iteration theory. The main idea of the general method of determining iterative roots comes from the articles of S. Łojasiewicz [81] and M. Kuczma [68]. The construction of all iterative roots of a Brouwer homeomorphism which is conjugate with a translation has been given in [B7].

If f is a homeomorphism of the plane onto itself, g is a continuous function satisfying equation (2.12) for a positive integer n, then g is also a homeomorphism of the plane onto itself (cf. [B7], Remark 1). Moreover, if f has no fixed points, then g also has no fixed points.

Let f be an orientation preserving homeomorphism of the plane onto itself and g be a continuous function satisfying equation (2.12) for an odd positive integer n. Then g is also an orientation preserving homeomorphism of the plane onto itself (cf. [B7], Remark 2). For even positive integers n equation (2.12) can also have orientation reversing homeomorphic solutions. But here we are interested only in orientation preserving homeomorphism of the plane onto itself which have no fixed points, so we only study the set of solutions g of equation (2.12) which consists of Brouwer homeomorphisms.

To determine iterative roots of a Brouwer homeomorphism embeddable in a flow we use the form of such a flow described in Theorems 2.28 i 2.29. The next lemma allows proving the main result of this section.

Lemma 2.33. ([A3], Lemma 4.6) *Let f be a Brouwer homeomorphism embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Let $\{C_\alpha : \alpha \in A\}$ and $\{U_\alpha : \alpha \in A\}$ be families of trajectories and maximal parallelizable regions occurring in Theorem 2.28, a $\{\varphi_\alpha : \alpha \in A\}$ be a family of homeomorphisms occurring in Theorem 2.29. Then for every $\alpha \in A$ and every $p \in C_\alpha$ there exists an $\varepsilon > 0$ such that the ball $B(p, \varepsilon)$ centered at p with radius ε has common points with exactly two elements of the family $\{G_\alpha : \alpha \in A\}$, where $G_\alpha := \varphi_\alpha^{-1}(\mathbb{R} \times [0, +\infty))$ in case $\alpha \in A^+$ and $G_\alpha := \varphi_\alpha^{-1}(\mathbb{R} \times (-\infty, 0])$ in case $\alpha \in A^-$.*

To prove this lemma, for each $\alpha \in A$ we define a family of trajectories by putting $S_\alpha := \{C_\alpha\} \cup \{C_{\alpha\frown i} : \alpha\frown i \in A\}$. For some $\alpha \in A$ this family may be equal to $\{C_\alpha\}$ (that is the case if the assumptions of Corollary 2.26 are fulfilled for the maximal parallelizable region U_α). Each of trajectories of the set $\{C_\alpha : \alpha \in A\}$ belongs to exactly two of the families S_α, namely S_1, S_{-1} for $\alpha \in \{1, -1\}$ and S_β, $S_{\beta\frown i}$ for $\alpha = \beta\frown i$.

Using Theorems 2.20 and 2.21 we get that for every $\alpha \in A$ the relation $|C_{p_1}, C_{p_2}, C_{p_3}|$ holds for any distinct trajectories C_{p_1}, C_{p_2}, C_{p_3} belonging to S_α. Let us fix an $\alpha \in A$. If the family S_α is finite, then for every $\alpha\frown i \in A$ and for every $p \in C_{\alpha\frown i}$ there exists a ball centered at p which is disjoint with each element of the family S_α different from $C_{\alpha\frown i}$, since trajectories of a flow of Brouwer homeomorphisms are closed sets.

In the case where the family S_α is not finite, to finish the proof we can use a theorem from a book of A. Becka (cf. [9], Lemma 11.6). It states that if S is an infinite subfamily of the family F of all trajectories of a plane flow without fixed points such that for distinct $C_1, C_2, C_3 \in S$ the relation $|C_1, C_2, C_3|$ holds, then S is countable and every compact subset of the plane has a common point only with a finite number of elements of the family S.

An analogous theorem can be found in a paper of W. Kaplana (cf. [58], Theorem 32). W. Kaplan obtains the same conclusion as A. Beck under the assumption that F is a family of homeomorphic images of a straight line which are closed sets in the plane such that for every $p \in \mathbb{R}^2$ there exists exactly one $C \in F$ such that $p \in C$ and there exists an open set U_p containing p which can be mapped homeomorphically on the open square $\{(x,y) \in \mathbb{R}^2 : |x| < 1, |y| < 1\}$ in such a way that the images of the intersections of elements of F with U_p are the sets $\{(x,y) \in \mathbb{R}^2 : |x| < 1, y = c\}$ for some $c \in \mathbb{R}$ such that $|c| < 1$.

The announced construction of all iterative roots of a Brouwer homeomorphism embeddable in a flow is given in the following theorem.

Theorem 2.34. ([A3], Theorem 4.7) *Let f be a Brouwer homeomorphism embeddable in a flow and n be a positive integer. Let $\{C_\alpha : \alpha \in A\}$ and $\{U_\alpha : \alpha \in A\}$ be families of trajectories and maximal parallelizable regions occurring in Theorem 2.28. For each $\alpha \in A$ let $\{f_\alpha^t : t \in \mathbb{R}\}$ be a flow defined on U_α such that $f_\alpha^1(x) = f(x)$ for $x \in U_\alpha$, $\alpha \in A$. Assume that $f_1^{\frac{1}{n}}(x) = f_{-1}^{\frac{1}{n}}(x)$ for every $x \in C_1 = C_{-1}$ and*

$$\lim_{k\to\infty} f_\alpha^{\frac{1}{n}}(x_k) = f_{\widehat{\alpha i}}^{\frac{1}{n}}(x) \tag{2.13}$$

for each $x \in C_{\widehat{\alpha i}}$ and every sequence $(x_k)_{k\in\mathbb{N}}$ of elements of U_α such that $\lim_{k\to\infty} x_k = x$. Then the function g given by the formula

$$g(x) = f_\alpha^{\frac{1}{n}}(x), \quad x \in G_\alpha, \quad \alpha \in A, \tag{2.14}$$

where $\{G_\alpha : \alpha \in A\}$ is the family occurring in Lemma 2.33, is a Brouwer homeomorphism satisfying equation (2.12). Moreover, each Brouwer homeomorphism g satisfying equation (2.12) can be obtained in this way.

The embeddability of the Brouwer homeomorphism f in a flow implies the existence of families $\{C_\alpha : \alpha \in A\}$ and $\{U_\alpha : \alpha \in A\}$ occurring in Theorem 2.28. In each of the regions U_α separately, we consider an arbitrary flow containing the homeomorphism $f|_{U_\alpha}$. We assume only that these flows satisfy condition (2.13).

The most important part of the proof of Theorem 2.34 is to show that the function g given by formula (2.14) is continuous. To prove the continuity of g we use the assumption that $\lim_{k\to\infty} f_\alpha^{\frac{1}{n}}(x_k) = f_{\widehat{\alpha i}}^{\frac{1}{n}}(x)$ for every point $x \in C_{\widehat{\alpha i}}$ and every sequence $(x_k)_{k\in\mathbb{N}}$ of elements of the region U_α such that $\lim_{k\to\infty} x_k = x$.

Next we can use the fact mentioned above that for a given homeomorphism f of the plane onto itself each continuous solution g of equation (2.12) is a homeomorphism of the plane onto itself. The homeomorphism g preserves orientation, since its restriction to the maximal parallelizable region U_1 is an element of the flow $\{f_1^t : t \in \mathbb{R}\}$ defined on this region.

2.6 Topological equivalence of flows of Brouwer homeomorphisms

In this section we present results concerning topological equivalence of flows of Brouwer homeomorphisms. These results describe the relations between the sets of regular points and strongly irregular points for topologically equivalent flows of Brouwer homeomorphisms. Moreover, we present a similar result for maximal parallelizable regions.

We start from a result about the sets of regular points for topologically equivalent flows of Brouwer homeomorphisms.

Theorem 2.35. ([A5, Theorem 4.1]) *Let $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ be topologically equivalent flows of Brouwer homeomorphisms. Assume that the equivalence is realized by a homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$. Then Φ maps the set of all regular points of $\{f^t : t \in \mathbb{R}\}$ onto the set of all regular points of $\{g^t : t \in \mathbb{R}\}$.*

In the proof of this theorem we need to show that for each equivalence class F_0 of the codivergency relation of the flow $\{f^t : t \in \mathbb{R}\}$ there exists an equivalence class G_0 of the codivergency relation of the flow $\{g^t : t \in \mathbb{R}\}$ such that $\Phi(\operatorname{int} F_0) = \operatorname{int} G_0$. Let us recall that speaking of an equivalence class of a flow we mean an equivalence class of the codivergency relation defined for any element of this flow different from the identity map. Then by Theorem 3.21 mentioned above (which says that the set of all regular points is equal to the union of the interiors of all equivalence classes) and the fact that Φ is a homeomorphism of the plane onto itself we get the assertion of Theorem 2.35.

To prove that $\Phi(\operatorname{int} F_0) = \operatorname{int} G_0$ let us fix a point $p \in \operatorname{int} F_0$. Then there exist $q_1, q_2 \in \operatorname{int} F_0$ belonging to different components of the complement of the trajectory C_p of the flow $\{f^t : t \in \mathbb{R}\}$. From the invariance of equivalence classes of the codivergency relation and Theorems 2.18 and 2.15 we get that the trajectories $C'_{\Phi(q_1)}$ and $C'_{\Phi(q_2)}$ of the flow $\{g^t : t \in \mathbb{R}\}$ are contained in the same equivalence class. Let us denote this class by G_0. Then by Corollary 2.14 we obtain that $\Phi(p) \in \operatorname{int} G_0$. Thus $\Phi(\operatorname{int} F_0) \subset \operatorname{int} G_0$. To finish the proof let us observe that replacing Φ by Φ^{-1} in our reasoning we obtain that $\Phi^{-1}(\operatorname{int} G_0) \subset \operatorname{int} F_0$. Thus we have $\Phi(\operatorname{int} F_0) = \operatorname{int} G_0$.

From Theorem 2.35 we obtain that each homeomorphism which realizes the topological equivalence of flows of Brouwer homeomorphisms $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ maps the set of all irregular points of $\{f^t : t \in \mathbb{R}\}$ onto the set of all irregular points of $\{g^t : t \in \mathbb{R}\}$. In fact, we can show even more. Namely, the analogous property holds for the sets of all strongly irregular points. We formulate this result for the first prolongational limit sets, but by Theorem 3.24, the set of all strongly irregular points of a flow of Brouwer homeomorphisms is equal to its first prolongational limit set.

Theorem 2.36. ([A4], Theorem 5.3) *Let $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ be topologically equivalent flows of Brouwer homeomorphisms. Assume that the equivalence is realized by a homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$. Then $\Phi(\mathrm{J}_f(\mathbb{R}^2)) = \mathrm{J}_g(\mathbb{R}^2)$, where $\mathrm{J}_f(\mathbb{R}^2)$ and $\mathrm{J}_g(\mathbb{R}^2)$ denote the first prolongational limit sets of the flows $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$, respectively.*

In the proof of this theorem we fix a point $p \in J_f(\mathbb{R}^2)$ and denote by H_0 the component of $\mathbb{R}^2 \setminus C_p$ having a common point with $J_f(p)$. It may happen that each of the two components of $\mathbb{R}^2 \setminus C_p$ satisfies this condition. Next we take a point $q \in J_f(p) \cap H_0$. To show that $\Phi(q) \in J_g(\Phi(p))$ we consider the following two cases: there exists an equivalence class G_0 of the codivergency relation such that $p \in \operatorname{bd} G_0$ and $G_0 \subset H_0$ and the case where the point p does not belong to the boundary of any equivalent class contained in H_0.

In the first of these cases we can directly use the results from [B13] (cf. Theorems 3.19 and 3.20). In the second case we consider a sequence of trajectories $(C_n)_{n\in\mathbb{N}}$ which are contained in the set of irregular points having the property that for every positive integer n we have $C_p|C_{n+1}|C_n$ and for each $p_0 \in C_p$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $p_0 = \lim_{n\to\infty} x_n$ and $x_n \in C_n$. For this sequence of trajectories we can apply the same reasoning as that given in [B13]. The crucial fact used here is that, if $q \in \mathrm{J}_f(p)$, then trajectories C_p and C_q have no disjoint invariant neighbourhoods, i.e. neighbourhoods which are equal to unions of trajectories.

The reasoning sketched above leads to the following result, which in turn implies Theorem 2.36.

Theorem 2.37. ([A6], Proposition 2.3) *Let $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ be topologically equivalent flows of Brouwer homeomorphisms. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism which realizes this equivalence. Then, if $q \in \mathrm{J}_f(p)$, then $\Phi(q) \in \mathrm{J}_g(\Phi(p))$ for any $p, q \in \mathbb{R}^2$, where $\mathrm{J}_f(\mathbb{R}^2)$ and $\mathrm{J}_g(\mathbb{R}^2)$ denote the first prolongational limit sets of the flows $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$, respectively.*

Now we proceed to results concerning maximal parallelizable regions of topologically equivalent flows of Brouwer homeomorphisms.

Theorem 2.38. ([A6], Proposition 2.1) *Let $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ be topologically equivalent flows of Brouwer homeomorphisms. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism which realizes this equivalence. Then, if U is a maximal parallelizable region of $\{f^t : t \in \mathbb{R}\}$, then $\Phi(U)$ is a maximal parallelizable region of $\{g^t : t \in \mathbb{R}\}$.*

In the proof of this theorem we use the fact that the homeomorphism which realizes the topological equivalence maps continuous section of a parallelizable region U of the flow $\{f^t : t \in \mathbb{R}\}$ onto continuous section of the region $\Phi(U)$. Hence $\Phi(U)$ is a parallelizable region of the flow $\{g^t : t \in \mathbb{R}\}$. If we suppose that $\Phi(U)$ is not a maximal parallelizable region of $\{g^t : t \in \mathbb{R}\}$ we get the contradiction with the assumption that U is a maximal parallelizable region of $\{f^t : t \in \mathbb{R}\}$.

The next theorem says that for topologically equivalent flows of Brouwer homeomorphisms we can take the same admissible class of finite sequences for the families of maximal parallelizable regions occurring in Theorem 2.28.

Theorem 2.39. ([A6], Theorem 2.4) *Let $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ be topologically equivalent flows of Brouwer homeomorphisms and $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism which realizes the equivalence. Let $\{C_{f,\alpha} : \alpha \in A\}$ be a family of trajectories and $\{U_{f,\alpha} : \alpha \in A\}$ a family of maximal parallelizable regions of $\{f^t : t \in \mathbb{R}\}$, for* *which conditions (2.4) - (2.9) hold and $U_{f,1} = U_{f,-1}$, $C_{f,1} = C_{f,-1}$, where $A = A^+ \cup A^-$ is an admissible class of finite sequences. Let $C_{g,\alpha} := \Phi(C_{f,\alpha})$ and $U_{g,\alpha} := \Phi(U_{f,\alpha})$ for $\alpha \in A$. Then $\{C_{g,\alpha} : \alpha \in A\}$ is a family of trajectories and $\{U_{g,\alpha} : \alpha \in A\}$ is a family of maximal parallelizable regions of $\{g^t : t \in \mathbb{R}\}$ such that conditions (2.4) - (2.9) are satisfied. Moreover, we have*

$$\Phi(C^{\alpha}_{f,\alpha\widehat{\ }i}) = C^{\alpha}_{g,\alpha\widehat{\ }i}, \quad \alpha\widehat{\ }i \in A, \tag{2.15}$$

where $C^{\alpha}_{f,\alpha\widehat{\ }i}$ and $C^{\alpha}_{g,\alpha\widehat{\ }i}$ are the unique trajectories contained in $\mathrm{J}_f(C_{f,\alpha\widehat{\ }i}) \cap U_{f,\alpha}$ and $\mathrm{J}_g(C_{g,\alpha\widehat{\ }i}) \cap U_{g,\alpha}$.

To show that each of the sets $\{U_{g,\alpha} : \alpha \in A\}$ is a maximal parallelizable region of $\{g^t : t \in \mathbb{R}\}$ we use Theorem 2.38. The fact that each of the sets $\{C_{g,\alpha} : \alpha \in A\}$ is a trajectory of $\{g^t : t \in \mathbb{R}\}$ contained in $\mathrm{J}_g(\mathbb{R}^2)$ we obtain from Theorem 2.36. Finally, we prove condition (2.15) using Theorem 2.37 and Corollary 2.24.

2.7 Topological conjugacy of flows of Brouwer homeomorphisms

In this section we present results which describe conditions for topological conjugacy of flows of Brouwer homeomorphisms. The general form of such flows is described in Theorem 2.28.

S. Andrea has proved a theorem ([5], Theorem 4.1) which says that a Brouwer homeomorphism is topologically conjugate with a translation if and only if it has exactly one equivalence class of the codivergency relation. Then all flows which contain such a homeomorphism are topologically congugate each other (cf. Theorem 3.11). In particular, they are topologically congugate with the flow of translations. The results presented below concern flows of Brouwer homeomorphisms which are not topologically congugate with the flow of translations.

Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. According to Theorem 2.28 there exists an admissible class of finite sequences A, a family of trajectories $\{C_{f,\alpha} : \alpha \in A\}$ and a family of maximal parallelizable regions $\{U_{f,\alpha} : \alpha \in A\}$, for which conditions (2.4) - (2.9) are fulfilled. Using Theorem 2.29 we obtain that for each region $U_{f,\alpha}$ there exists a parallelizing homeomorphism which maps trajectories of the flow $\{f^t : t \in \mathbb{R}\}$ onto horizontal straight lines in the Cartesian coordinate system with coordinates denoted by (t, s).

Using such families we give a condition under which there exists a homeomorphism which realizes the topological conjugacy between flows $\{f^t : t \in \mathbb{R}\}$, $\{g^t : t \in \mathbb{R}\}$ and describe the construction of such a homeomorphism. The main step of this construction is to extend the homeomorphism Ψ which will conjugate the flows from the maximal parallelizable region $U_{f,\alpha}$ onto the trajectories $C_{f,\alpha\widehat{\ }i}$ for $\alpha\widehat{\ }i \in A$. Here we use Lemma 2.33 which says that every point $p \in C_{f,\alpha\widehat{\ }i}$ has a neighbourhood contained in $U_{f,\alpha} \cup U_{f,\alpha\widehat{\ }i}$.

For each $\alpha\widehat{\ }i \in A$ denote by $C^{\alpha}_{f,\alpha\widehat{\ }i}$ the uniquely determined trajectory contained in $U_{f,\alpha} \cap \mathrm{J}(C_{f,\alpha\widehat{\ }i})$ (cf. Theorem 2.24). According to the definition of a parallelizing homeomorphism $\varphi_{f,\alpha\widehat{\ }i}$, the trajectory $C_{f,\alpha\widehat{\ }i}$ corresponds to the value $s = 0$ in the coordinate system given by this homeomorphism.

Theorem 2.31 says that the trajectory $C^{\alpha}_{f,\alpha\widehat{\ }i}$ corresponds to one of the values $c^{\alpha}_{f,\alpha\widehat{\ }i}$, $d^{\alpha}_{f,\alpha\widehat{\ }i}$ in the coordinate system given by the parallelizing homeomorphism $\varphi_{f,\alpha}$ depending on how this trajectory is situated in relation to the trajectories $C_{f,\alpha}$ and $C_{f,\alpha\widehat{\ }i}$, where $c^{\alpha}_{f,\alpha\widehat{\ }i}$, $d^{\alpha}_{f,\alpha\widehat{\ }i}$ are the constants occurring in Theorem 2.29.

Let $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ be topologically equivalent flows of Brouwer homeomorphisms, and $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism which realizes the equivalence. Let A be an admissible class of finite sequences, $\{C_{f,\alpha} : \alpha \in A\}$ be a family of trajectories and a $\{U_{f,\alpha} : \alpha \in A\}$ a family of maximal parallelizable regions of $\{f^t : t \in \mathbb{R}\}$ occurring in Theorem 2.28. Let $\{\varphi_{f,\alpha} : \alpha \in A\}$, where $\varphi_{f,\alpha} : U_{f,\alpha} \to \mathbb{R}^2$, be a family of parallelizing homeomorphisms occurring in Theorem 2.29.

Using the properties of the homeomorphism Φ realizing the topological equivalence of these flows we can determine corresponding families $\{C_{g,\alpha} : \alpha \in A\}$ and $\{U_{g,\alpha} : \alpha \in A\}$ for the flow $\{g^t : t \in \mathbb{R}\}$. To this end let us put $C_{g,\alpha} := \Phi(C_{f,\alpha})$ and $U_{g,\alpha} := \Phi(U_{f,\alpha})$ for $\alpha \in A$. Then by Theorem 2.39, $\{C_{g,\alpha} : \alpha \in A\}$ is a family of trajectories and $\{U_{g,\alpha} : \alpha \in A\}$ is a family of maximal parallelizable regions of $\{g^t : t \in \mathbb{R}\}$ for which conditions (2.4) - (2.9) are satisfied. Let $\{\varphi_{g,\alpha} : \alpha \in A\}$, where $\varphi_{g,\alpha} : U_{g,\alpha} \to \mathbb{R}^2$, be a family of parallelizing homeomorphisms described in Theorem 2.29.

For $\alpha\widehat{\ }i \in A^+$ let

$$b^{\alpha}_{f,\alpha\widehat{\ }i} := \begin{cases} c^{\alpha}_{f,\alpha\widehat{\ }i}, & \text{if } C_{f,\alpha}|C^{\alpha}_{f,\alpha\widehat{\ }i}|C_{f,\alpha\widehat{\ }i} \text{ or } C_{f,\alpha} = C^{\alpha}_{f,\alpha\widehat{\ }i}, \\ d^{\alpha}_{f,\alpha\widehat{\ }i}, & \text{if } |C_{f,\alpha}, C^{\alpha}_{f,\alpha\widehat{\ }i}, C_{f,\alpha\widehat{\ }i}|. \end{cases} \tag{2.16}$$

Then

$$\varphi_{f,\alpha}(C^{\alpha}_{f,\alpha\widehat{\ }i}) = \mathbb{R} \times \{b^{\alpha}_{f,\alpha\widehat{\ }i}\}.$$

Similarly, let

$$b^{\alpha}_{g,\alpha\widehat{\ }i} := \begin{cases} c^{\alpha}_{g,\alpha\widehat{\ }i}, & \text{if } C_{g,\alpha}|C^{\alpha}_{g,\alpha\widehat{\ }i}|C_{g,\alpha\widehat{\ }i} \text{ or } C_{g,\alpha} = C^{\alpha}_{g,\alpha\widehat{\ }i}, \\ d^{\alpha}_{g,\alpha\widehat{\ }i}, & \text{if } |C_{g,\alpha}, C^{\alpha}_{g,\alpha\widehat{\ }i}, C_{g,\alpha\widehat{\ }i}|. \end{cases} \tag{2.17}$$

Then

$$\varphi_{g,\alpha}(C^{\alpha}_{g,\alpha\widehat{\ }i}) = \mathbb{R} \times \{b^{\alpha}_{g,\alpha\widehat{\ }i}\}.$$

Hence

$$(\varphi_{g,\alpha} \circ \Phi \circ \varphi^{-1}_{f,\alpha})(\mathbb{R} \times \{b^{\alpha}_{f,\alpha\widehat{\ }i}\}) = \mathbb{R} \times \{b^{\alpha}_{g,\alpha\widehat{\ }i}\},$$

since, by Theorem 2.39 we have $\Phi(C^{\alpha}_{f,\alpha\widehat{\ }i}) = C^{\alpha}_{g,\alpha\widehat{\ }i}$.

For the class A^- we obtain an analogous result. The only difference is the definition of $b^{\alpha}_{f,\alpha\widehat{\ }i}$ and $b^{\alpha}_{g,\alpha\widehat{\ }i}$. In this case we put

$$b^{\alpha}_{f,\alpha\widehat{\ }i} := \begin{cases} d^{\alpha}_{f,\alpha\widehat{\ }i}, & \text{if } C_{f,\alpha}|C^{\alpha}_{f,\alpha\widehat{\ }i}|C_{f,\alpha\widehat{\ }i} \text{ or } C_{f,\alpha} = C^{\alpha}_{f,\alpha\widehat{\ }i}, \\ c^{\alpha}_{f,\alpha\widehat{\ }i}, & \text{if } |C_{f,\alpha}, C^{\alpha}_{f,\alpha\widehat{\ }i}, C_{f,\alpha\widehat{\ }i}|. \end{cases} \tag{2.18}$$

and

$$b^{\alpha}_{g,\alpha\widehat{\ }i} := \begin{cases} d^{\alpha}_{g,\alpha\widehat{\ }i}, & \text{if } C_{g,\alpha}|C^{\alpha}_{g,\alpha\widehat{\ }i}|C_{g,\alpha\widehat{\ }i} \text{ or } C_{g,\alpha} = C^{\alpha}_{g,\alpha\widehat{\ }i}, \\ c^{\alpha}_{g,\alpha\widehat{\ }i}, & \text{if } |C_{g,\alpha}, C^{\alpha}_{g,\alpha\widehat{\ }i}, C_{g,\alpha\widehat{\ }i}|. \end{cases} \tag{2.19}$$

Thus, for the topologically equivalent flows $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ we have

$$b^{\alpha}_{f,\alpha\widehat{\ }i} = c^{\alpha}_{f,\alpha\widehat{\ }i} \quad \Leftrightarrow \quad b^{\alpha}_{g,\alpha\widehat{\ }i} = c^{\alpha}_{g,\alpha\widehat{\ }i}, \tag{2.20}$$

since every homeomorphism preserves the considered ternary relations in the set of all trajectories. Moreover,

$$b^{\alpha}_{f,\alpha\widehat{\ }i} < b^{\alpha}_{f,\alpha\widehat{\ }j} \quad \Leftrightarrow \quad b^{\alpha}_{g,\alpha\widehat{\ }i} < b^{\alpha}_{g,\alpha\widehat{\ }j} \tag{2.21}$$

for $\alpha\widehat{\ }i, \alpha\widehat{\ }j \in A$, $i \neq j$, since for each $\alpha \in A$ the homeomorphism $\varphi_{g,\alpha} \circ \Phi \circ \varphi^{-1}_{f,\alpha}$ is strictly increasing with respect to the second variable.

Now we present the main result concerning the problem of the topological conjugacy of flows of Brouwer homeomorphisms.

Theorem 2.40. ([A6], Theorem 3.2) *Let $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ be topologically equivalent flows of Brouwer homeomorphisms. Assume that for each $\alpha\widehat{\ }i \in A$ there exists a continuous function $\gamma_{\alpha\widehat{\ }i} : I^{\alpha}_{f,\alpha\widehat{\ }i} \to \mathbb{R}$ and an increasing homeomorphism $\beta_{\alpha\widehat{\ }i} : I^{\alpha}_{f,\alpha\widehat{\ }i} \to I^{\alpha}_{g,\alpha\widehat{\ }i}$ such that*

$$\mu_{f,\alpha\widehat{\ }i}(s) = (\mu_{g,\alpha\widehat{\ }i} \circ \beta_{\alpha\widehat{\ }i})(s) + \gamma_{\alpha\widehat{\ }i}(s), \qquad s \in I^{\alpha}_{f,\alpha\widehat{\ }i}, \tag{2.22}$$

and $\lim_{s \to b^{\alpha}_{f,\alpha\widehat{\ }i}} \beta_{\alpha\widehat{\ }i}(s) = b^{\alpha}_{g,\alpha\widehat{\ }i}$, $\lim_{s \to b^{\alpha}_{f,\alpha\widehat{\ }i}} \gamma_{\alpha\widehat{\ }i}(s) = a^{\alpha}_{g,\alpha\widehat{\ }i}$ for some $a^{\alpha}_{g,\alpha\widehat{\ }i} \in \mathbb{R}$, where $\mu_{f,\alpha\widehat{\ }i}$, $\mu_{g,\alpha\widehat{\ }i}$ are continuous functions and $c^{\alpha}_{f,\alpha\widehat{\ }i}$, $d^{\alpha}_{f,\alpha\widehat{\ }i}$, $c^{\alpha}_{g,\alpha\widehat{\ }i}$, $d^{\alpha}_{g,\alpha\widehat{\ }i}$ are constants occurring in Theorem 2.29, and $I^{\alpha}_{f,\alpha\widehat{\ }i} := (c^{\alpha}_{f,\alpha\widehat{\ }i}, c^{\alpha}_{f,\alpha\widehat{\ }i} + \varepsilon^{\alpha}_{f,\alpha\widehat{\ }i})$, $I^{\alpha}_{g,\alpha\widehat{\ }i} := (c^{\alpha}_{g,\alpha\widehat{\ }i}, c^{\alpha}_{g,\alpha\widehat{\ }i} + \varepsilon^{\alpha}_{f,\alpha\widehat{\ }i})$ in case $b^{\alpha}_{g,\alpha\widehat{\ }i} = c^{\alpha}_{g,\alpha\widehat{\ }i}$ or $I^{\alpha}_{f,\alpha\widehat{\ }i} := (d^{\alpha}_{f,\alpha\widehat{\ }i} - \varepsilon^{\alpha}_{f,\alpha\widehat{\ }i}, d^{\alpha}_{f,\alpha\widehat{\ }i})$, $I^{\alpha}_{g,\alpha\widehat{\ }i} := (d^{\alpha}_{g,\alpha\widehat{\ }i} - \varepsilon^{\alpha}_{g,\alpha\widehat{\ }i}, d^{\alpha}_{g,\alpha\widehat{\ }i})$ in case $b^{\alpha}_{g,\alpha\widehat{\ }i} = d^{\alpha}_{g,\alpha\widehat{\ }i}$ for some $\varepsilon^{\alpha}_{f,\alpha\widehat{\ }i} > 0$ and $\varepsilon^{\alpha}_{g,\alpha\widehat{\ }i} > 0$ with $b^{\alpha}_{f,\alpha\widehat{\ }i}$ and $b^{\alpha}_{g,\alpha\widehat{\ }i}$ defined by (2.16) and (2.17) or by (2.18) and (2.19), respectively. Moreover, we assume that for $\alpha\widehat{\ }i, \alpha\widehat{\ }j \in A$, $i \neq j$ we have $\beta_{\alpha\widehat{\ }i} = \beta_{\alpha\widehat{\ }j}$ in the case where $C^{\alpha}_{f,\alpha\widehat{\ }i} = C^{\alpha}_{f,\alpha\widehat{\ }j}$ and $C_{f,\alpha\widehat{\ }i}$, $C_{f,\alpha\widehat{\ }j}$ are contained in the same component of $\mathbb{R} \setminus C^{\alpha}_{f,\alpha\widehat{\ }i}$. Then the flows $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ are topologically conjugate.

The construction of a homeomorphism realizing the topological conjugacy of flows $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ is given by induction. By Theorem 2.39 we can take the same set of indices A for the both flows, since they are topologically equivalent. For each positive integer n let us define $A^+_n := \{\alpha \in A^+ : |\alpha| = n\}$ oraz $A^-_n := \{\alpha \in A^- : |\alpha| = n\}$, where $|\alpha|$ denotes the length of the sequence α. By the definition of an admissible class of finite sequences we obtain that if $A^+_n = \emptyset$ for some n, then $A^+_m = \emptyset$ for all $m > n$. Similarly, if $A^-_n = \emptyset$ for some n, then $A^-_m = \emptyset$ for all $m > n$.

From the definition of parallelizing homeomorphisms we infer that $C_{f,\alpha} = \varphi^{-1}_{f,\alpha}(\mathbb{R} \times \{0\})$ and $C_{g,\alpha} = \varphi^{-1}_{g,\alpha}(\mathbb{R} \times \{0\})$. For each $\alpha \in A$ put $G_{f,\alpha} := \varphi^{-1}_{f,\alpha}(\mathbb{R} \times [0, +\infty))$

if $\alpha \in A^+$ or $G_{f,\alpha} := \varphi_{f,\alpha}^{-1}(\mathbb{R} \times (-\infty, 0])$ if $\alpha \in A^-$. Denote by $H_{f,\alpha}$ the component of $\mathbb{R}^2 \setminus C_{f,\alpha}$ containing $G_{f,\alpha} \setminus C_{f,\alpha}$. Hence $C_{f,\alpha\frown i} \subset H_{f,\alpha}$ for every $\alpha\frown i \in A$. Similarly, let $G_{g,\alpha} := \varphi_{g,\alpha}^{-1}(\mathbb{R} \times [0, +\infty))$ if $\alpha \in A^+$ or $G_{g,\alpha} := \varphi_{g,\alpha}^{-1}(\mathbb{R} \times (-\infty, 0])$ if $\alpha \in A^-$. Denote by $H_{g,\alpha}$ the component of $\mathbb{R}^2 \setminus C_{g,\alpha}$ containing $G_{g,\alpha} \setminus C_{g,\alpha}$. Then $C_{g,\alpha\frown i} \subset H_{g,\alpha}$ for every $\alpha\frown i \in A$.

For each positive integer n such that $A_n^+ \neq \emptyset$ we define $U_{f,n}^+$ by putting $U_{f,n}^+ := G_{f,1}$ in the case where $n = 1$ or by $U_{f,n}^+ := U_{f,n-1}^+ \cup \bigcup_{\alpha \in A_n^+} G_{f,\alpha}$ in the case where $n > 1$. Similarly, we define $U_{g,n}^+$ by putting $U_{g,n}^+ := G_{g,1}$ in case $n = 1$ or $U_{g,n}^+ := U_{g,n-1}^+ \cup \bigcup_{\alpha \in A_n^+} G_{g,\alpha}$ in case $n > 1$. In an analogous way we define $U_{f,n}^-$ and $U_{g,n}^-$ if $A_n^- \neq \emptyset$.

Now we proceed to describing the announced construction of a homeomorphism realizing the topological conjugacy of flows $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$. We start from start from defining this homeomorphism on the maximal parallelizable region $U_{f,1}$. Since $U_{f,1} = U_{f,1}^+ \cup C_{f,1} \cup U_{f,1}^-$ and $U_{g,1} = U_{g,1}^+ \cup C_{g,1} \cup U_{g,1}^-$, the homeomorphism defined in the first step of the construction will be used in the construction of the homeomorphism realizing the topological conjugacy for the both components $H_{f,1}$ and $H_{f,-1}$ of the set $\mathbb{R}^2 \setminus C_{f,1}$.

Next we show how to extend the homeomorphism defined on $U_{f,1}$ to a homeomorphism realizing the topological conjugacy of the considered flows on $U_{f,1} \cup H_{f,1}$. Here we use the fact that the set $C_{f,1} \cup H_{f,1}$ is a union of the ascending sequence of sets $(U_{f,n}^+)_{n \in \mathbb{N}}$. The range of this extension will be equal to $U_{g,1} \cup H_{g,1}$. For the component $H_{f,-1}$ the homeomorphism realizing the topological conjugacy of the considered flows can be defined in an analogous way.

Since $U_{f,1}$ i $U_{g,1}$ are parallelizable regions, there exists a homeomorphism $\Psi_1 : U_{f,1} \to U_{g,1}$ realizing the topological conjugacy of the flows $\{f^t|_{U_{f,1}} : t \in \mathbb{R}\}$ oraz $\{g^t|_{U_{g,1}} : t \in \mathbb{R}\}$. We can take this homeomorphism in such a way that $\Psi_1(C_{f,1}) = C_{g,1}$ and $\Psi_1(C_{f,1\frown i}^1) = C_{g,1\frown i}^1$ for $1\frown i \in A^+$ and $\Psi_1(C_{f,-1\frown i}^{-1}) = C_{g,-1\frown i}^{-1}$ for $-1\frown i \in A^-$.

Let us fix a positive integer n such that $A_{n+1}^+ \neq \emptyset$. Assume that we have defined a homeomorphism $\Psi_n : U_{f,n}^+ \to U_{g,n}^+$ realizing the topological conjugacy of the flows $\{f^t|_{U_{f,n}^+} : t \in \mathbb{R}\}$ and $\{g^t|_{U_{g,n}^+} : t \in \mathbb{R}\}$ such that for each $\alpha\frown i \in A^+$ with $|\alpha| \leq n$ we have $\Psi_n(C_{f,\alpha}) = C_{g,\alpha}$ and $\Psi_n(C_{f,\alpha\frown i}^\alpha) = C_{g,\alpha\frown i}^\alpha$. Moreover, for each $\alpha \in A^+$ such that $|\alpha| \leq n$ the restriction $\Psi_\alpha : G_{f,\alpha} \to G_{g,\alpha}$ of the homeomorphism Ψ_n to the set $G_{f,\alpha}$ has the form $\Psi_\alpha := \varphi_{g,\alpha}^{-1} \circ \psi_\alpha \circ (\varphi_{f,\alpha}|_{G_{f,\alpha}})$, where

$$\psi_\alpha(t, s) = (\eta_\alpha + t, \theta_\alpha(s)), \qquad (t, s) \in \mathbb{R} \times [0, +\infty),$$

with an increasing homeomorphism $\theta_\alpha : [0, +\infty) \to [0, +\infty)$ having the property that $\theta_\alpha(b_{f,\alpha\frown i}^\alpha) = b_{g,\alpha\frown i}^\alpha$ for $\alpha\frown i \in A$ and θ_α restricted to the interval $I_{f,\alpha\frown i}^\alpha$ is equal to $\beta_{\alpha\frown i}$. The existence such a homeomorphism θ_α follows from condition (2.21). By the definition of the homeomorphism Ψ_1 we get $\eta_1 = 0$, and all other values of η_α for $\alpha \in A^+$ takich, że $|\alpha| \leq n$, has been determined in the subsequent steps of the construction.

Let us fix an $\alpha\frown i \in A_{n+1}^+$. Denote by (t, s) the coordinates of points belonging to do $U_{f,\alpha}$ in the chart $\varphi_{f,\alpha}$ and by (t', s') the coordinates of points belonging to $U_{f,\alpha\frown i}$ in the chart $\varphi_{f,\alpha\frown i}$. Then by (2.11) we obtain that $t' = \mu_{f,\alpha\frown i}(s)+t$ and $s' = \nu_{f,\alpha\frown i}(s)$ for $t \in \mathbb{R}$ and $s \in (c^{\alpha}_{f,\alpha\frown i}, d^{\alpha}_{f,\alpha\frown i})$. Similarly, denote by (u,v) the coordinates of points belonging to $U_{g,\alpha}$ in the chart $\varphi_{g,\alpha}$ oraz przez (u',v') the coordinates of points belonging to $U_{g,\alpha\frown i}$ in the chart $\varphi_{g,\alpha\frown i}$. Then $u' = \mu_{g,\alpha\frown i}(v) + u$ and $v' = \nu_{g,\alpha\frown i}(v)$ for $u \in \mathbb{R}$, $v \in (c^{\alpha}_{g,\alpha\frown i}, d^{\alpha}_{g,\alpha\frown i})$.

The relationship between the coordinates of points belonging to $U_{f,\alpha}$ and $U_{g,\alpha}$ may be expressed by using the the homeomorphism ψ_α. The condition $(u,v) = \psi_\alpha(t,s)$ means that $u = \eta_\alpha + t$ and $v = \theta_\alpha(s)$ for $t \in \mathbb{R}$ and $s \in [0,+\infty)$. For the sets $U_{f,\alpha} \cap U_{f,\alpha\frown i}$ and $U_{g,\alpha} \cap U_{g,\alpha\frown i}$ we change the systems of coordinates. In the new systems we have

$$u' = \mu_{g,\alpha\frown i}(\beta_{\alpha\frown i}(\nu^{-1}_{f,\alpha\frown i}(s'))) + \eta_\alpha + t' - \mu_{f,\alpha\frown i}(\nu^{-1}_{f,\alpha\frown i}(s'))$$

and

$$v' = \nu_{g,\alpha\frown i}(\beta_{\alpha\frown i}(\nu^{-1}_{f,\alpha\frown i}(s'))).$$

Using the assumptions of our theorem we get $u' = -\gamma_{\alpha\frown i}(\nu^{-1}_{f,\alpha\frown i}(s')) + \eta_\alpha + t'$.

Let

$$\eta_{\alpha\frown i}(s') := -\gamma_{\alpha\frown i}(\nu^{-1}_{f,\alpha\frown i}(s')) + \eta_\alpha$$

and

$$\xi_{\alpha\frown i}(s') := \nu_{g,\alpha\frown i}(\beta_{\alpha\frown i}(\nu^{-1}_{f,\alpha\frown i}(s')))$$

for $s' \in (d_{f,\alpha\frown i}, 0)$, where $d_{f,\alpha\frown i} = \nu_{f,\alpha\frown i}(c^{\alpha}_{f,\alpha\frown i} + \varepsilon^{\alpha}_{f,\alpha\frown i})$ or $d_{f,\alpha\frown i} = \nu_{f,\alpha\frown i}(d^{\alpha}_{f,\alpha\frown i} - \varepsilon^{\alpha}_{f,\alpha\frown i})$. Then by the assumptions of our theorem we have $\lim_{s'\to 0} \eta_{\alpha\frown i}(s') = -a^{\alpha}_{g,\alpha\frown i} + \eta_\alpha$ and $\lim_{s'\to 0} \xi_{\alpha\frown i}(s') = 0$, since after changing coordinate system the condition $s \to b^{\alpha}_{f,\alpha\frown i}$ gives $s' \to 0$. Thus the function $\phi^{\alpha}_{\alpha\frown i} : \mathbb{R} \times (d_{f,\alpha\frown i}, 0) \to \mathbb{R} \times (d_{g,\alpha\frown i}, 0)$ defined by the formula

$$\phi^{\alpha}_{\alpha\frown i}(t', s') := (\eta_{\alpha\frown i}(s') + t', \xi_{\alpha\frown i}(s'))$$

can be extended to a homeomorphism defined on $\mathbb{R} \times (d_{f,\alpha\frown i}, 0]$ by putting $\eta_{\alpha\frown i}(0) := -a^{\alpha}_{g,\alpha\frown i} + \eta_\alpha$ and $\xi_{\alpha\frown i}(0) := 0$.

After changing coordinate systems the restriction of the homeomorphism Ψ_α to the region $G_{f,\alpha} \cap U_{f,\alpha\frown i}$ has the form

$$\Psi_\alpha|_{G_{f,\alpha}\cap U_{f,\alpha\frown i}} = \varphi^{-1}_{g,\alpha} \circ \phi^{\alpha}_{\alpha\frown i} \circ (\varphi_{f,\alpha}|_{G_{f,\alpha}\cap U_{f,\alpha\frown i}}).$$

Thus by the extension of the homeomorphism $\phi^{\alpha}_{\alpha\frown i}$ we get the extension of the homeomorphism Ψ_α to $G_{f,\alpha} \cup C_{f,\alpha\frown i}$ which maps $C_{f,\alpha\frown i}$ onto $C_{g,\alpha\frown i}$, since $\xi_{\alpha\frown i}(0) = 0$.

We define $\Psi_{\alpha\frown i} : G_{f,\alpha\frown i} \to G_{g,\alpha\frown i}$ by the formula $\Psi_{\alpha\frown i} := \varphi^{-1}_{g,\alpha\frown i} \circ \psi_{\alpha\frown i} \circ (\varphi_{f,\alpha\frown i}|_{G_{f,\alpha\frown i}})$, where

$$\psi_{\alpha\frown i}(t,s) = (\eta_{\alpha\frown i} + t, \theta_{\alpha\frown i}(s)), \qquad (t,s) \in \mathbb{R} \times [0,+\infty),$$

with $\eta_{\alpha\frown i} := \eta_{\alpha\frown i}(0)$ and an increasing homeomorphism $\theta_{\alpha\frown i} : [0,+\infty) \to [0,+\infty)$ having the property that $\theta_{\alpha\frown i}(b^{\alpha\frown i}_{f,\alpha\frown i\frown j}) = b^{\alpha\frown i}_{g,\alpha\frown i\frown j}$ for $\alpha\frown i\frown j \in A$ and the restriction of $\theta_{\alpha\frown i}$ to the interval $I^{\alpha\frown i}_{f,\alpha\frown i\frown j}$ is equal to $\beta_{\alpha\frown i\frown j}$.

The extension of the homeomorphism Ψ_n to the set $U^+_{f,n+1}$ we define in the following way

$$\Psi_{n+1}(p) \quad := \begin{cases} \Psi_n(p), & p \in U^+_{f,n}, \\ \Psi_{\alpha\widehat{\ }i}(p), & p \in G_{f,\alpha\widehat{\ }i},\ \alpha\widehat{\ }i \in A^+_{n+1}. \end{cases}$$

Then Ψ_{n+1} is a homeomorphism realizing the topological conjugacy of flows $\{f^t|_{U^+_{f,n+1}} : t \in \mathbb{R}\}$ and $\{g^t|_{U^+_{g,n+1}} : t \in \mathbb{R}\}$.

At the end of this section we present a result which says that the assumptions made to construct a homeomorphism realizing the topological conjugacy of flows of Brouwer homeomorphisms are necessary conditions for the existence such a homeomorphism.

Theorem 2.41. ([A6], Proposition 3.3) *Let $\{f^t : t \in \mathbb{R}\}$ and $\{g^t : t \in \mathbb{R}\}$ be topologically conjugate flows of Brouwer homeomorphisms. Then for each $\alpha\widehat{\ }i \in A$ there exist a continuous function $\gamma_{\alpha\widehat{\ }i} : I^\alpha_{f,\alpha\widehat{\ }i} \to \mathbb{R}$ and an increasing homeomorphism $\beta_{\alpha\widehat{\ }i} : I^\alpha_{f,\alpha\widehat{\ }i} \to I^\alpha_{g,\alpha\widehat{\ }i}$ such that relation (2.22) holds, i.e.*

$$\mu_{f,\alpha\widehat{\ }i}(s) = (\mu_{g,\alpha\widehat{\ }i} \circ \beta_{\alpha\widehat{\ }i})(s) + \gamma_{\alpha\widehat{\ }i}(s), \qquad s \in I^\alpha_{f,\alpha\widehat{\ }i},$$

where $\mu_{f,\alpha\widehat{\ }i}$, $\mu_{g,\alpha\widehat{\ }i}$, $c^\alpha_{f,\alpha\widehat{\ }i}$, $d^\alpha_{f,\alpha\widehat{\ }i}$, $c^\alpha_{g,\alpha\widehat{\ }i}$, $d^\alpha_{g,\alpha\widehat{\ }i}$ are those occurring in Theorem 2.29, and $I^\alpha_{f,\alpha\widehat{\ }i} := (c^\alpha_{f,\alpha\widehat{\ }i}, c^\alpha_{f,\alpha\widehat{\ }i} + \varepsilon^\alpha_{f,\alpha\widehat{\ }i})$, $I^\alpha_{g,\alpha\widehat{\ }i} := (c^\alpha_{g,\alpha\widehat{\ }i}, c^\alpha_{g,\alpha\widehat{\ }i} + \varepsilon^\alpha_{f,\alpha\widehat{\ }i})$ in case $b^\alpha_{g,\alpha\widehat{\ }i} = c^\alpha_{g,\alpha\widehat{\ }i}$ or $I^\alpha_{f,\alpha\widehat{\ }i} := (d^\alpha_{f,\alpha\widehat{\ }i} - \varepsilon^\alpha_{f,\alpha\widehat{\ }i}, d^\alpha_{f,\alpha\widehat{\ }i})$, $I^\alpha_{g,\alpha\widehat{\ }i} := (d^\alpha_{g,\alpha\widehat{\ }i} - \varepsilon^\alpha_{g,\alpha\widehat{\ }i}, d^\alpha_{g,\alpha\widehat{\ }i})$ in case $b^\alpha_{g,\alpha\widehat{\ }i} = d^\alpha_{g,\alpha\widehat{\ }i}$ for some $\varepsilon^\alpha_{f,\alpha\widehat{\ }i} > 0$ and $\varepsilon^\alpha_{g,\alpha\widehat{\ }i} > 0$, with $b^\alpha_{f,\alpha\widehat{\ }i}$, $b^\alpha_{g,\alpha\widehat{\ }i}$ defined by (2.16) and (2.17) or by (2.18) and (2.19), respectively. Moreover $\lim_{s\to b^\alpha_{f,\alpha\widehat{\ }i}} \beta_{\alpha\widehat{\ }i}(s) = b^\alpha_{g,\alpha\widehat{\ }i}$, $\lim_{s\to b^\alpha_{f,\alpha\widehat{\ }i}} \gamma_{\alpha\widehat{\ }i}(s) = a^\alpha_{g,\alpha\widehat{\ }i}$ for some $a^\alpha_{g,\alpha\widehat{\ }i} \in \mathbb{R}$.

In the presented below a sketch of the proof of this result we will show how, under the assumption that the flows are topologically conjugate, to define the functions $\gamma_{\alpha\widehat{\ }i}$ and $\beta_{\alpha\widehat{\ }i}$. Let $\{C_{f,\alpha} : \alpha \in A\}$, $\{U_{f,\alpha} : \alpha \in A\}$, $\{\varphi_{f,\alpha} : \alpha \in A\}$ and $\{C_{g,\alpha} : \alpha \in A\}$, $\{U_{g,\alpha} : \alpha \in A\}$, $\{\varphi_{g,\alpha} : \alpha \in A\}$ be the families described above. Denote by (t, s) the coordinates of points belonging to $U_{f,\alpha}$ in the chart $\varphi_{f,\alpha}$ and by (u, v) the coordinates of points belonging to $U_{g,\alpha}$ in the chart $\varphi_{g,\alpha}$. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism which realizes the topological conjugacy. Let us fix an arbitrary $\alpha\widehat{\ }i \in A$. We will consider the case where $b^\alpha_{g,\alpha\widehat{\ }i} = c^\alpha_{g,\alpha\widehat{\ }i}$ (the second case is similar).

As in the proof of Theorem 2.40 we can find an $\varepsilon^\alpha_{f,\alpha\widehat{\ }i} > 0$ such that the interval $(c^\alpha_{f,\alpha\widehat{\ }i}, c^\alpha_{f,\alpha\widehat{\ }i} + \varepsilon^\alpha_{f,\alpha\widehat{\ }i})$ does not contain any $b^\alpha_{f,\alpha\widehat{\ }j}$ for $\alpha\widehat{\ }j \in A$. Denote by $\phi^\alpha_{\alpha\widehat{\ }i}$ the restriction of the function $\varphi_{g,\alpha} \circ \Phi \circ \varphi^{-1}_{f,\alpha}$ to the set $\mathbb{R} \times (c^\alpha_{f,\alpha\widehat{\ }i}, c^\alpha_{f,\alpha\widehat{\ }i} + \varepsilon^\alpha_{f,\alpha\widehat{\ }i})$. Then $\phi^\alpha_{\alpha\widehat{\ }i} : \mathbb{R} \times (c^\alpha_{f,\alpha\widehat{\ }i}, c^\alpha_{f,\alpha\widehat{\ }i} + \varepsilon^\alpha_{f,\alpha\widehat{\ }i}) \to \mathbb{R} \times (c^\alpha_{g,\alpha\widehat{\ }i}, c^\alpha_{g,\alpha\widehat{\ }i} + \varepsilon^\alpha_{g,\alpha\widehat{\ }i})$ and $(u, v) = \phi^\alpha_{\alpha\widehat{\ }i}(t, s)$. Define the function $\beta_{\alpha\widehat{\ }i} : (c^\alpha_{f,\alpha\widehat{\ }i}, c^\alpha_{f,\alpha\widehat{\ }i} + \varepsilon^\alpha_{f,\alpha\widehat{\ }i}) \to (c^\alpha_{g,\alpha\widehat{\ }i}, c^\alpha_{g,\alpha\widehat{\ }i} + \varepsilon^\alpha_{g,\alpha\widehat{\ }i})$ by the formula

$$\beta_{\alpha\widehat{\ }i}(s) = v.$$

Then $\beta_{\alpha\widehat{\ }i}$ is an increasing homeomorphism, since Φ is a homeomorphism which maps trajectories of $\{f^t : t \in \mathbb{R}\}$ onto trajectories of $\{g^t : t \in \mathbb{R}\}$ and $\Phi(C^\alpha_{f,\alpha\widehat{\ }i}) = C^\alpha_{g,\alpha\widehat{\ }i}$.

Let $K_{f,\alpha} := \varphi_{f,\alpha}^{-1}(\{0\} \times \mathbb{R})$ and $K_{f,\alpha\frown i} := \varphi_{f,\alpha\frown i}^{-1}(\{0\} \times \mathbb{R})$. Then $\mu_{f,\alpha\frown i}(s)$ describes the time needed for the flow $\{f^t : t \in \mathbb{R}\}$ to move along the trajectory $\varphi_{f,\alpha}^{-1}(\mathbb{R} \times \{s\})$ from the section $K_{f,\alpha\frown i}$ to the section $K_{f,\alpha}$ for each $s \in (c^{\alpha}_{f,\alpha\frown i}, c^{\alpha}_{f,\alpha\frown i} + \varepsilon^{\alpha}_{f,\alpha\frown i})$. Put $L_{g,\alpha} := \Phi(K_{f,\alpha})$ and $L_{g,\alpha\frown i} := \Phi(K_{f,\alpha\frown i})$. Then $L_{g,\alpha}$ is a section in $U_{g,\alpha}$ and $L_{g,\alpha\frown i}$ is a section in $U_{g,\alpha\frown i}$, since Φ is a homeomorphism which maps trajectories of $\{f^t : t \in \mathbb{R}\}$ onto trajectories of $\{g^t : t \in \mathbb{R}\}$. By the assumption that $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ realizes the topological conjugacy we get that $\mu_{f,\alpha\frown i}(s)$ is equal to the time needed for the flow $\{g^t : t \in \mathbb{R}\}$ to move along the trajectory $\varphi_{g,\alpha}^{-1}(\mathbb{R} \times \{\beta_{\alpha\frown i}(s)\})$ from the section $L_{g,\alpha\frown i}$ to the section $L_{g,\alpha}$ for each $s \in (c^{\alpha}_{f,\alpha\frown i}, c^{\alpha}_{f,\alpha\frown i} + \varepsilon^{\alpha}_{f,\alpha\frown i})$, since $v = \beta_{\alpha\frown i}(s)$.

Let $K_{g,\alpha} := \varphi_{g,\alpha}^{-1}(\{0\}\times\mathbb{R})$ and $K_{g,\alpha\frown i} := \varphi_{g,\alpha\frown i}^{-1}(\{0\}\times\mathbb{R})$. For every $s \in (c^{\alpha}_{f,\alpha\frown i}, c^{\alpha}_{f,\alpha\frown i} + \varepsilon^{\alpha}_{f,\alpha\frown i})$ denote by $\tau_{g,\alpha}(s)$ the time needed for the flow $\{g^t : t \in \mathbb{R}\}$ to move along the trajectory $\varphi_{g,\alpha}^{-1}(\mathbb{R} \times \{\beta_{\alpha\frown i}(s)\})$ from $K_{g,\alpha}$ to $L_{g,\alpha}$ and by $\tau_{g,\alpha\frown i}(s)$ the time needed to move along this trajectory from $K_{g,\alpha\frown i}$ to $L_{g,\alpha\frown i}$. Then for each $s \in (c^{\alpha}_{f,\alpha\frown i}, c^{\alpha}_{f,\alpha\frown i}+\varepsilon^{\alpha}_{f,\alpha\frown i})$ we have

$$\mu_{g,\alpha\frown i}(\beta_{\alpha\frown i}(s)) = \mu_{f,\alpha\frown i}(s) + \tau_{g,\alpha\frown i}(s) - \tau_{g,\alpha}(s).$$

Define $\gamma_{\alpha\frown i} : (c^{\alpha}_{f,\alpha\frown i}, c^{\alpha}_{f,\alpha\frown i} + \varepsilon^{\alpha}_{f,\alpha\frown i}) \to \mathbb{R}$ by putting

$$\gamma_{\alpha\frown i}(s) := \tau_{g,\alpha}(s) - \tau_{g,\alpha\frown i}(s), \quad s \in (c^{\alpha}_{f,\alpha\frown i}, c^{\alpha}_{f,\alpha\frown i} + \varepsilon^{\alpha}_{f,\alpha\frown i}).$$

Then

$$\mu_{f,\alpha\frown i}(s) = (\mu_{g,\alpha\frown i} \circ \beta_{\alpha\frown i})(s) + \gamma_{\alpha\frown i}(s)$$

for each $s \in (c^{\alpha}_{f,\alpha\frown i}, c^{\alpha}_{f,\alpha\frown i} + \varepsilon^{\alpha}_{f,\alpha\frown i})$.

Chapter 3

Overview of remaining scientific achievements

The purpose of this chapter is to present the outline of my scientific interests and research based on a brief description of the results contained in the papers written after doctoral degree that are not part of the indicated scientific achievement. The results have been organized due to the issues considered. In the first section we discuss supplementary results concerning Brouwer homeomorphisms. However, in the second section of this chapter we present results from other areas of my research. This chapter does not contain a discussion on the contents of the survey papers [B18], [B20] and [B23].

3.1 Brouwer homeomorphisms - supplementary results

The results presented in this section describe selected properties of Brouwer homeomorphisms which are embeddable in a flow. In particular, we discuss results obtained for Sperner homeomorphisms and Reeb homeomorphisms which provide the simplest examples of Brouwer homeomorphisms.

3.1.1 Iterative roots of a Sperner homeomorphism

In paper [B7] we find all continuous solutions g of the equation

$$g^n(x) = f(x) \qquad \text{for} \qquad x \in \mathbb{R}^2, \tag{3.1}$$

where $n \in \mathbb{Z}$, $n > 1$, a f is a homeomorphism of the plane onto itself satisfying the condition

(S) every Jordan domain B meets at most a finite number of its images $f^n(B)$, $n \in \mathbb{Z}$, where by a Jordan domain is meant the union of a Jordan curve C and the inside of C (i.e. the bounded component of $\mathbb{R}^2 \setminus C$).

A homeomorphism f of the plane onto itself satisfying condition (S) is called a *Sperner homeomorphism.* Condition (S) can be considered on an arbitrary invariant simply connected set H (in this case we take $B \subset H$ in (S)). Then we say that f is a Sperner homeomorphism on H.

From condition (S) we obtain directly that f has no fixed points. A homeomorphism satisfying condition (S) can preserve or reverse orientation. Thus a Sperner homeomorphism which preserves orientation is a Brouwer homeomorphism.

Emanuel Sperner [105] has showed a theorem which says that if a Brouwer homeomorphism f satisfies condition (S), then f is topologically conjugate with the translation with vector $(1, 0)$ (cf. [105], Satz 27). Since condition (S) holds in the case where f is a translation, the converse of the Sperner theorem is also true.

D. Betten [13] has proved that an orientation reversing homeomorphism of the plane onto itself satisfies condition (S) if and only if it is topologically conjugate with the glide reflection S_1 given by

$$S_1(x_1, x_2) = (x_1 + 1, -x_2) \qquad \text{for} \quad (x_1, x_2) \in \mathbb{R}^2. \tag{3.2}$$

The glide reflection S_1 plays here a similar role as the translation with vector $(1, 0)$ for orientation preserving Sperner homeomorphisms, since if a homeomorphism of the plane onto itself f is topologically conjugate with a glide reflection, then it is topologically conjugate with S_1 (cf. [B7], Lemma 1).

We start this presentation of results regarding equation (3.1) from a result which says that the class of Sperner homeomorphisms is closed with respect to taking iterative roots.

Theorem 3.1. ([B7], Proposition 3) *Let f be a homeomorphism of the plane onto itself, g be a continuous function such that $g^n = f$ for some $n \in \mathbb{Z}$, $n > 1$. If f satisfies condition (S), then g is a homeomorphism of the plane onto itself satisfying condition (S).*

The form of all continuous iterative roots of a Sperner homeomorphism is described in the following two theorems.

Theorem 3.2. ([B7], Theorem 3) *Let f an orientation preserving Sperner homeomorphism. Then*

(a) for every even $n \in \mathbb{Z}$, $n > 0$ function g is a continuous solution of equation (3.1) if and only if it can be expressed in either of the forms

$$g = \varphi^{-1} \circ T_{\frac{1}{n}} \circ \varphi \tag{3.3}$$

or

$$g = \varphi^{-1} \circ S_{\frac{1}{n}} \circ \varphi, \tag{3.4}$$

where φ is a homeomorphic solution of the Abel equation

$$\varphi(f(x)) = \varphi(x) + (1,0) \qquad \textit{for} \quad x \in \mathbb{R}^2, \tag{3.5}$$

and $T_{\frac{1}{n}}$, $S_{\frac{1}{n}}$ are given by

$$T_{\frac{1}{n}}(x_1, x_2) := (x_1 + \frac{1}{n}, x_2) \qquad \textit{for} \quad (x_1, x_2) \in \mathbb{R}^2, \tag{3.6}$$

$$S_{\frac{1}{n}}(x_1, x_2) := (x_1 + \frac{1}{n}, -x_2) \qquad \textit{for} \quad (x_1, x_2) \in \mathbb{R}^2. \tag{3.7}$$

(b) for every odd $n \in \mathbb{Z}$, $n > 1$, function g is a continuous solution of equation (3.1)if and only if it has the form (3.3), where φ is a homeomorphic solution of (3.5), and $T_{\frac{1}{n}}$ is given by (3.6).

Theorem 3.3. ([B7], Theorem 4) *Let f be an orientation reversing Sperner homeomorphism and let n be an odd integer greater than 1. Then function g is a continuous solution of equation (3.1) if and only if it has the form (3.4), where φ is a homeomorphic solution of equation*

$$\varphi(f(x)) = S_0(\varphi(x)) + (1,0) \qquad \textit{dla} \quad x \in \mathbb{R}^2, \tag{3.8}$$

and $S_{\frac{1}{n}}$ is given by (3.7).

Under the assumptions that f reverses orientation and n is even, there are no solutions of equation (3.1) (cf. [B7], Remark 3).

A direct construction of continuous solutions of equation (3.1) is described in paper [B5] (cf. [B5], Theorem 1 and Theorem 2). This construction is based on the condition

(B) there exists a line K (i.e. a homeomorphic image of a straight line which is a closed set) such that

$$K \cap f(K) = \emptyset, \tag{3.9}$$

$$U^0 \cap f(U^0) = \emptyset, \tag{3.10}$$

$$\bigcup_{n \in \mathbb{Z}} f^n(U^0) = \mathbb{R}^2, \tag{3.11}$$

where $U^0 = M^0 \cup f(K)$ and M^0 is the strip between K and $f(K)$.

This condition is equivalent to condition (S) for each homeomorphism of the plane onto itself (cf. [B1], Theorem 1 and [B7], Theorem 2).

In paper [B5] it is also discussed the issue of a selection of the line K occurring in condition (B) during construction of solutions of equation (3.1). The problem arise with the fact that for Sperner homeomorphisms f, g satisfying $f = g^n$ for a positive integer $n > 1$ and a line K occurring in (B), the condition $K \cap g(K) = \emptyset$ may not be fulfilled.

3.1.2 Codivergency relation for a Brouwer homeomorphism embeddable in a flow

In this subsection we present results selected from papers [B8], [B9], [B12]. They concern the codivergency relation defined for a Brouwer homeomorphism. The definition and selected properties of this relation are described in the main chapter of this report. The results presented here are obtained under the additional assumption that a Brouwer homeomorphism is embeddable in a flow.

In paper [B8] the fundamental properties of the codivergency relation are presented. The most important one concerns trajectories contained in the same equivalence class of the considered relation.

Theorem 3.4. ([B8], Proposition 2.2) *Let f be Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Let C_1, C_2 be distinct trajectories of this flow. If C_1, C_2 are contained in the same equivalence class of the codivergency relation, then each point of the strip between C_1 and C_2 belongds to this equivalence class.*

In the next subsection we present a result from paper [B10] which concerns the strip between trajectories which are contained in different equivalence classes of the codivergency relation (cf. Theorem 3.13).

Paper [B8] contains also a result saying that the restriction of each element of a flow of Brouwer homeomorphisms different from the identity to any equivalence class G_0 with nonempty interior is a Sperner homeomorphism on G_0.

Theorem 3.5. ([B8], Theorem 3.1) *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Let G_0 be an equivalence class of the codivergency relation which does not consists of just one trajectory. Then the restriction $f|_{G_0}$ of the homeomorphism f to G_0 is a Sperner homeomorphism on G_0.*

In paper [B9] one can find further properties of the codivergency relation. We start from an extension of a result from paper [B8] (cf. [B8], Lemma 2.1). The definition of the 3-argument relation defined on the set of all trajectories of the considered flow occurring in the statement of this result can be found in the main part of this report.

Theorem 3.6. ([B9], Proposition 2.1) *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. For any distinct trajectories C_1, C_2, C_3 of the flow $\{f^t : t \in \mathbb{R}\}$, if $|\ C_1, C_2, C_3|$, then each of this trajectories is contained in a different equivalence class of the codivergency relation.*

The next result presented here concerns trajectories contained in the boundary of equivalence classes of the codivergency relation.

Theorem 3.7. ([B9], Proposition 2.3) *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Let G_0 be an equivalence class of the codivergency relation and let $p \in G_0 \cap \operatorname{bd} G_0$. Then the trajectory C_p of the point p is contained in $G_0 \cap \operatorname{fr} G_0$.*

Using Theorems 3.6 and 3.7 we prove a result describing the boundary of an equivalence class of the codivergency relation.

Theorem 3.8. ([B9], Proposition 2.4, Corollary 2.7) *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then the boundary of each equivalence class of the codivergency relation is a union of a family of trajectories of the flow $\{f^t : t \in \mathbb{R}\}$. Moreover, each equivalence class can contain at most two trajectories that are contained in its boundary.*

Paper [B9] also contains a result which is an extension of Theorem 3.5.

Theorem 3.9. ([B9], Theorem 4.2) *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then for each equivalence class G_0 of the codivergency relation there exists a simply connected region H invariant under $\{f^t : t \in \mathbb{R}\}$ such that $G_0 \subset H$ and $f\,|_H$ is a Sperner homeomorphism on H.*

In paper [B12] one can find further results describing properties of trajectories contained in the boundary of equivalence classes of the codivergency relation for Brouwer homeomorphisms which are embeddable in a flow.

It has been proved there that if a point p belongs to the boundary of an equivalence class G_0, then for each equivalence class G such that $G \setminus C_p \neq \emptyset$, where C_p denotes the trajectory of p, the set $\operatorname{cl} G \setminus C_p$ is contained in one of the components of the set $\mathbb{R}^2 \setminus C_p$ (cf. [B12], Corollary 4).

The main result of this paper reads as follows.

Theorem 3.10. ([B12], Theorem 6) *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Let G_0 be an equivalence class of the codivergency relation which does not consists of just one trajectory and let $p \in \operatorname{bd} G_0$. Then $p \notin \operatorname{bd} G$ for each equivalence class $G \neq G_0$ contained in the component of the set $\mathbb{R}^2 \setminus C_p$ which contains $\operatorname{cl} G_0 \setminus C_p$.*

From the above theorem we obtain two corollaries describing properties of trajectories contained in the boundary of equivalence classes of the codivergency relation. The first of them says that for each point p, if the equivalence class containing p does not consists of just one trajectory, then p can belong to the boundary of at most two equivalence classes, since each of the components of the complement of the trajectory of p can contain at most one equivalence class containing p in its boundary (cf. [B12], Corollary 7). The second corollary concerns a point whose trajectory is an equivalence class. Then such a point can belong to the boundary of at most three equivalence classes - at most two equivalence classes contained in the complement of the trajectory of p (at most one in each of the components of the complement of this trajectory) and the class which is equal to the trajectory of p (cf. [B12], Corollary 7).

3.1.3 Maximal parallelizable regions of a flow of Brouwer homeomorphisms

In this subsection we present results from papers [B6], [B10], [B11]. They concern properties of maximal parallelizable regions of a flow of Brouwer homeomorphisms. The definition of a maximal parallelizable region and the most important results concerning parallelizable regions are presented in the main chapter of this report.

In paper [B6] it has been studied some flows of Brouwer homeomorphisms which are not topologically conjugate with the flow of translations. The aim of this study was to determine the relationship between homeomorphisms realizing topological conjugacy with the flow of translations on overlapping parallelizable regions. The obtained form of the transition maps between parallelizing homeomorphisms has been later translated into the general case described in the main chapter of this report.

Now we define a class of plane flows called Reeb flows which have the simplest structure of trajectories in the class of all flows of Brouwer homeomorphisms which are not topologically conjugate with the flow of translations. In the definition we use the notion of topological equivalence of flows.

Let

$$\begin{aligned} P_0 &:= \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}, \\ P_1 &:= \{(x,y) \in \mathbb{R}^2 : x < 0, y > 0\}, \\ P_2 &:= \{(x,y) \in \mathbb{R}^2 : x > 0, y < 0\}, \\ L_x &:= \{(x,0) \in \mathbb{R}^2 : x > 0\}, \\ L_y &:= \{(0,y) \in \mathbb{R}^2 : y > 0\} \end{aligned}$$

and $H := P_0 \cup P_1 \cup P_2 \cup L_x \cup L_y$.

We say that a flow $\{f^t : t \in \mathbb{R}\}$ of Brouwer homeomorphisms is a *Reeb flow* if it is topologically equivalent to the flow $\{h^t : t \in \mathbb{R}\}$, where for each $t \in \mathbb{R}$ the homeomorphism $h^t : H \to H$ is defined by

$$h^t(x,y) := \begin{cases} (2^t x, 2^{-t} y) & \text{if} \quad (x,y) \in P_0 \cup L_x \cup L_y, \\ (x, 2^{-t} y) & \text{if} \quad (x,y) \in P_1, \\ (2^t x, y) & \text{if} \quad (x,y) \in P_2. \end{cases}$$

From this definition we obtain that there exists one-to-one correspondence between trajectories of the flows $\{f^t : t \in \mathbb{R}\}$ and $\{h^t : t \in \mathbb{R}\}$, since a homeomorphism realizing the topological equivalence of flows maps trajectories one of the flows onto trajectories the other one. Each Brouwer homeomorphism belonging to a Reeb flow has three equivalence classes of the codivergency relation corresponding to the sets P_0, $P_1 \cup L_y$, $P_2 \cup L_x$, respectively. The first of them is an open set, the other two are closed sets. In general, a Brouwer homeomorphism with exactly three equivalence classes of the codivergency relation is called a *Reeb homeomorphism*. In this section we are interested in Reeb homeomorphisms which are embeddable in a flow.

Trajectories of the flow $\{f^t : t \in \mathbb{R}\}$ corresponding to the sets L_x and L_y are boundary trajectories of equivalence classes of the codivergency relation and are contained in the first prolongational limit set of the flow. A Reeb flow has two maximal parallelizable regions corresponding to the sets $P_1 \cup L_y \cup P_0$, $P_2 \cup L_x \cup P_0$. Each of them is a union of two equivalence classes of the codivergency relation. Moreover, each of the trajectories corresponding to the sets L_x and L_y is equal to the boundary of one of these maximal parallelizable regions and is contained in the other one.

Before presenting a result concerning Reeb flows we give a theorem which describes the form of flows containing a Sperner homeomorphism.

Theorem 3.11. ([B3], Theorem 1) *Let $D \subset \mathbb{R}^2$ be a simply connected region. Then a flow $\{f^t : t \in \mathbb{R}\}$ defined on D is topologically conjugate with the flow of translation if and only if f^1 is a Sperner homeomorphism on D.*

In other words, each flow $\{f^t : t \in \mathbb{R}\}$ such that f^1 is a Sperner homeomorphism has the form

$$f^t(x) = \varphi^{-1}(\varphi(x) + (t, 0)) \qquad \text{for} \quad x \in D,\ t \in \mathbb{R}, \tag{3.12}$$

where $\varphi : D \to \mathbb{R}^2$ is a homeomorphic solution of the Abel equation. Homeomorphic solution of the Abel equation depends on an arbitrary function defined on a suitable strip. Their construction is given in paper [B1]. Moreover, from (3.12) we obtain that each element of this flow different from the identity is a Sperner homeomorphism, since the existence of homeomorphic solution of equation (3.5) is equivalent to condition (S). Therefore, for every $t \in \mathbb{R} \setminus \{0\}$ the element $f^t|_H$ of the flow $\{f^t|_H : t \in \mathbb{R}\}$ being a restriction of the flow occurring in Theorem 3.9 is a Sperner homeomorphism.

In the statements of results contained in paper [B6] we use Kaplan diagrams (cf. Beck [19], Chapter 11). Each flow $\{f^t : t \in \mathbb{R}\}$ of Brouwer homeomorphisms has its Kaplan diagram made up by polygonoids (i.e. generalized polygons with a finite or infinite number of sides) inscribed in a disc and chords of the disc which are parallel to the sides of the polygonoids. These chords (including the sides of the polygonoids) correspond to trajectories of the flow. The only exception is that each of these polygonoids has exactly one open side which does not correspond to any trajectory of the flow. The Kaplan diagram for a Reeb flow contains only one polygonoid which is a triangle. Each of the segments which are cut off by the sides of the triangle represents an equivalence class of the codivergency relation. Moreover, one of these classes is open, the other two are closed sets.

Now we proceed to a result which describes the form of a Reeb flow (in paper [B6] it has been also studied two types of flows with 5 classes of the codivergency relation).

Theorem 3.12. ([B6], Theorem 1) *Let $\{f^t : t \in \mathbb{R}\}$ be a Reeb flow. Denote by G_0 the equivalence class of the codivergency relation which is an open set, and by G_1, G_2 the other two classes. Then there exist homeomorphisms φ_1, φ_2 mapping $G_0 \cup G_1$,*

$G_0 \cup G_2$ onto $\mathbb{R}^2$, respectively, such that

$$f^t(x) = \begin{cases} \varphi_1^{-1}(\varphi_1(x) + (t,0)), & x \in G_0 \cup G_1, \\ \varphi_2^{-1}(\varphi_2(x) + (t,0)), & x \in G_0 \cup G_2, \end{cases} \tag{3.13}$$

and the function $\psi = \varphi_2 \circ (\varphi_1|_{G_0})^{-1} : \varphi_1(G_0) \longrightarrow \varphi_2(G_0)$ can be represented in the form

$$\psi(x,y) = (x + \alpha(y), \beta(y)), \qquad (x,y) \in \varphi_1(G_0), \tag{3.14}$$

where $\alpha : (0,+\infty) \to \mathbb{R}$ is a continuous function, and β is a homeomorphism of the interval $(0,+\infty)$ onto itself.

Relation (3.13) can be obtained by using Theorem 3.11 for the simply connected regions $G_0 \cup G_1$ and $G_0 \cup G_2$. Next, solving an appropriate functional equation we get formula (3.14).

In paper [B10] we study parallelizable regions of a flow of Brouwer homeomorphisms by using properties of equivalence classes of the codivergency relation which are contained in these regions.

We start from the result announced in the previous subsection which asserts that the strip between two trajectories contained in different equivalence classes of the codivergency relation must contain a point which does not belong to any of these classes.

Theorem 3.13. ([B10], Theorem 2.2) *Let $p, q \in \mathbb{R}^2$ belong to different equivalence classes G_1, G_2 of the codivergency relation. Then there exists a point r belonging to the strip D_{pq} between trajectories C_p, C_q of points p, q, respectively, such that $r \notin G_1 \cup G_2$.*

Using this theorem we prove the main result of paper [B10] connected to the existence of common boundary trajectories of equivalence classes contained in a parallelizable region.

Theorem 3.14. ([B10], Theorem 3.3) *Let M_0 be a parallelizable region of a flow of Brouwer homeomorphisms. Assume that G_1, G_2 are equivalence classes of the codivergency relation such that $G_1 \cup G_2 \subset M_0$ and $\operatorname{bd} G_1 \cap \operatorname{bd} G_2 \neq \emptyset$. Let $p \in G_1$, $q \in G_2$. Then there exists a point $z \in D_{pq}$ such that $z \in \operatorname{bd} M_0$, where D_{pq} denotes the strip between trajectories C_p, C_q of points p, q. Moreover, $z \notin G_1 \cup G_2$.*

Paper [B11] contains results concerning maximal parallelizable regions of a flow of Brouwer homeomorphisms. Describing the boundary of such regions one can use the notion of the first prolongatonal limit set. Namely, if M is a maximal parallelizable region, then $J(M) = \operatorname{bd} M$ (cf. McCann [86], Proposition 2.6).

We start this overview of results contained in paper [B11] by presenting a result mentioned in the main chapter of this report.

Theorem 3.15. ([B11], Corollary 2) *Let M be a parallelizable region and $p \in \operatorname{bd} M$. Then $\operatorname{cl} M \setminus C_p$ is contained in one of the components of $\mathbb{R}^2 \setminus C_p$.*

Next result presented here shows that the relationship between maximal parallelizable regions and equivalence classes of the codivergency relation can be generalized from Reeb flows to each flow of Brouwer homeomorphisms.

Theorem 3.16. ([B11], Theorem 4) *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Then each maximal parallelizable region M of the flow $\{f^t : t \in \mathbb{R}\}$ is a union of equivalence classes of the codivergency relation.*

Another of results from this paper implies that a point contained in the interior of an equivalence class cannot belong to the boundary of an maximal parallelizable region.

Theorem 3.17. ([B11], Proposition 5) *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Then if q belongs to the interior of an equivalence class of the codivergency relation, then $q \notin J(\mathbb{R}^2)$.*

It is worth pointing out that the converse of the above theorem is not valid. An example of a Brouwer homeomorphism embeddable in a flow for which there are boundary points of an equivalence class which do not belong to $J(\mathbb{R}^2)$ can be found in a paper of R. McCann (cf. [86], Example 3.10).

According to Theorem 3.17 we have that each point of $M \cap J(\operatorname{bd} M)$ belongs to the boundary of an equivalence class contained in M. The set $J(\operatorname{bd} M)$ can also contain points which does not belong to M, i.e. elements of the boundary of a maximal parallelizable region M can be elements of the first prolongatonal limit set of points which does not belong to M. From the following result we obtain that it can happen only for points of a trajectory contained in the boundary of M which is an equivalence class itself.

Theorem 3.18. ([B11], Proposition 8) *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Let M be a maximal parallelizable region, $p \in \operatorname{bd} M$ and G_0 be an equivalence class containing p which is not equal to a trajectory. Then $p \notin J(q)$ for every q belonging to the component of $\mathbb{R}^2 \setminus C_p$ which does not contain M.*

3.1.4 First prolongational limit set of a flow of Brouwer homeomorphisms

In this subsection we present results from papers [B13], [B16] and [B19]. They mainly concern the relationship between the first prolongational limit set of a flow $\{f^t : t \in \mathbb{R}\}$ of Brouwer homeomorphisms and the equivalence classes of the codivergency relation defined for a Brouwer homeomorphism f which is an element of this flow. We also show that the first prolongational limit set of a flow of Brouwer homeomorphisms is equal to the set of all strongly irregular points of any Brouwer homeomorphism belonging to this flow.

Let us remind that by Theorem 3.17, interior points of equivalence classes cannot belong to the first prolongational limit set. In this subsection we present a result which says that the union of the interiors of all equivalence classes of the codivergency relation is equal to the set of all regular points of a Brouwer homeomorphism f which is an element of the considered flow. Thus, in a natural way, the question arises whether the first prolongational limit set of the considered flow is equal to the set of all irregular points of f.

In paper [B13] we describe properties of points belonging to the first prolongational limit set using the codivergency relation. We start from the result in which the assumption $q \in J(p)$ implies that p and q belong to the boundary of the same equivalence class of the codivergency relation.

Theorem 3.19. ([B13], Theorem 2.1) *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Let G_0 be an equivalence class of the codivergency relation which does not consists of just one trajectory. Let $p \in \operatorname{bd} G_0$ and H_0 be the component of $\mathbb{R}^2 \setminus C_p$ which contains $\operatorname{cl} G_0 \setminus C_p$. Then for every $q \in H_0$, if $q \in J(p)$, then $q \in \operatorname{bd} G_0$.*

Under an additional assumption about boundary points of the considered equivalence class, the converse of Theorem 3.19 is also valid.

Theorem 3.20. ([B13], Theorem 3.2) *Let $\{f^t : t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Let G_0 be an equivalence class of the codivergency relation which does not consists of just one trajectory. Let $p \in \operatorname{bd} G_0$, $q \in \operatorname{bd} G_0$ and $C_p \neq C_q$. Assume that p and q belong to the same component of $\mathbb{R}^2 \setminus C_r$ for some $r \in G_0$. Then $p \in J(q)$.*

The foregoing theorem is not true if we omit the assumption that $p, q \in \operatorname{bd} G_0$ belong to the same component of the complement of a trajectory contained in G_0.

Paper [B16] shows the relationship between equivalence classes of the codivergency relation defined for a Brouwer homeomorphism f embeddable in a flow and the sets of all regular and irregular points of f.

We start from a theorem which describes the set of all regular points of a Brouwer homeomorphism embeddable in a flow.

Theorem 3.21. ([B16], Proposition 2.1) *Let f be a Brouwer homeomorphism embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then the set of all regular points of f is equal to the union of the interiors of all equivalence classes of the codivergency relation.*

Using Theorem 3.21 we show a result concerning the invariance of the set of all regular points.

Corollary 3.22. ([B16], Proposition 3.1) *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Let p be a regular point. Then each point of the trajectory $C_p = \{f^t(p) : t \in \mathbb{R}\}$ is a regular point.*

Now we proceed to results concerning the set of all irregular points of a Brouwer homeomorphism which is embeddable in a flow. From Theorem 3.17 contained in the previous subsection we obtain that each element of the the first prolongational limit set has to belong to the boundary of an equivalence class of the codivergency relation. Thus by Theorem 3.21 we have that the first prolongational limit set is contained in the set of all irregular points.

The following result describes the relationship between the set of all strongly irregular points of a Brouwer homeomorphism f and the first prolongational limit set of a flow containing f.

Theorem 3.23. ([B16], Proposition 3.1) *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Let p be a strongly irregular point. Then $J(p) \neq \emptyset$.*

The above theorem says that the set of all strongly irregular points is a subset of the first prolongational limit set, since

$$\mathrm{J}(p) \neq \emptyset \Leftrightarrow p \in \mathrm{J}(\mathbb{R}^2).$$

It turns out that these two sets are equal (cf. Theorem 3.24). Thus the set of all boundary points of equivalence classes of the codivergency relation which do not belong to the first prolongational limit set is equal to the set of all weakly irregular points.

Now we present the theorem mentioned above which implies that for each Brouwer homeomorphism which is embeddable in a flow the set of all strongly irregular points is equal to the first prolongational limit set of the flow.

Theorem 3.24. ([B19], Corollary 3) *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then $P^+(p) = \mathrm{J}^+(p)$ and $P^-(p) = \mathrm{J}^-(p)$ for every $p \in \mathbb{R}^2$.*

The main part of the proof of this result is the reasoning which leads to the inclusion $\mathrm{J}^+(p) \subset P^+(p)$ (cf. [B19], Theorem 2). In the proof of this inclusion we fix an arbitrary point $q \in \mathrm{J}^+(p)$. To prove that $q \in P^+(p)$ we show that for any Jordan domain B containg p in its interior we have $q \in \omega_f(B)$. The crucial role play here arcs K and L such that $K \subset B$, $p \in K$, $q \in L$ having at most one common point with every trajectory of the flow, i.e. arcs which are continuous sections of the flow. Using the assumption that $q \in \mathrm{J}^+(p)$ we obtain that there exists a sequence of positive integers k_n tending to infinity and a sequence of points $w_n \in f^{k_n}(K) \cap L$ tending to q. Putting $z_n := f^{-k_n}(w_n)$ we get a sequence of poits contained in the Jordan domain B such that $f^{k_n}(z_n) \to q$, which means that $q \in \omega_f(B)$.

From Theorem 3.24 we obtain corollaries which concern the set of all strongly irregular points of Brouwer homeomorphism which is embeddable in a flow and the first prolongational limit set of a flow of Brouwer homeomorphisms.

Corollary 3.25. ([B19], Corollary 4) *Let f be a Brouwer homeomorphism which is embeddable in a flow. Then, for each flow containing f, the first prolongational limit set is the same.*

Corollary 3.26. ([B19], Corollary 5) *Let f be a Brouwer homeomorphism which is embeddable in a flow $\{f^t : t \in \mathbb{R}\}$. Then the set of all strongly irregular points of f^t is the same for all $t \in \mathbb{R} \setminus \{0\}$.*

The latter corollary is also true if we replace the set of all strongly irregular points by the set of all irregular points. It follows from the fact that the set of all irregular points is equal to the closure of set of all strongly irregular points (cf. Theorem 2.5).

At the end of this section we present the role of Theorems 3.22 and 3.24 in showing that for each flow of Brouwer homeomorphisms the sets of all regular, strongly irregular and weakly irregular points are invariant under each element of this flow, i.e. if a point belongs to one of these sets, then the trajectory of this point is contained in the same of these three sets.

The invariance of the set of all regular points follows directly from Theorem 3.22. By Theorem 3.24 we obtain that the set of all strongly irregular points is invariant, since the first prolongational limit set is invariant. Thus the set of all weakly irregular points is invariant. It follows from the fact that the other two of the considered three sets are invariant and the union of these disjoint sets is equal to the whole plane.

3.2 Other results

In this part of this chapter we present results that are not directly related to the main topic of this report. They concern mainly solutions and stability of functional equations, including problem of finding iterative roots and conditions that guarantee the topological conjugacy of continuous piecewise monotone maps of an interval. Moreover, one can find here the results on integral, difference and differential equations and their stability in Ulam sense.

3.2.1 Solutions of the d'Alembert differential equation

In paper [B14] we study the d'Alembert partial differential equation

$$uu_{xy} - u_x u_y = 0. \tag{3.15}$$

We treat equation (3.15) as a submanifold Σ in the space of jets $J^2(\mathbb{R}^2, \mathbb{R})$. In this space, we consider the canonical exterior differential system $(\mathcal{I}, \omega)$, where $\mathcal{I}$ is a differential ideal generated by the 1-forms

$$\begin{aligned} \theta_1 = &\ du - u_x dx - u_y dy, \\ \theta_2 = &\ du_x - u_{xx} dx - u_{xy} dy, \\ \theta_3 = &\ du_y - u_{xy} dx - u_{yy} dy, \end{aligned}$$

and $\omega = dx \wedge dy$ is the 2-form which gives an independence condition. This means that x and y are independent variables, and the lifts of graphs of maps $u : \mathbb{R}^2 \to$

$\mathbb{R}$ to $J^2(\mathbb{R}^2, \mathbb{R})$ are integral manifolds. Canonical exterior differential system with independence condition is called a *contact system.*

Let I be a submodule of the module of 1-forms $\Omega^1(\Sigma)$ generated by θ_1, θ_2, θ_3, i.e.

$$I = \operatorname{span}\{\theta_1, \theta_2, \theta_3\}.$$

Let

$$J = \operatorname{span}\{\theta_1, \theta_2, \theta_3, dx, dy\}.$$

For the contact system we will write (I, J) instead of $(\mathcal{I}, \omega)$. The pair (I, J) is a linear Pfaffian system, i.e. $d\theta_i \equiv 0 \mod J$ for $1 \leq i \leq 3$, since each system of partial differential equations expressed as a contact system on $\Sigma \subset J^k(\mathbb{R}^n, \mathbb{R}^m)$ is a linear Pfaffian system.

The existence of analytic solutions of equation (3.15) we obtain using the Cartan-Kähler theorem for linear Pfaffian system which can be found in a book of T.A. Ivey and J.M. Landsberg (cf. [55], p. 176). To prove the involutivity requirement we change the system of coordinates, and from equation (3.15) we obtain the equation

$$uu_{yy} + uu_{xy} - u_x u_y - u_y^2 = 0. \tag{3.16}$$

After using the Cartan-Kähler theorem we come back to the variables x and y of the original problem. Using these variables we can formulate the main result of paper [B14] in the following way.

Theorem 3.27. ([B14], Corollary 5.2) *For every point $(x_0, y_0) \in \mathbb{R}^2$ there exists a unique analytic solution u of equation (3.15) defined in a neighbourhood of (x_0, y_0) satisfying the conditions*

$$\begin{aligned} u(x_0, y_0) &= u_0, \\ u_x(x + x_0, -x + y_0) &= f(x), \qquad |x| < \varepsilon \\ u_y(x + x_0, -x + y_0) &= g(x), \end{aligned}$$

for some $\varepsilon > 0$, where $u_0 \neq 0$ is an arbitrary constant and f, g are arbitrary analytic functions.

3.2.2 Plane involutions

In paper [B15], we determine all solutions of the Babbage functional equation

$$\varphi^2 = \mathrm{id} \tag{3.17}$$

belonging to the class of 2-dimensional rational maps $\varphi_\lambda : \mathbb{R}^2 \setminus (L_1 \cup L_2) \to \mathbb{R}^2 \setminus (L_1 \cup L_2)$ of the form

$$\varphi_\lambda(x, y) := \left(\frac{a_1 x + a_2 y + a_3}{a_4 x + a_5 y + a_6}, \; \frac{b_1 y + b_2 x + b_3}{b_4 y + b_5 x + b_6} \right), \tag{3.18}$$

where $\lambda = (a_1, ..., a_6, b_1, ..., b_6) \in \mathbb{R}^{12}$, L_1, L_2 denote the corresponding straight lines which consist of points of the plane that make the denominators of $\varphi_\lambda(x, y)$ equal to zero.

The method used involves transforming equation (3.17) into a multivariate polynomial system. Next, we decompose the algebraic set of zeros of this system into its irreducible components by computing the reduced primary decomposition of the ideal generated by this system. In this way, we give necessary and sufficient conditions on the 12 parameters $a_i, b_i, i = 1, 2, ..., 6$ for the function φ_λ to be an involution, i.e. to satisfy equation (3.17).

Inserting (3.18) into (3.17) and comparing the coefficients on the both sides of equation (3.17), we obtain a polynomial system of the form

$$f_1 = f_2 = ... = f_{18} = 0, \tag{3.19}$$

where each of the polynomials f_j, for $j = 1, 2, ..., 18$, is a homogeneous polynomial of degree 3 of variables $a_i, b_i, i = 1, 2, ..., 6$. Thus solving equation $\varphi_\lambda^2 = \mathrm{id}$ is equivalent to seeking the common zeros of system (3.19).

We split the process of determining solutions of the obtained polynomial system into 4 steps by distinguishing the following cases:

$(H1)$ $a_4 = a_5 = a_6 - 1 = b_4 = b_5 = b_6 - 1 = 0$;

$(H2)$ $a_4 = a_5 = a_6 - 1 = 0$ and $b_4^2 + b_5^2 \neq 0$;

$(H3)$ $b_4 = b_5 = b_6 - 1 = 0$ and $a_4^2 + a_5^2 \neq 0$;

$(H4)$ $(a_4^2 + a_5^2)(b_4^2 + b_5^2) \neq 0$.

For solving the first three of them we use the Gröbner bases approach. The last one has been solved by the technique of pseudo-division described in a book of D.E. Knuth ([23, pp. 368–369]). Here we only present the results concerning the first three cases.

For case (H1) we have the following result

Theorem 3.28. ([B15], Theorem 1) *Let $\varphi_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ be of the form*

$$\varphi_\lambda(x, y) := (a_1 x + a_2 y + a_3, \quad b_1 y + b_2 x + b_3)\,.$$

Then φ_λ is an involution if and only if φ_λ has one of the following five forms up to the conjugacy $H(x, y) = (y, x)$:

(1) $(x, y) \mapsto (x, \quad y)$.

(2) $(x, y) \mapsto (-x + a_3, \quad -y + b_3)$.

(3) $(x, y) \mapsto (x, \quad -y + b_2 x + b_3)$.

(4) $(x,y) \mapsto (-x+a_3, \quad y+b_2x-\frac{a_3b_2}{2})$.

(5) $(x,y) \mapsto \left(a_1x+a_2y+a_3, \quad -a_1y+\frac{1-a_1^2}{a_2}x-\frac{a_3(a_1+1)}{a_2}\right)$.

In the proof of this result we consider the ideal $I_1 := \langle f_1, ..., f_{24}\rangle$ with the additional polynomials

$$\begin{aligned} f_{19} &:= a_4, \quad f_{20} := a_5, \quad f_{21} := a_6 - 1, \\ f_{22} &:= b_4, \quad f_{23} := b_5, \quad f_{24} := b_6 - 1 \end{aligned}$$

corresponding to the conditions occurring in (H1). We get the primary decomposition of the ideal I_1 by means of the routine *minAssGTZ* from the library *primdec.lib* of Singular which is based on the algorithm by Gianni, Trager and Zacharias and gives the list of the minimal associated prime ideals of a proper ideal. Using this routine with the lex order

$$b_6 > b_5 > b_4 > a_6 > a_5 > a_4 > b_3 > b_2 > b_1 > a_3 > a_2 > a_1$$

for the ring $\mathbb{Q}[b_6, b_5, b_4, a_6, a_5, a_4, b_3, b_2, b_1, a_3, a_2, a_1]$ we obtain the reduced primary decomposition of I_1 which consists of ideals I_{11}, I_{12}, I_{13}, where

$$\begin{aligned} I_{11} &= \langle a_1-1, a_2, a_3, b_1-1, b_2, b_3, a_4, a_5, a_6-1, b_4, b_5, b_6-1\rangle, \\ I_{12} &= \langle a_1+1, a_2, b_1+1, b_2, a_4, a_5, a_6-1, b_4, b_5, b_6-1\rangle, \\ I_{13} &= \langle b_1+a_1, b_2a_2+a_1^2-1, b_3a_1-b_3-b_2a_3, \\ &\quad\; b_3a_2+a_3a_1+a_3, a_4, a_5, a_6-1, b_4, b_5, b_6-1\rangle. \end{aligned}$$

Since I_{11}, I_{12}, I_{13} are minimal associated primes of I_1, we have $\mathbf{V}(I_1) = \mathbf{V}(I_{11}) \cup \mathbf{V}(I_{12}) \cup \mathbf{V}(I_{13})$, where by $\mathbf{V}(I)$ is denoted the set of zeros of an ideal I. Moreover, the calculated sets of generators of the ideals I_{11}, I_{12}, I_{13} are their reduced Gröbner bases.

Let us observe that I_{11} and I_{12} give the forms (1) and (2) of φ_λ, respectively, of Theorem 3.28. Next we show that the ideal I_{13} corresponds to the forms (3), (4) in case $a_2 = 0$, and to (5) in case $a_2 \neq 0$.

For case (H2) we obtain the reduced primary decomposition which consists of two ideals.

Theorem 3.29. ([B15], Theorem 2) *Let $\varphi_\lambda : \mathbb{R}^2 \setminus L_2 \to \mathbb{R}^2 \setminus L_2$ be of the form*

$$\varphi_\lambda(x,y) := \left(a_1x+a_2y+a_3, \quad \frac{b_1y+b_2x+b_3}{b_4y+b_5x+b_6}\right), \quad b_4^2+b_5^2 \neq 0.$$

Then φ_λ is an involution if and only if φ_λ has one of the following two forms:

(1) $(x,y) \mapsto \left(x, \quad \frac{b_1y+b_2x+b_3}{b_4y-b_1}\right)$.

(2) $(x, y) \mapsto \left(-x + a_3, \;\; \frac{b_1 y + b_3}{b_4 y - b_1}\right)$.

In the proof ot this result we use the routine *minAssGTZ* of Singular for the ideal $I_2 := \langle f_1, ..., f_{21}, g_3 \rangle$, where g_3 is a polynomial given by the formula $g_3 := 1 - c_3(b_4^2 + b_5^2)^2$ with an additional variable c_3. Let us observe that an appropriate result for case (H3) can be directly obtained from Theorem 3.29.

3.2.3 Piecewise monotone interval maps

In papers [B17] and [B22] the problem of the topological conjugacy of continuous piecewise strictly monotone interval maps is considered.

We say that continuous functions $f : I \to I$ and $g : J \to J$, where I and J are non-degenerate closed intervals, are *topologically conjugate* if there exists a homeomorphism $\varphi : I \to J$ such that

$$\varphi \circ f \;\; = \;\; g \circ \varphi. \tag{3.20}$$

Let $I := [a, b]$, where $a < b$. Let r be a non-negative integer. A map $f : I \to I$ is said to be *piecewise strictly monotone* or *r-modal* if it is continuous and there exists a partition $a = t_0 < t_1 < \cdots < t_r < ... < t_{r+1} = b$ such that f is strictly monotone on each of the intervals $[t_i, t_{i+1}]$ for $i = 0, 1, \ldots, r$, and f is not monotone in any neighbourhood of t_i for $i = 1, \ldots, r$. The points $t_1, \ldots, t_r$, are called *turning points* of f.

Let $\mathcal{S}_r(I)$ be the family of all r-modal maps from I into itself, where r is a non-negative integer. Given an $f \in \mathcal{S}_r(I)$ we denote by $N(f)$ the number of turning points of f. One can observe that

$$0 = N(f^0) \le N(f) \le N(f^2) \le \cdots \le N(f^n) \le N(f^{n+1}) \le \cdots ,$$

where f^n is the n-th iterate of f. Let $H(f)$ denote the least non-negative integer n such that $N(f^n) = N(f^{n+1})$, if it exists. If there is no non-negative integer n such that $N(f^n) = N(f^{n+1})$, then we put $H(f) := \infty$. The number $H(f)$ is called the *nonmonotonicity height* of f.

In paper [B17] we consider the family

$$\begin{aligned}\mathcal{S}_r^1(I) := \big\{ f \in \mathcal{S}_r(I) : f(x) < x \text{ for } x \in (t_0, t_1], f(t_1) \ge f(t_{r+1}), f(t_{2i}) = \\ f(t_0) \,\text{for}\, i = 0, \ldots, \left\lfloor \frac{r}{2} \right\rfloor, f(t_{2j-1}) = f(t_1) \,\text{for}\, j = 2, \ldots, \left\lfloor \frac{r+1}{2} \right\rfloor \big\}.\end{aligned}$$

From the definition of $\mathcal{S}_r^1(I)$ we obtain that for each map $f \in \mathcal{S}_r^1(I)$ we have $H(f) = 1$.

The main result of paper [B17] gives a necessary and sufficient condition for functions $f \in \mathcal{S}_r^1(I)$ and $g \in \mathcal{S}_r^1(J)$ to be topologically conjugate.

Theorem 3.30. ([B17], Theorem 1) *Let r be a non-negative integer, $I = [a, b]$ and $J = [c, d]$ for some $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$. Assume that $f \in \mathcal{S}_r^1(I)$ and $g \in \mathcal{S}_r^1(J)$ have turning points $(t_i)_{i=1}^r$ and $(s_i)_{i=1}^r$, respectively, and $t_0 = a$, $t_{r+1} = b$, $s_0 = c$, $s_{r+1} = d$. Then f and g are topologically conjugate if and only if one of the following condition holds*

(i) there exists a positive integer m such that $f(t_{r+1}) = f^m(t_1)$, $g(s_{r+1}) = g^m(s_1)$;

(ii) $f(t_{r+1}) = t_0$, $g(s_{r+1}) = s_0$;

(iii) there exists a positive integer m such that $f(t_{r+1}) \in (f^{m+1}(t_1), f^m(t_1))$, $g(s_{r+1}) \in (g^{m+1}(s_1), g^m(s_1))$.

Furthermore, for cases (i) and (ii), any homeomorphism $\phi_0 : [f(t_1), t_1] \to [g(s_1), s_1]$ such that

$$\phi_0(t_1) = s_1, \quad \phi_0(f(t_1)) = g(s_1)$$

can be uniquely extended onto I to a homeomorphic solution of equation (3.20), as well as for case (iii) with the additional condition

$$\phi_0(f_0^{-m}(f(t_{r+1}))) = g_0^{-m}(g(s_{r+1})),$$

where $f_0 := f|_{[t_0,t_1]}$, $g_0 := g|_{[s_0,s_1]}$.

As a corollary we obtain a result concerning the topological conjugacy of a function $f \in \mathcal{S}_r^1(I)$ with its iterates.

Corollary 3.31. ([B17], Corollary 1) *Let r be a non-negative integer, $I = [a, b]$ for some $a, b \in \mathbb{R}$ such that $a < b$. Assume that $f \in \mathcal{S}_r^1(I)$ satisfies one of the conditions $f(t_{r+1}) = t_0$, $f(t_{r+1}) = f(t_1)$. Then f and f^n are topologically conjugate for every positive integer n.*

Using this result we prove that each iterative root $g \in \mathcal{S}_r^1(I)$ of $f \in \mathcal{S}_r^1(I)$, i.e. a solution of the functional equation $g^n = f$ can be obtained by the formula

$$g = \phi^{-1} \circ f \circ \phi,$$

where ϕ is a homeomorphism realizing the topological conjugacy between f and f^n (cf. [B17], Corollary 2). Therefore, the construction described in Theorem 3.30 can be used to obtain iterative roots of a map $f \in \mathcal{S}_r^1(I)$ satisfying the assumptions of Corollary 3.31.

In paper [B22] we consider the following class of r-modal maps

$$\mathcal{M}_r(I) := \big\{f \in \mathcal{S}_r(I) : f([t_0, t_1]) \subseteq [t_0, t_1],\ f(x) < x \text{ for } x \in (t_1, t_{r+1})\big\}.$$

and its subclasses of the form

$$\mathcal{M}_r^H(I) := \{f \in \mathcal{M}_r(I) : H = H(f)\}$$

for $H \in \mathbb{Z}$, $H > 0$. Let us note that $\mathcal{S}_r^1(I) \subset \mathcal{M}_r^1(I)$.

By the definition of the nonmonotonicity height we obtain that for any $f \in \mathcal{M}_r(I)$, if $H(f)$ is finite, then $H(f)$ is equal to the smallest nonnegative integer n having the property that for all $x \in I$ the condition $f^n(x) \in I_0$ holds, where $I_0 = [t_0, t_1]$.

In our procedure for constructing homeomorphic solution of equation (3.20) for functions $f \in \mathcal{M}_r^H(I)$ and $g \in \mathcal{M}_r^H(J)$ we need to compare for any $x_1, x_2 \in I$ the smallest nonnegative integers n_1 and n_2 such that $f^{n_1}(x_1) \in I_0$ and $f^{n_2}(x_2) \in I_0$. Therefore, in paper [B22] we introduce a notion of *nonmonotonicity height* of x under f by

$$H_f(x) := \inf\{n \in \mathbb{N} : f^n(x) \in I_0\},$$

where $I_0 = [t_0, t_1]$, for all $f \in \mathcal{M}_r(I)$ and $x \in I$.

The relationship between the notions of the nonmonotonicity height of f and the nonmonotonicity height of x under f is described in the following result.

Theorem 3.32. ([B22], Proposition 2.2) *For all $f \in \mathcal{M}_r(I)$*

$$H(f) = \sup\{H(f, x) : x \in I\}.$$

Moreover, if $H := H(f) < \infty$, then

$$H(f) = \max\{H(f, p_i) : i = 1, \dots, N(f^H) + 1\},$$

where p_i are the turning points of f^H for $i = 1, \dots, N(f^H)$ and $p_{N(f^H)+1} = t_{r+1}$.

In this respect it is worth noting that $H(f)$ is not necessarily equal to $\max\{H(f, t_i) : i = 0, 1, ..., r + 1\}$ (cf. [B22], Remark 2.1).

For all $f \in \mathcal{M}_r(I)$ and $x \in I$, we define a sequence of nonnegative integers $I_f(x) = (i_k(x))_{k \in \mathbb{N}}$ in the following way

$$i_k(x) := \begin{cases} l & \text{if} \quad f^k(x) \in I_l \setminus \{t_1, \dots, t_r\},\ l \in \{0, \dots r\}, \\ m & \text{if} \quad f^k(x) \in I_m \cap I_{m+1} = \{t_m\},\ m \in \{0, \dots r-1\}. \end{cases}$$

Note that for each $x \in I$ the sequence $(i_k(x))_{k \in \mathbb{N}}$ is nonincreasing.

The iterative sequences starting from turning points of functions $f \in \mathcal{M}_r^H(I)$ and $g \in \mathcal{M}_r^H(J)$ plays an important role in the main result of paper [B22] presented below.

Theorem 3.33. ([B22], Theorem 4.1) *Let $f \in \mathcal{M}_r^H(I)$ and $g \in \mathcal{M}_r^H(J)$. Then f and g are topologically conjugate if and only if $I_f(t_i) = I_g(s_i)$ for $i = 0, 1, ..., r + 1$ and there exists a strictly increasing function $\varphi_0 : I_0 \to J_0$ such that φ_0 is a topological conjugacy between f_0 and g_0 with*

$$\text{(i)} \qquad \varphi_0 \circ f^{m_i}(t_i) = g^{m_i}(s_i), \quad i = 0, 1, ..., r + 1 \text{ if } H < \infty, \tag{3.21}$$

$$\text{(ii)} \qquad \varphi_0 \circ f^{m_i}(t_i) = g^{m_i}(s_i), \quad i = 0, 1, ..., r \quad \text{if } H = \infty, \tag{3.22}$$

where $m_i := H(t_i) = H(s_i)$. Furthermore, there exists a unique extension φ from φ_0 such that φ realizes the topological conjugacy between f and g.

3.2.4 Approximate solutions of the Volterra integral equation

Paper [B21] contains results on approximate solutions of the following generalization of the Volterra integral equation

$$\psi(x) = \int_a^x N(x,t,\psi(\alpha(x,t))\,dt + G(x), \qquad x \in I, \tag{3.23}$$

where I is a real interval of the form $[a,\infty)$ or $[a,b]$ or $[a,b)$ with some real $a<b$, $\int$ denotes the Bochner integral, and $G : I \to B$, $N : I \times I \times B \to B$ and $\alpha : I \times I \to I$ are given continuous functions, and ψ mapping I into B is an unknown continuous function. Moreover, as the general hypothesis in this paper, we assume that there is a continuous function $L : I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|N(x,t,u_1) - N(x,t,u_2)\| \le L(x,t,\|u_1-u_2\|), \qquad x,t \in I,\ u_1,u_2 \in B \tag{3.24}$$

and

$$L(x,t,s_1) \le L(x,t,s_2), \qquad x,t \in I,\ 0 \le s_1 \le s_2. \tag{3.25}$$

Let $\mathcal{C}_B(I)$ denotes the space of all continuous functions mapping I into B, $\mathcal{C}_{\mathbb{R}_+}(I)$ stands for the space of all continuous functions mapping I into $\mathbb{R}_+ = [0,+\infty)$. Let $\mathcal{T} : \mathcal{C}_B(I) \to B^I$, $\Lambda : \mathcal{C}_{\mathbb{R}_+}(I) \to {\mathbb{R}_+}^I$ are operators given by

$$\mathcal{T}f(x) := \int_a^x N\big(x,t,f(\alpha(x,t))\big)\,dt + G(x), \qquad f \in \mathcal{C}_B(I), x \in I, \tag{3.26}$$

$$\Lambda f(x) := \int_a^x L\big(x,t,\eta(\alpha(x,t))\big)\,dt, \qquad \eta \in \mathcal{C}_{\mathbb{R}_+}(I), x \in I, \tag{3.27}$$

where C^D denotes the family of all functions mapping a nonempty set D into a nonempty set C.

For all $\varepsilon \in {\mathbb{R}_+}^I$ and $\gamma \in {\mathbb{R}_+}^I$ we define

$$L_\varepsilon(\gamma) := \inf\{s \in \mathbb{R}_+ : \gamma(x) \le s\varepsilon(x) \text{ for } x \in I\},$$

where we assume that $\inf \emptyset = \infty$. Further, for every $\phi \in B^I$ we define a function $\|\phi\| \in {\mathbb{R}_+}^I$ by

$$\|\phi\|(x) := \phi(x), \qquad x \in I.$$

Let us yet recall that a function $h \in {\mathbb{R}_+}^{\mathbb{R}_+}$ is subadditive provided

$$h(s+t) \le h(s) + h(t), \qquad s,t \in \mathbb{R}_+.$$

The main result of paper [B21] presented below says that for each approximate solution of equation (3.23) there exist an exact solution of this equation belonging to the class of continuous functions. Moreover, it gives an approximation of the distance between these solutions.

Theorem 3.34. ([B21], Theorem 1) *Let $\varphi \in \mathcal{C}_B(I)$, where $I = [a, b)$ or $I = [a, b]$ with some reals $a < b$. Let $\varepsilon \in \mathcal{C}_{\mathbb{R}_+}(I)$ be given by*

$$\varepsilon(x) := \left\| \varphi(x) - \int_a^x N\big(x, t, \varphi(\alpha(x,t))\big)\, dt - G(x) \right\|, \quad x \in I. \tag{3.28}$$

Assume that

$$\sigma_0(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \qquad x \in I. \tag{3.29}$$

Then $\mathcal{T}(\mathcal{C}_B(I)) \subset \mathcal{C}_B(I)$, the limit

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n \varphi(x) \tag{3.30}$$

exists for each $x \in I$ and the function $\psi \in B^I$, defined in this way, is a continuous solution of equation (3.23) *with*

$$\|\psi(x) - \varphi(x)\| \le \sigma_0(x), \quad x \in I. \tag{3.31}$$

Moreover, if the function $L(x, t, \cdot)$ is subadditive for every $x, t \in I$, then for each function $\eta : I \to \mathbb{R}_+$ with

$$\lim_{n\to\infty} \Lambda^n \eta(x) = 0, \qquad x \in I, \tag{3.32}$$

ψ is the unique solution of (3.23) *such that*

$$L_{\sigma_0+\eta}(\|\psi - \varphi\|) < \infty. \tag{3.33}$$

The above theorem concerns the case where the interval I is finite. For proving an analogous result for the case of infinite I we need an additional assumption about subadditivity of the function $L(x, t, \cdot)$ (cf. [B21], Corollary 2). This assumption is already expressed in the last statement of the above theorem and guarantees some kind of uniqueness of solutions of (3.23).

From Theorem 3.34 we get the following corollary, used in the example ending this subsection.

Corollary 3.35. ([B21], Corollary 3) *Let $I = [a, b)$ or $I = [a, b]$ with some real $a < b$, $\varphi : I \to B$ be continuous and $\varepsilon : I \to \mathbb{R}_+$ be given by* (3.28). *Assume that there exist continuous functions $\varepsilon_0 : I \to \mathbb{R}_+$, $L : I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$ and $K : I \to [0, 1)$ such that $\varepsilon \le \varepsilon_0$,* (3.24) *and* (3.25) *are valid and*

$$\int_a^x L(x, t, K(\alpha(x,t))^n \varepsilon_0(\alpha(x,t)))\, dt \le K(x)^{n+1} \varepsilon_0(x), \quad x \in I, n \in \mathbb{N}. \tag{3.34}$$

Then $\psi : I \to B$, given by (3.30), *is a continuous solution of* (3.23) *with*

$$\|\psi(x) - \varphi(x)\| \le \frac{\varepsilon_0(x)}{1 - K(x)}, \quad x \in I. \tag{3.35}$$

Let us consider the following equation

$$\psi(x) = \int_a^x (x-t)\,\psi(t)\,dt + G(x), \quad x \in [0,1], \tag{3.36}$$

where $G : [0,1] \to \mathbb{R}$ is a continuous function. This equation is of the form (3.23), where $I = [0,1]$, $B = \mathbb{R}$, $N(x,t,u) = (x-t)\,u$ for $x,t \in [0,1]$, $u \in \mathbb{R}$ and $\alpha(x,t) = t$ for $x,t \in [0,1]$. Equation (3.36) has a unique solution ψ as a linear Volterra equation of convolution type. In the case where $G(x) = x$ we can obtain the solution in an explicit form, namely $\psi(x) = \sinh x$.

Let $\varphi : [0,1] \to \mathbb{R}$ be a continuous function and c be a positive real constant such that

$$|\varphi(x) - \int_a^x (x-t)\,\varphi(t)\,dt - G(x)| \le c, \quad x \in [0,1],$$

i.e. we take $\varepsilon_0(x) = c$ for $x \in [0,1]$. One can observe that condition (3.34) holds with K given by $K(x) = \frac{1}{2}$ for $x \in [0,1]$. On account of Corollary 3.35 we have

$$|\varphi(x) - \psi(x)| \le 2\,c, \quad x \in [0,1],$$

where ψ is the unique solution of (3.36). Moreover, if $|G(x) - x| < c$ for $x \in [0,1]$, then

$$|\varphi(x) - \sinh x| \le 4\,c, \quad x \in [0,1].$$

3.2.5 Queueing model for a LAN gateway

In paper [B24] we investigate the functional equation

$$\begin{aligned}(M(x,y) - xy)\,P(x,y) = {} & (1-y)(M(x,0) + \widehat{r}_1\xi_2 xy)P(x,0) \\ & + (1-x)(M(0,y) + \widehat{r}_2\xi_1 xy)P(0,y) \\ & - (1-x)(1-y)M(0,0)P(0,0),\end{aligned} \tag{3.37}$$

where $r_j, s_j \in (0,1)$ for $j = 1,2$ are fixed and

$$M(x,y) = (\widehat{r}_1 + r_1\widehat{s}_1 y + \xi_1 xy)(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 xy) \tag{3.38}$$

with $\xi_j = r_j s_j$ for $j = 1,2$ and $\widehat{q} = 1 - q$ for every $q \in \mathbb{R}$. The unknown function P is defined for $x, y \in \overline{D}$, where $\overline{D} := \{z \in \mathbb{C} : |z| \le 1\}$.

Equation (3.37) appears in investigations of a two-dimensional queueing model for the LAN gateway. It belongs to a class of functional equations which arose in connection with numerous issues in network communication. The general form of all equations in that class is

$$\begin{aligned}C_1(x,y)P(x,y) = {} & C_2(x,y)P(x,0) + C_3(x,y)P(0,y) \\ & + C_4(x,y)P(0,0) + C_5(x,y),\end{aligned} \tag{3.39}$$

where C_j, for $j = 1, ..., 5$, are given functions in two complex variables x, y. Putting in equation (3.39)

$$\begin{aligned} C_1(x,y) &= (\widehat{r}_1 + r_1\widehat{s}_1 y + \xi_1 xy)(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 xy) - xy, \\ C_2(x,y) &= (1-y)\widehat{r}_1(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 xy), \\ C_3(x,y) &= (1-x)\widehat{r}_2(\widehat{r}_1 + r_1\widehat{s}_1 y + \xi_1 xy), \\ C_4(x,y) &= -(1-x)(1-y)\widehat{r}_1\,\widehat{r}_2, \\ C_5(x,y) &= 0, \end{aligned} \tag{3.40}$$

we obtain equation (3.37).

We start from a general observation concerning equation (3.37) considered in T^2 for any $T \subset \mathbb{C}$ such that $0 \in T$. Write

$$\begin{gathered} \mathcal{K} := \{(x,y) \in T^{\,2} : C_1(x,y) = 0\}, \\ \mathcal{K}_0 := \{x \in T : (x,0) \in \mathcal{K}\}, \qquad \mathcal{K}^0 := \{x \in T : (0,x) \in \mathcal{K}\}. \end{gathered}$$

The next theorem provides a useful description of all solutions $P : T^2 \to \mathbb{C}$ (in particular, also analytic solutions, for $T = \overline{D}$) of equation (3.37).

Theorem 3.36. ([B24], Theorem 2.1) *If a function $P : T^2 \to \mathbb{C}$ satisfies equation* (3.37)*, then there exist functions $f, g : T \to \mathbb{C}$ such that $f(0) = g(0)$,*

$$C_2(x,y)f(x) + C_3(x,y)g(y) + C_4(x,y)g(0) = 0, \qquad (x,y) \in \mathcal{K}, \tag{3.41}$$

and

$$P(x,y) = \frac{C_2(x,y)f(x) + C_3(x,y)g(y) + C_4(x,y)g(0)}{C_1(x,y)}, \qquad (x,y) \in T^{\,2} \setminus \mathcal{K}, \tag{3.42}$$

where C_1, C_2, C_3, C_4 have the forms described by (3.40)*. In particular,*

$$P(x,0) = f(x), \qquad P(0,y) = g(y), \qquad x \in T \setminus \mathcal{K}_0, y \in T \setminus \mathcal{K}^0. \tag{3.43}$$

Moreover, if $T = \overline{D}$, then every function $P : T^{\,2} \to \mathbb{C}$ fulfilling (3.42)*, with some continuous functions $f, g : T \to \mathbb{C}$ such that $f(0) = g(0)$ and* (3.41) *holds, satisfies equation* (3.37)*.*

Theorem 3.36 shows that the main issue in solving equation (3.37) in the class of analytic (or continuous) functions $P : \overline{D}^2 \to \mathbb{C}$ is to find all pairs of suitable (analytic or continuous) functions $f, g : \overline{D} \to \mathbb{C}$ satisfying condition (3.41) (such functions are uniquely determined for each P in view of (3.43)). So, in the remaining parts of the paper we focus on condition (3.41).

Notice that in case $T = \overline{D}$ we have

$$\mathcal{K} := \{(x, y) \in \overline{D}^{\,2} : M(x, y) = xy\}.$$

The condition $M(x, y) = xy$ can be written in the form

$$(\widehat{r}_1 + r_1\widehat{s}_1 y + \xi_1 xy)(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 xy) = xy, \tag{3.44}$$

which means that, for each fixed x, it is a quadratic equation (with respect to y) of the form

$$a(x)y^2 + b(x)y + c(x) = 0,$$

where

$$\begin{aligned} a(x) &\equiv \xi_1\xi_2 x^2 + r_1\widehat{s}_1\xi_2 x, \\ b(x) &\equiv r_2\widehat{s}_2\xi_1 x^2 + (\widehat{r}_1\xi_2 + r_1 r_2\widehat{s}_1\widehat{s}_2 + \widehat{r}_2\xi_1 - 1)x + r_1\widehat{s}_1\widehat{r}_2, \\ c(x) &\equiv \widehat{r}_1 r_2\widehat{s}_2 x + \widehat{r}_1\widehat{r}_2. \end{aligned} \tag{3.45}$$

Since $a(x) \neq 0$ for $x \notin \{0, -\widehat{s}_1/s_1\}$, there exist functions $y_1, y_2 : \mathbb{C} \setminus \{0, -\widehat{s}_1/s_1\} \to \mathbb{C}$ with

$$a(x)y^2 + b(x)y + c(x) = a(x)(y - y_1(x))(y - y_2(x)).$$

Write

$$\mathcal{K}_j := \{(x, y_j(x)) : x \in \widehat{D}\} \cap \overline{D}^2 = \{(x, y_j(x)) : x \in \widehat{D}, |y_j(x)| \leq 1\},$$

$$\overline{D}_j := \{x \in \widehat{D} : (x, y_j(x)) \in \mathcal{K}_j\} = \{x \in \widehat{D} : |y_j(x)| \leq 1\}$$

for $j = 1, 2$, where $\widehat{D} := \overline{D} \setminus \{0\}$. Then $\mathcal{K}_j = \{(x, y_j(x)) : x \in \overline{D}_j\}$ for $j = 1, 2$ and

$$\mathcal{K} \subset \mathcal{K}_1 \cup \mathcal{K}_2 \cup \{(-\widehat{s}_1/s_1, \widetilde{y}_0), (0, \widetilde{y})\}.$$

Consider now the particular case where

$$s_1 < 1/2, \quad r_2 < \frac{1 - r_1}{2 - s_2}. \tag{3.46}$$

Then, by the Vieta formulas, we get

$$|y_1(x)y_2(x)| = \left|\frac{c(x)}{a(x)}\right| > 1, \qquad x \in \widehat{D} = \overline{D} \setminus \{0\}.$$

Now, without loss of generality we can choose the values of the functions y_1 and y_2 in such a way that $|y_1(x)| \leq |y_2(x)|$ for $x \in \widehat{D}$. Hence $|y_2(x)| > 1$ for $x \in \widehat{D}$ and therefore $\overline{D}_2 = \emptyset$.

Next we show that, in this particular case, condition (3.41) holds if and only if functions $f, g : \overline{D} \to \mathbb{C}$ satisfy the condition

$$f(x) = \frac{\widehat{r}_2(1-x)g(0)}{\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 x y_1(x)} - \frac{\widehat{r}_2(\widehat{r}_1 + r_1\widehat{s}_1 y_1(x) + \xi_1 x y_1(x))(1-x)g(y_1(x))}{\widehat{r}_1(1-y_1(x))(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 x y_1(x))} \tag{3.47}$$

for $x \in \overline{D}_1$, $x \neq 1$. This means that we can use Theorem 3.36 to obtain a description of all continuous or analytic solutions $P : \overline{D}^2 \to \mathbb{C}$ of equation (3.37).

The main result of this paper gives such a description of all continuous solutions of equation (3.37).

Theorem 3.37. ([B24], Theorem 4.1) *Assume that condition* (3.46) *is valid. A continuous function $P : \overline{D}^2 \to \mathbb{C}$ satisfies equation* (3.37) *if and only if there exists a continuous function $g : \overline{D} \to \mathbb{C}$ such that*

$$P(x,y) = \frac{C_2(x,y)f(x) + C_3(x,y)g(y) + C_4(x,y)g(0)}{C_1(x,y)}, \quad (x,y) \in \overline{D}^{\,2}, y \neq y_1(x), \tag{3.48}$$

where f is given by (3.47)*. In particular, $f(x) = P(x,0)$ and $g(x) = P(0,x)$ for $x \in \overline{D}$.*

3.2.6 Solutions and stability of a generalized Fréchet functional equation

Paper [B25] contains results concerning the following functional equation with constant coefficients

$$A_1F(x+y+z) + A_2F(x) + A_3F(y) + A_4F(z) = A_5F(x+y) + A_6F(x+z) + A_7F(y+z), \tag{3.49}$$

where $A_1, \dots, A_7 \in \mathbb{K}$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, considered for functions $F : X \to Y$, where $(X, +)$ is a commutative monoid (i.e. a semigroup with a neutral element denoted by 0) and Y is a Banach space over the field $\mathbb{K}$. This equation is a generalization of the Fréchet functional equation

$$F(x+y+z) + F(x) + F(y) + F(z) = F(x+y) + F(x+z) + F(y+z). \tag{3.50}$$

It is known that, if X is a group and Y is an abelian group divisible by 2, then the general solution of equation (3.50) is the sum of a quadratic and an additive function.

We start our study the set all of solutions of equation (3.49) from the case where $F(0) = 0$ which plays a crucial role in determining this set.

Theorem 3.38. ([B25], Proposition 3) *If a nonzero function $F : X \to Y$, with $F(0) = 0$, satisfies equation* (3.49)*, then*

$$\begin{cases} A_2 = -A_1 + A_5 + A_6, \\ A_3 = -A_1 + A_5 + A_7, \\ A_4 = -A_1 + A_6 + A_7. \end{cases} \tag{3.51}$$

From the above theorem we obtain a result concerning additive solutions of equation (3.49).

Corollary 3.39. ([B25], Corollary 5) *A nonzero additive function $a : X \to Y$ satisfies equation* (3.49) *if and only if relations* (3.51) *hold.*

The main result of paper [B25] describes the set of solution of equation (3.49) in the considered case and refers to situations where equation (3.49) cannot be reduced to equation (3.50) by dividing both sides by a nonzero element of the field $\mathbb{K}$.

Theorem 3.40. ([24], Theorem 7) *If $A_i \neq A_j$ for some $i, j \in \{1, ..., 7\}$, then each solution $F : X \to Y$ of equation* (3.49)*, with $F(0) = 0$, is an additive function.*

From Theorem 3.40 we get the following description of the set of solution of equation (3.49) in the case where $F(0) = 0$.

Corollary 3.41. ([B25], Corollary 9) *If $A_i \neq A_j$ for some $i, j \in \{1, ..., 7\}$ and condition*

$$A_1 + A_2 + A_3 + A_4 \neq A_5 + A_6 + A_7,$$

holds, then each solution of equation (3.49) *is an additive function.*

Now we proceed to the general case, without the assumption that $F(0) = 0$.

Corollary 3.42. ([B25], Corollary 10) *Assume that $A_i \neq A_j$ for some $i, j \in \{1, ..., 7\}$. If $F : X \to Y$ is a solution of equation* (3.49)*, then*

$$F(x) = a(x) + c, \quad x \in X, \tag{3.52}$$

where $a : X \to Y$ is an additive function and $c = F(0)$.

Paper [B25] also contains a stability result for equation (3.49).

Theorem 3.43. ([B25], Theorem 13) *Let $A_2 + A_3 + A_4 \neq 0$ and*

$$\beta_0 := \left| \frac{A_5 + A_6 + A_7 - A_1}{A_2 + A_3 + A_4} \right| < 1.$$

Let $L\colon X^3 \to [0, \infty)$ satisfy the condition

$$L(kx, ky, kz) \leq c_k L(x, y, z), \qquad (x, y, z) \in X^3 \setminus \{(0, 0, 0)\},\ k \in \{2, 3\} \tag{3.53}$$

with some $c_2, c_3 \in [0,\infty)$ *such that* $\beta := b_2c_2 + b_3c_3 < 1$, *where*

$$b_2 := \left|\frac{A_5 + A_6 + A_7}{A_2 + A_3 + A_4}\right|, \qquad b_3 := \left|\frac{A_1}{A_2 + A_3 + A_4}\right|. \tag{3.54}$$

If $f\colon X \to Y$ *fulfils the condition*

$$\begin{aligned} \|A_1 f(x+y+z) + A_2 f(x) + A_3 f(y) + A_4 f(z) - A_5 f(x+y) \\ - A_6 f(x+z) - A_7 f(y+z)\| \le L(x,y,z), \qquad (x,y,z) \in X^3, \end{aligned} \tag{3.55}$$

then there exists a unique function $F : X \to Y$ *satisfying* (3.49) *such that* $F(0) = 0$ *and*

$$\|f(x) - F(x)\| \le \rho_L(x), \qquad x \in X, \tag{3.56}$$

where

$$\rho_L(x) := \frac{L(x,x,x)}{|A_2 + A_3 + A_4|(1 - \gamma(x))}, \qquad x \in X, \tag{3.57}$$

with

$$\gamma(x) := \begin{cases} \beta & \text{if} \quad x \neq 0, \\ \beta_0 & \text{if} \quad x = 0. \end{cases}$$

In the proof of the above theorem, for an approximate solution $f : X \to Y$ of equation (3.49) the existence of a solution $F : X \to Y$ of this equation is obtained by using a fixed point theorem proved in a paper of J. Brzdęk, J. Chudziak and Z. Páles (cf. [24]).

3.2.7 Fixed points of operators and the Ulam type stability

In papers [B26] and [B27] we give fixed point theorems and show their application to prove the stability in Ulam sense of the considered types of equations. The stability of an equation is proved by using an operator defined for this equation in an appropriate way. Fixed point of such operators turn out to be exact solutions that meet the imposed conditions.

Various definitions of the Ulam type stability have been used in the literature for particular equations, but the following one describes our considerations in papers [B26] and [B27]. Given a metric space (X, d), where d is a metric or one of its generalizations defined in X, a set $S \neq \emptyset$, nonempty classes of functions $\mathcal{D}_0 \subset \mathcal{D} \subset X^S$ and $\mathcal{E} \subset (\mathbb{R}_0^+)^S$, and operators $\mathcal{T} : \mathcal{D} \to X^S$, $\mathcal{S} : \mathcal{E} \to (\mathbb{R}_0^+)^S$, we say that the equation

$$\mathcal{T}(\psi) = \psi$$

is $\mathcal{S}$–stable in $\mathcal{D}_0$ provided for any $\psi \in \mathcal{D}_0$ and $\delta \in \mathcal{E}$ with

$$d\big(\mathcal{T}(\psi)(t), \psi(t)\big) \le \delta(t), \qquad t \in S,$$

there is a solution $\phi \in \mathcal{D}$ of the equation such that

$$d\big(\phi(t), \psi(t)\big) \le (\mathcal{S}\delta)(t), \qquad t \in S,$$

where A^B denotes the family of all functions mapping a set B into a set A.

In paper [B26], we prove a fixed point theorem for a linear operator of polynomial form of order 3 motivated by the Ulam type stability problem of the equation

$$p_3\mathcal{L}^3\psi + p_2\mathcal{L}^2\psi + p_1\mathcal{L}\psi = \psi, \tag{3.58}$$

where $\mathcal{L} : X \to X$ is a given linear operator of a complex linear space X and $p_1, p_2, p_3 \in \mathbb{C}$, $p_3 \neq 0$. Given linear operator $\mathcal{L} : X \to X$ we define the operator $\mathcal{P} : X \to X$ by the formula

$$\mathcal{P}\psi := p_3\mathcal{L}^3\psi + p_2\mathcal{L}^2\psi + p_1\mathcal{L}\psi, \qquad \psi \in X. \tag{3.59}$$

Thus equation (3.58) can be written in the form $\mathcal{P}\psi = \psi$.

In our consideration, an important role is played by the roots of the characteristic polynomial of equation (3.58), i.e. the roots of the polynomial

$$P(x) = p_3x^3 + p_2x^2 + p_1x - 1. \tag{3.60}$$

By Viete's formulas we obtain that if $a_1, a_2, a_3 \in \mathbb{C}$ are the roots of the characteristic polynomial of the equation (3.58), then $a_i \neq 0$ for $i \in \{1, 2, 3\}$ and

$$p_3 = \frac{1}{a_1a_2a_3}, \qquad -p_2 = \frac{1}{a_1a_2} + \frac{1}{a_1a_3} + \frac{1}{a_2a_3}, \qquad p_1 = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}.$$

We are looking for fixed points of the operator $\mathcal{P}$ under the assumption that is an extended complex normed space. Let us recall that a pair $(X, \|\cdot\|)$ is an *extended complex normed space* if X is a complex linear space and $\|\cdot\|$ is a function mapping X into $[0, \infty]$ (i.e. $\|\cdot\|$ may take the value $+\infty$) such that, for every $\alpha \in \mathbb{C}$ and $x, y \in X$ with $\|x\|, \|y\| \in [0, \infty)$,

$$\|x + y\| \le \|x\| + \|y\|, \qquad \|\alpha x\| = |\alpha|\, \|x\|,$$

and the equality $\|x\| = 0$ means that x is the zero vector. An extended complex normed space X is called an *extended complex Banach space*, if every Cauchy sequence of elements of X is convergent (in X).

The main result of paper [B26] concerns fixed points of the operator $\mathcal{P}$ defined above.

Theorem 3.44. ([B26], Theorem 2) *Let X be an extended complex Banach space. Let $a_1, a_2, a_3 \in \mathbb{C}$ be the roots of polynomial* (3.60) *such that*

$$a_i \neq a_j, \qquad i, j \in \{1, 2, 3\}, i \neq j.$$

Assume that a linear operator $\mathcal{L}$ satisfies the Lipschitz condition

$$\|\mathcal{L}f - \mathcal{L}g\| \ \leq \ L\|f - g\|, \qquad f, g \in X,$$

with some positive constant L such that

$$L < \min\{|a_1|, |a_2|, |a_3|\}. \tag{3.61}$$

Then for every $\varphi \in X$ with

$$\varepsilon := \big\|\mathcal{P}\varphi - \varphi\big\| < \infty, \tag{3.62}$$

where $\mathcal{P}$ is given by fotmula (3.59)*, the operator $\mathcal{P}$ has a unique fixed point $\psi \in X$ such that*

$$\|\varphi - \psi\| < \infty.$$

Moreover,

$$\|\varphi - \psi\| \leq C\varepsilon,$$

where

$$C = \left(\frac{1}{|a_2 - a_1|\,|a_3 - a_1|\,(|a_1| - L)} + \frac{1}{|a_1 - a_2|\,|a_3 - a_2|(|a_2| - L)} + \frac{1}{|a_1 - a_3|\,|a_2 - a_3|\,(|a_3| - L)}\right)|a_1||a_2||a_3|. \tag{3.63}$$

In the proof of this result we apply the classical Diaz–Margolis fixed point alternative (cf. [30]) for strictly contractive operators $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : X \to X$ defined by

$$\mathcal{T}_j f := \frac{1}{a_j}\mathcal{L}f, \qquad f \in X, j = 1, 2, 3.$$

For each $j = 1, 2, 3$, the unique fixed point F_j of $\mathcal{T}_j$ is an eigenvector of $\mathcal{L}$.

Let Y be a complex Banach space and S be a nonempty set. In the space Y^S we consider the supremum norm

$$\|f\| := \sup_{s \in S}\|f(s)\|, \qquad f \in Y^S,$$

being a natural example of an extended norm (it will be called the supremum extended norm). Using Theorem 3.44 for this space we obtain an Ulam type stability result for equation (3.58).

Theorem 3.45. ([B26], Theorem 3) *Let Y be a complex Banach space and S be a nonempty set. Let $\mathcal{C}$ be a linear subspace of Y^S closed with respect to the supremum extended norm and $\mathcal{L} : \mathcal{C} \to \mathcal{C}$ be a linear operator. Assume that $a_i \neq a_j$ for $i, j \in \{1, 2, 3\}$, $i \neq j$, and $\mathcal{L}$ satisfies the Lipschitz condition*

$$\|\mathcal{L}f - \mathcal{L}g\| \ \leq \ L\|f - g\|, \qquad f, g \in \mathcal{C}, \tag{3.64}$$

with a positive constant $L < \min\{|a_1|, |a_2|, |a_3|\}$. Then, for every function $\varphi \in \mathcal{C}$ with

$$\varepsilon := \left\|p_3\mathcal{L}^3\varphi + p_2\mathcal{L}^2\varphi + p_1\mathcal{L}\varphi - \varphi\right\| < \infty,$$

there is a unique solution $\psi \in \mathcal{C}$ of equation (3.58) with $\|\varphi - \psi\| < \infty$. Moreover,

$$\|\varphi - \psi\| \leq C\varepsilon,$$

where C is given by (3.63).

In paper [B27] we prove a fixed point theorem for an operator acting in a dq-metric space. Let us remind that a pair (X, d), where X is a nonempty set and $d\colon X \times X \to [0, +\infty)$, is called a *dq-metric space*, if the function d satisfies the following conditions:

(A1) if $d(x, y) = d(y, x) = 0$, tthen $x = y$,

(A2) $d(x, y) \leqslant d(x, z) + d(z, y)$

for all $x, y, z \in X$.

Let (X, d) be a dq-metric space. We say that $x \in X$ is a *limit* of a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of the space X, if

$$\lim_{n\to\infty} \max\{d(x_n, x), d(x, x_n)\} = 0.$$

From condition (A2) we obtain the uniqueness of the limit. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of the space X is called a *Cauchy sequence*, if

$$\lim_{N\to\infty} \sup_{m,n\geqslant N} d(x_n, x_m) = 0.$$

A dq-metric space (X, d) is said to be *complete*, if every Cauchy sequence of elements of X has a limit in X.

For a dq-metric space (X, d) and a nonempty set E we define the function $d_f : X^E \times X^E \to {\mathbb{R}_+}^E$ by putting

$$d_f(\xi, \mu)(t) := d(\xi(t), \mu(t)), \qquad \xi, \mu \in X^E, t \in E. \tag{3.65}$$

Analogously, as in the classical metric spaces, if $(\chi_n)_{n\in\mathbb{N}}$ is a sequence of elements of X^E, then a function $\chi \in X^E$ is said to be *a pointwise limit* of the sequence of functions $(\chi_n)_{n\in\mathbb{N}}$, if

$$\lim_{n\to\infty} \max\left\{d_f(\chi,\chi_n)(t), d_f(\chi_n,\chi)(t)\right\} = 0, \qquad t \in E.$$

A function $\chi \in Y^E$ is called a *uniform limit* of the sequence $(\chi_n)_{n\in\mathbb{N}}$, if

$$\lim_{n\to\infty} \sup_{t\in E} \max\left\{d_f(\chi,\chi_n)(t), d_f(\chi_n,\chi)(t)\right\} = 0.$$

A nonempty subset $\mathcal{F}$ of X^E is called *p-closed* (*u-closed*, respectively), if every function $\chi \in X^E$ which is a pointwise (uniform, respectively) limit of a sequence $(\chi_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.

For functions $f, g \in \mathbb{R}^E$, we write $f \le g$, if $f(t) \le g(t)$ for all $t \in E$. Let $\emptyset \ne \mathcal{C} \subset X^E$, $\Lambda\colon {\mathbb{R}_+}^E \to {\mathbb{R}_+}^E$ and $\omega \in {\mathbb{R}_+}^E$. We say that an operator $\mathcal{T}\colon \mathcal{C} \to X^E$ is $(\omega,\Lambda)-$*contractive*, provided

$$d_f(\mathcal{T}\xi, \mathcal{T}\mu) \le \Lambda\delta$$

for all $\xi, \mu \in \mathcal{C}$ and $\delta \in {\mathbb{R}_+}^E$ such that

$$\delta \le \omega, \qquad d_f(\xi,\mu) \le \delta.$$

Moreover, in order to simplify some formulas we denote by Λ_0 the identity operator in ${\mathbb{R}_+}^E$, i.e. $\Lambda_0\delta = \delta$ for each $\delta \in {\mathbb{R}_+}^E$.

The main result of paper [B27] is the following fixed point theorem.

Theorem 3.46. ([B27], Theorem 2) *Let (X,d) be a dq-metric space, E be a nonempty set, and $d_f : X^E \times X^E \to {\mathbb{R}_+}^E$ be the function defined by (3.65). Let $\emptyset \ne \mathcal{C} \subset X^E$, $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$ and $\Lambda_n\colon {\mathbb{R}_+}^E \to {\mathbb{R}_+}^E$ for $n \in \mathbb{N}$. Assume that there exist functions $\varepsilon_1, \varepsilon_2 \in {\mathbb{R}_+}^E$ and $\varphi \in \mathcal{C}$ such that*

$$\varepsilon_j^*(t) := \sum_{i=0}^{\infty} \Lambda_i\varepsilon_j(t) < \infty, \qquad t \in E, j = 1,2, \tag{3.66}$$

$$d_f(\mathcal{T}\varphi,\varphi) \le \varepsilon_1, \qquad d_f(\varphi,\mathcal{T}\varphi) \le \varepsilon_2, \tag{3.67}$$

$$\liminf_{n\to\infty} \Lambda_1\Big(\sum_{i=n}^{\infty} \Lambda_i\varepsilon_j\Big)(t) = 0, \qquad t \in E, j = 1,2. \tag{3.68}$$

Let $\varepsilon^*(t) := \max\{\varepsilon_1(t), \varepsilon_2(t)\}$ for $t \in E$. If $\mathcal{T}^n$ is an $(\varepsilon^*, \Lambda_n)-$contractive operator for $n \in \mathbb{N}$ and one of the following two conditions is valid

(i) *$\mathcal{C}$ is p-closed;*

(ii) *$\mathcal{C}$ is u-closed and the sequence $\left(\sum_{i=0}^{n}\Lambda_i\varepsilon_j\right)_{n\in\mathbb{N}}$ tends uniformly to ε_j^* on the set E for $j=1,2$,*

then for each $t\in E$ there exists the limit

$$\psi(t) := \lim_{n\to\infty}\mathcal{T}^n\varphi(t), \tag{3.69}$$

and a function $\psi\in\mathcal{C}$ defined in this way, is a fixed point of the operator $\mathcal{T}$ with

$$d_f(\mathcal{T}^n\varphi,\psi)\le\sum_{i=n}^{\infty}\Lambda_i\varepsilon_1, \qquad d_f(\psi,\mathcal{T}^n\varphi)\le\sum_{i=n}^{\infty}\Lambda_i\varepsilon_2, \qquad n\in\mathbb{N}. \tag{3.70}$$

Moreover, the following two statements are valid:

(a) *for every sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers with $\lim_{n\to\infty}k_n=\infty$, the function ψ is the unique fixed point of the operator $\mathcal{T}$ such that*

$$d_f(\mathcal{T}^{k_n}\varphi,\psi)\le\sum_{i=k_n}^{\infty}\Lambda_i\varepsilon_j, \qquad d_f(\psi,\mathcal{T}^{k_n}\varphi)\le\sum_{i=k_n}^{\infty}\Lambda_i\varepsilon_l, \qquad n\in\mathbb{N},$$

for some $j,l\in\{1,2\}$;

(b) *if*

$$\liminf_{n\to\infty}\Lambda_n\varepsilon_j^*(t)=0, \qquad j=1,2,\ t\in E, \tag{3.71}$$

then ψ is the unique fixed point of the operator $\mathcal{T}$ with

$$d_f(\varphi,\psi)\le\varepsilon_1^*, \qquad d_f(\psi,\varphi)\le\varepsilon_2^*,$$

and for any $j,l\in\{1,2\}$

$$\psi(t)=\lim_{n\to\infty}\mathcal{T}^{k_n}\xi(t), \qquad t\in E \tag{3.72}$$

with $\xi\in\mathcal{C}$, $d_f(\xi,\psi)\le\varepsilon_j^*$ and $d_f(\psi,\xi)\le\varepsilon_l^*$, for each sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers with $\lim_{n\to\infty}\Lambda_{k_n}\varepsilon_m^*(t)=0$ for $t\in E$ and $m\in\{j,l\}$.

Now, we show how one can derive some Ulam stability outcomes from the results of the above theorem. Let us consider the functional equation of the form

$$\Phi(t,\psi(f_1(t)),...,\psi(f_j(t)))=\psi(t), \qquad t\in E, \tag{3.73}$$

where $E\neq\emptyset$, $j\in\mathbb{N}$, $f_i:E\to E$ for $i=1,\dots,j$, $\Phi:E\times X^j\to X$ are given, and $\psi:E\to X$ is an unknown function.

Let

(H1) $L_i : E \to \mathbb{R}_+$ for $i = 1, \dots, j$, satisfy the condition

$$d(\Phi(t, w_1, ..., w_j), \Phi(t, z_1, ..., z_j)) \leq \sum_{k=1}^{j} L_k(t) d(w_k, z_k)$$

for $t \in E$ and $(w_1, ..., w_j), (z_1, ..., z_j) \in X^j$ such that $d(z_i, w_i) \leq e$ for $i = 1, \dots, j$,

with a constant e such that $e > 0$ or $e = \infty$. Then using Theorem 3.46 we obtain a stability result for equation (3.73).

Corollary 3.47. ([B27], Corollary 6) *Let (X, d) be a dq-metric space, $E \neq \emptyset$, $j \in \mathbb{N}$, $\Phi : E \times X^j \to X$ and $\varepsilon_1, \varepsilon_2 : E \to \mathbb{R}_+$. Assume that $L_i : E \to \mathbb{R}_+$ for $i = 1, \dots, j$ are functions for which condition* (H1) *holds with $e := \sup\{\varepsilon_j^*(t) : t \in E, j = 1, 2\}$, where*

$$\varepsilon_j^*(t) := \sum_{i=0}^{\infty} \Lambda^i \varepsilon_j(t) < \infty, \qquad t \in E,\ j = 1, 2.$$

Let $\Lambda : \mathbb{R}_+^E \to \mathbb{R}_+^E$ be given by the formula

$$\Lambda\delta(t) := \sum_{k=1}^{j} L_k(t) \delta(f_k(t)), \qquad \delta \in \mathbb{R}_+^E,\ t \in E,$$

for some $f_1, \dots, f_j : E \to E$. Then, if $\varphi : E \to X$ satisfies conditions

$$d(\Phi(t, \varphi(f_1(t)), ..., \varphi(f_j(t))), \varphi(t)) \leq \varepsilon_1(t), \qquad t \in E,$$

$$d(\varphi(t), \Phi(t, \varphi(f_1(t)), ..., \varphi(f_j(t)))) \leq \varepsilon_2(t), \qquad t \in E,$$

then there exists the limit

$$\psi(t) := \lim_{n \to \infty} \mathcal{T}^n \varphi(t) \tag{3.74}$$

for all $t \in E$, where $\mathcal{T}$ is given by the formula

$$\mathcal{T}\varphi(t) := \Phi(t, \varphi(f_1(t)), ..., \varphi(f_j(t))), \qquad \varphi \in X^E,\ t \in E,$$

and the function $\psi : E \to X$ defined by (3.74) *is the unique solution of the functional equation* (3.73) *such*

$$d(\varphi(t), \psi(t)) \leq \varepsilon_1^*(t), \qquad d(\psi(t), \varphi(t)) \leq \varepsilon_2^*(t), \qquad t \in E.$$

The difference equation

$$\psi(i) = \Phi(i, \psi(i+1)), \qquad i \in \mathbb{N}, \tag{3.75}$$

where $\Phi : \mathbb{N} \times X \to X$ is given and $\psi : \mathbb{N} \to X$ is unknown, has been used in paper [B27] as an example of application of Corollary 3.47. This equation is, in fact, a particular case of equation (3.73), where $E = \mathbb{N}$, $j = 1$ and $f_1(i) = i + 1$ for $i \in \mathbb{N}$.

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