Author's review of his research, achievements and publications

1. Name: **Robert Rałowski**
2. Obtained diplomas and academic degrees:
   - M. Sc. in Physics, Wrocław University of Technology, 1989,
   - Ph. D. in Physics, Uniwersity of Wrocław, 1998.
3. Employment in scientific institutions:
   - Assistant Lecturer at Institute of Low Temperatures and Structural Research, Polish Academy of Sciences, 1990 – 1992,
   - Ph.D. Student at Institute of Theoretical Physics, University of Wrocław, 1992–1997,
   - Assistant Lecturer at Institute of Mathematics and Computer Science, Wrocław University of Technology, 1997 – 1999,
   - Assistant Professor at Institute of Mathematics and Computer Science, Wrocław University of Technology, 1999 – 31.10.2014,
   - Assistant Professor at Department of Computer Science, Faculty of Fundamental Problems of Technology, Wrocław University of Technology, since 1.11.2014.
4. Achievement resulting from Article 16 Paragraph 2 of the Act of 14 March 2003 on Academic Degrees and Title and on Degrees and Title in the Field of Art is a series of publications under the title:

Nonmeasurable subsets in Polish spaces.

List of publications included in the above-mentioned achievement

[H1] J. Cichoń, M. Morayne, R. Rałowski, Cz. Ryll-Nardzewski, Sz. Żeberski, On nonmeasurable unions, Topology and its Applications, 154 (2007), pp.884-893.

[H2] R. Rałowski, Sz. Żeberski, Complete nonmeasurability in regular families, Houston Journal of Mathematics, 34 (3) (2008), pp. 773-780,

[H3] R. Rałowski, Remarks on nonmeasurable unions of big point families, Mathematical Logic Quarterly, vol. 55, nr 6 (2009), pp. 659-665.

[H4] R. Rałowski, Nonmeasurability in Banach spaces, Far East Journal of Mathematical Sciences, vol. 36, nr 2 (2010), pp. 125-131.

[H5] R. Rałowski and Sz. Żeberski, On nonmeasurable images, Czechoslovak Mathematical Journal, 60(135) (2010), pp. 424-434.

[H6] R. Rałowski, Sz. Żeberski, Completely nonmeasurable unions, Central European Journal of Mathematics, 8(4) (2010), pp. 683-687.

[H7] R. Rałowski, Sz. Żeberski, Generalized Łuzin sets, Houston Journal of Mathematics, electronic edition vol. 39, no. 3 , 2013, pp. 983-993.

[H8] R. Rałowski, Families of sets with nonmeasurable unions with respect to ideals defined by trees, Archive for Mathematical Logic, 54 (2015), no. 5-6, 649-658.

Discussion of the scientific and the results achieved on the basis of the above-mentioned works

Motivation and description of the field

Henri Lebesgue in his work [Leb] from 1904 posed the following problem: *is there a non-negative function defined on all subsets of the interval* $[0,1]$, $m : \mathscr{P}([0,1]) \to [0,1]$, *which is*

(1) *translationally invariant, i.e.*
$$(\forall X, Y \in \mathscr{P}([0,1]))(\forall t \in \mathbb{R})\ (Y = (X+t) \mod 1 \longrightarrow m(X) = m(Y)),$$
(2) *σ-additive, i.e. if $\mathcal{F} \in [\mathscr{P}([0,1])]^\omega$ is a countable family of pairwise disjoint sets, then* $m(\bigcup \mathcal{F}) = \sum_{A \in \mathcal{F}} m(A)$,
(3) $m([0,1]) = 1$*?*

Here $\mathscr{P}(X)$ denotes the power set of the set X and $[X]^{<\kappa} = \{A \in \mathscr{P}(X) : |A| < \kappa\}$. The definitions of $[X]^\kappa$, $[X]^{\leq\kappa}$ are analogous.

In 1905, Giuseppe Vitali showed in [Vitali] that such a function on $\mathscr{P}([0,1])$ does not exist. For this purpose, using the axiom of choice, Vitali built nonmeasurable a selector for the family of all equivalence classes with respect to the congruence: $a \sim b \longleftrightarrow a - b \in \mathbb{Q}$.

Stefan Banach and Kazimierz Kuratowski [BaKu], assuming the continuum hypothesis **CH**, gave a negative answer to the analogous problem posed by Lebesgue, where condition (1) was replaced by the condition that m vanishes on all singletons.

Let X be an infinite set, then the function $m : \mathscr{P}(X) \to [0,1]$ is a non-trivial κ-additive measure if:

(1) $(\forall x \in X)\ m(\{x\}) = 0$,
(2) $(\forall \mathcal{F} \in [\mathscr{P}(X)]^{<\kappa}(\forall A, B \in \mathcal{F})((A \neq B \longrightarrow A \cap B = \emptyset) \longrightarrow (m(\bigcup \mathcal{F}) = \sum_{A \in \mathcal{F}} m(A)))$,
(3) $m(X) = 1$.

Measure m we call σ-additive if it is ω_1-additive.

Let κ be cardinal number such that $|X| = \kappa$. If there is nontrivial κ-additive measure on the set X, which fulfils the above conditions, then the cardinal number κ we call a real measurable cardinal. A real measurable cardinal is weakly inaccessible, i.e. it is an uncountable limit cardinal, which is regular. An uncountable cardinal number κ is a measurable cardinal if there is a κ-complete nonprincipial ultrafilter $\mathcal{U}$ on κ. Each such ultrafilter generates a two-valued κ-additive measure on κ defined as follows:
$$(\forall A \in \mathscr{P}(\kappa))\ m(A) = \begin{cases} 1 & A \in \mathcal{U} \\ 0 & A \notin \mathcal{U} \end{cases}.$$
A measurable cardinal κ is strongly inaccessible i.e. it is a regular, uncountable limit cardinal such that, for each cardinal number $\lambda < \kappa$, $2^\lambda < \kappa$. The concept of a measurable cardinal number was introduced by Stanisław Ulam see, [Ulam]. In the same article, the author proved the following theorem, which opened a very important branch of set theory, namely the theory of large cardinals.

Theorem 1 (Ulam, 1930)**.** *If there is a nontrivial σ-additive measure on a set X, then either there exists a measurable κ which is not greater than $|X|$, or there exists a real measurable cardinal which is not grater than $2^{\aleph_0}$.*

Robert Solovay proved [So2] that the existence of a measurable cardinal κ, implies that there exists a forcing notion $\mathbb{P}$, such that in a generic extension $V[G]$ (where $G \subseteq \mathbb{P}$ is a generic filter over model V) $\kappa = 2^{\aleph_0}$ and κ is real-measurable.

In [So1] Solovay showed that the existence of a strongly inaccessible cardinal implies that it is consistent with the Zermelo-Frankel theory and depending choices axiom (**ZF**+ **DC**) that every subset of $\mathbb{R}$ is measurable with respect to the Lebesgue measure, and has the Baire property. Moreover, every uncountable subset of $\mathbb{R}$ contains a perfect set. The axiom **DC** says that for every set X and arbitrary relation $R \subseteq X \times X$ such that $dom(R) = X$, there exists a sequence $(x_n)_{n\in\omega} \in X^\omega$ such that $(x_n, x_{n+1}) \in R$ for every $n \in \omega$.

The transfinite recursion theorem, which is a consequence of the axiom of choice **AC**, is the main tool for the construction of nonmeasurable sets with respect to the Lebesgue measure, and sets which do not have the Baire property. Using transfinite recursion, we can obtain a Bernstein set $A \subset X$ in an uncountable Polish space X. Recall that A is a Bernstein set if for each perfect subset $P \subseteq X$ we have $A \cap P \neq \emptyset$ and $A^c \cap P \neq \emptyset$.

Nikolai Lusin (Wacław Sierpiński) proved that **CH** implies the existence of a so called Lusin (Sierpiński, resp.) set in $\mathbb{R}$. A set is a Lusin (Sierpiński) set if is uncountable and and has a countable intersection with every set of first Baire category (Lebesgue measure zero, resp). Of course, the definition of a Lusin set extends naturally to every Polish space.

A consequence of the axiom of choice **AC** is the existence of a nonprincipial ultrafilter $\mathscr{U} \subseteq \mathscr{P}(\omega)$ on ω, which allowed Sierpiński to construct a non-measurable subset with respect to the Haar measure on the Cantor space $2^\omega = \{0,1\}^\omega$, defined as follows

$$\{x \in 2^\omega : \ \{n \in \omega : \ x(n) = 1\} \in \mathscr{U}\}.$$

Jacek Cichoń and Przemysław Szczepaniak [CS] gave a new method for construction of a nonmeasurable subsets of Euclidean space $\mathbb{R}^n$. They used a linear isomorphism $\mathbb{R}^n$ and $\mathbb{R}^m$ over the field of rational numbers. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is such an isomorphism for different numbers m, n, then for every set $A \subseteq \mathbb{R}^n$ such that $int(A) \neq \emptyset$ and $int(A^c) \neq \emptyset$, the image $f[A] \subseteq \mathbb{R}^m$ is a nonmeasurable set with respect to the m-dimensional Lebesgue measure.

The existence of Bernstein and Vitali sets and the existence of nonprincipial ultrafilter on ω used by Sierpiński to construct a nonmeasurable set on the Cantor space are provable within **ZFC** theory. This is not the case for Lusin and Sierpinski sets, which can be constructed, e.g., in any model in which the continuum hypothesis is satisfied; but, e.g., Martin's Axiom and the negation of the continuum hypothesis $\mathbf{MA} + \neg\mathbf{CH}$ prohibit the construction of these sets. However, if to any ground model V, such that $V \models \mathbf{CH}$, we add ω_2 Cohen independent reals $\mathcal{C}_{\omega_2} = \{c_\xi \in 2^\omega : \xi < \aleph_2\}$, then in the generic extension $V[\mathcal{C}_{\omega_2}]$ there exists a Lusin set with the cardinality $\aleph_2 = \mathfrak{c}$. Moreover, the set $\mathcal{C}_{\omega_1} = \{c_\xi : \xi < \aleph_1\} \notin \mathcal{M}$ is a Lusin set. Therefore $\mathcal{C}_{\omega_1}$ is not measurable in the sense of Baire. Similarly, if to any model $V \models \mathbf{CH}$, we add $\aleph_2$ Solovay independent reals, we get a nonmeasurable set with respect to the Lebesgue measure of cardinality less than $\mathfrak{c}$ in the generic extension.

In [Ku] Kazimierz Kuratowski, assuming **CH**, proved that for each family $\mathcal{A} \subseteq \mathcal{M}$ of pairwise disjoint sets of first category, such that $\bigcup \mathcal{A} \notin \mathcal{M}$, there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ whose union is not measurable in the sense of Baire.

One of the well known theorems on nonmeasurable unions of sets is the following theorem proved by Jan Brzuchowski, Jacek Cichoń, Edward Grzegorek and Czesław Ryll-Nardzewski [BCGR].

Theorem 2. *Let $\mathcal{I}$ be a σ-ideal with a Borel base on a Polish space X, containing all singletons. Then for every point-finite family $\mathcal{A} \subseteq \mathcal{I}$ such that $\bigcup \mathcal{A} \notin \mathcal{I}$, there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$, such that the union $\bigcup \mathcal{A}'$ is not $\mathcal{I}$-measurable, i.e. does not belong to the σ-field of sets generated by the σ-ideal $\mathcal{I}$ and the σ-field of all Borel sets* $\mathrm{Bor}(X)$.

Lev Bukovsky proved this theorem for partitions of $\mathbb{R}$ into the first Baire category sets or into Lebesgue measure zero sets. His paper [Bu] appeared in the same issue of the Bulletin of the Polish Academy of Sciences as the paper [BCGR]. The author used a nonelementary method of generic ultrapower for Cohen forcing in the case of the first category sets and Solovay forcing of adding a one random real for the measure case.

The last two assertions cannot be extended to the point-countable families $\mathcal{A} \subseteq \mathcal{N}$, i.e. those for which we have

$$(\forall x \in X)\ (\{A \in \mathcal{A} : \ x \in A\} \in [\mathcal{A}]^{\leq\omega}).$$

Namely, David Fremlin [Frem], by adding ω_2 independent Cohen reals to the ground model L which is the Gödel constructible universe, constructed a point-countable family $\mathcal{A} \subseteq \mathcal{N}$ of Lebesgue measure zero sets, such that $\bigcup \mathcal{A} = \mathbb{R}$ and for each subfamily $\mathcal{B} \subseteq \mathcal{A}$ the union $\bigcup \mathcal{B}$ is measurable with respect to the Lebesgue measure.

In the particular case of measure, it is not known whether for each partition of the segment $[0,1]$ into sets of Lebesgue measure zero, we can select a subfamily whose union is completely nonmeasurable, i.e. has the inner measure equal to 0 and the outer measure equal to 1. A partial result was obtained by David Fremlin and Stevo Todorćević in [FrTod]. The authors showed that for any partition of the interval $[0,1]$ into sets of Lebesgue measure zero and for any $\epsilon > 0$ one can select a subfamily the union of which has inner measure less than ϵ and outer measure greater than $1-\epsilon$.

A special case of the set-theoretical union of families, are algebraic sums of subsets of $(G,+)$. If $A, B \in \mathscr{P}(G)$, then we define the algebraic sum of A and B as follows

$$A + B = \{a + b \in G : \ (a,b) \in A \times B\}.$$

In this report we consider only uncountable Polish Abelian groups.

Sierpiński [Sier] showed that there are two subsets of X, Y of the real line $\mathbb{R}$ such that $X+Y$ is not Lebesgue measurable.

A pair $(\mathcal{I}, \mathcal{A})$ has the perfect set property if every set $B \in \mathcal{A} \setminus \mathcal{I}$ contains a perfect set. ($(\mathcal{N}, \mathcal{LM})$ and $(\mathcal{M}, \mathcal{BP})$ are examples of such pairs, where $\mathcal{LM}$ is the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$, and $\mathcal{BP}$ is the algebra of all subsets of $\mathbb{R}$ having the Baire property). Recently, Marcin Kysiak proved in [Kys1] that if $\mathcal{I}$ is a σ-ideal on the real line, $\mathcal{I}$ contains all singletons, $\mathcal{A} \subseteq \mathscr{P}(\mathbb{R})$ and the pair $(\mathcal{I}, \mathcal{A})$ has the perfect set property, then for any subset $A \subseteq \mathbb{R}$ such that $A + A \notin \mathcal{I}$ there exists $X \subseteq A$ for which $X + X \notin \mathcal{A}$.

As a corollary we obtain the theorem of Ciesielski, Fejzić and Freiling [CFF] which says that if $A \subseteq \mathbb{R}$ is a subset of $\mathbb{R}$ such that $A + A$ has positive outer measure, then there exists a set $X \subseteq A$ such that $X + X$ is Lebesgue nonmeasurable. The analogous theorem for the σ-ideal of first category sets $\mathcal{M}$ also holds and it follows from the fact that $(\mathcal{M}, \mathcal{BP})$ has the perfect set property. In [CFF] the authors proved the following theorem: if $A + A \notin \mathcal{N}$ for $A \subseteq \mathbb{R}$, then there exists a subset $X \in \mathscr{P}(A)$ of Lebesgue measure zero such that $X + X$ is Lebesgue nonmeasurable. The analogous theorem for the Baire category holds, too.

Jacek Cichoń and Andrzej Jasiński proved the following theorem in [CJ].

Theorem 3. *If $\mathcal{I}$ is a translationally invariant σ-ideal on the real line $\mathbb{R}$ having a co-analytic base, then the following two properties are equivalent:*

- $(\exists A, B \in \mathcal{I})\ (A + B \notin \mathcal{I})$,
- $(\exists A, B \in \mathcal{I})\ (A + B \notin Bor(\mathbb{R})[\mathcal{I}])$.

$Bor(\mathbb{R})[\mathcal{I}]$ denotes the σ-algebra generated by all Borel sets and all elements of $\mathcal{I}$.

Using the structure of Vitali set, Jacek Cichoń, Alexander Kharazishvili and Bogdan Węglorz proved that if G is an uncountable, analytic proper subgroup of the real line, then there exist measurable and nonmeasurable (in the sense of Lebesgue) selectors in the quotient group $\mathbb{R}/G$.

In the Polish space theory a crucial role is played by the following cardinal coefficients.

Let $\mathcal{F}$ be a family of subsets of a Polish space X. Let

$$\begin{aligned}
&\mathrm{add}(\mathcal{F}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{F} \ \wedge\ \textstyle\bigcup \mathcal{A} \notin \mathcal{F}\},\\
&\mathrm{non}(\mathcal{F}) = \min\{|A| : A \subseteq X \ \wedge\ A \notin \mathcal{F}\},\\
&\mathrm{cov}(\mathcal{F}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{F} \wedge \textstyle\bigcup \mathcal{A} = X\},\\
&\mathrm{cov}_h(\mathcal{F}) = \min\{|\mathcal{A}| :\ \mathcal{A} \subseteq \mathcal{F} \wedge (\exists B \in Bor(X) \setminus \mathcal{F})\ B \subseteq \bigcup \mathcal{A}\},\\
&\mathrm{cof}(\mathcal{F}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{F} \ \wedge\ (\forall B \in \mathcal{F})(\exists A \in \mathcal{A})\ B \subseteq A\},\\
&\mathrm{Cof}(\mathcal{F}) = \min\{|\mathcal{A}| :\ \mathcal{A} \subseteq \mathcal{F} \ \wedge\ (\forall B \in \mathcal{F})(\exists A \in \mathcal{A})\ \ A \subseteq B\}.
\end{aligned}$$

Additionally, the following two cardinal numbers describe the smallest size of unbounded and dominating families, respectively, on the Baire space ω^ω:

$$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \omega^\omega \wedge (\forall x \in \omega^\omega)(\exists y \in \mathcal{B})\ \neg(y \leq^* x)\}$$

$$\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \subseteq \omega^\omega \wedge (\forall x \in \omega^\omega)(\exists y \in \mathcal{D})\ x \leq^* y\}$$

(where $f \leq^* g$ stands for $(\exists m \in \omega)(\forall n \geq m) f(n) \leq g(n)$). These two cardinals are related to the previous coefficients for the σ-ideals of sets of first Baire category and of Lebesgue measure zero. The relations between these cardinals are the content of the following **Cichoń diagram**:

$$\begin{tikzcd}
& \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
& & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\omega_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}), \arrow[uu] &
\end{tikzcd}$$

where $\to$ stands for the inequality $\leq$ between cardinals. Additionally, we known that

$$\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}, \ \ \mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}.$$

In [CKP] Cichoń Kamburelis and Pawlikowski proved that if quotient algebra $Bor(X)[I]/I$ is c.c.c., then $\mathrm{cof}(\mathcal{I}) = \mathrm{Cof}(Bor(X)[\mathcal{I}] \setminus \mathcal{I})$, what gives an equality between those cardinal coefficients for σ-ideals $\mathcal{M}$ and $\mathcal{N}$ on the real line $\mathbb{R}$.

The monograph [BartJud] as well as the article [BJS] are closely related to the Cichoń diagram. In particular, the authors presented models of **ZFC** where all coefficients are either ω_1, ω_2 and all admissible (by the diagram) schemes are realised.

Description of scientific achievement.

Further we will use the standard set-theoretical notation, e.g. ω will denote the smallest infinite ordinal, $\alpha, \beta, \gamma, \xi, \eta$ will denote infinite ordinals, κ, λ will denote infinite cardinal numbers, $\mathscr{P}(X)$ will denote the power set of X, $[X]^{<\kappa}$, $[X]^{\leq\kappa}$, $[X]^\kappa$ will denote the families of all subsets of X of size less than, less or equal to, equal to κ, respectively. A separable topological space X is a Polish space if it is complete and metrizable. We will consider only uncountable Polish spaces. By $Bor_\tau(X)$ we will denote the σ-algebra of Borel sets on a topological space (X, τ), and, if the topology is clear from a context, this family is simply denoted by $Bor(X)$. By $\mathcal{M}, \mathcal{N}$ we will denote σ-ideals of sets of the first category sets (on an uncountable Polish space) and of sets of Lebesgue measure zero (on a Euclidean space $\mathbb{R}^n$), respectively. We say that $\mathcal{I} \subseteq \mathscr{P}(X)$ is an ideal with a Borel base on a Polish space X if $(\forall A \in \mathcal{I})(\exists B \in Bor(X))\ A \subseteq B \wedge B \in \mathcal{I}$.

Let $\mathcal{I}$ be a fixed σ-ideal on a Polish space X, then the σ-algebra $Bor(X)[\mathcal{I}]$ generated by the family $Bor(X) \cup \mathcal{I}$ is called the σ-algebra of measurable sets relative to the σ-ideal. Then such σ-algebra can be written as $\{B \triangle I :\ (B, I) \in Bor(X) \times \mathcal{I}\}$, where $\triangle$ stands for the symmetric difference of two sets. Every subset of a Polish space X is **$\mathcal{I}$-measurable** if and only if it is a member of the σ- algebra $Bor(X)[\mathcal{I}]$. Similarly, a mapping which is $Bor(X)[\mathcal{I}]$-measurable will be called for short $\mathcal{I}$-measurable. Each family $\mathcal{A} \subseteq \mathscr{P}(X)$ is called **$\mathcal{I}$-summable** if for any subfamily $\mathcal{A}' \subseteq \mathcal{A}$ the union $\bigcup \mathcal{A}$ is $\mathcal{I}$-measurable. For example, every family of open sets of a Polish space is $\mathcal{I}$-summable for any σ-ideal with a Borel base. Each subset of the real line is $\mathcal{N}$-nonmeasurable iff it is not Lebesgue measurable. Analogously, each subset of a Polish space is $\mathcal{M}$-nonmeasurable if it does not have the Baire property.

One of the main notions I focused on in my research is the complete nonmeasurability with respect to an ideal containing all singletons (defined on a Polish space).

Definition 1. *Let $\mathcal{I} \subseteq \mathscr{P}(X)$ be a fixed σ-ideal with a Borel base, containing all singletons in a Polish space X. A set $A \in \mathscr{P}(X)$ is* **completely $\mathcal{I}$-nonmeasurable** *when*

$$(\forall B \in Bor(X) \setminus \mathcal{I})\ (A \cap B \neq \emptyset \wedge A^c \cap B \neq \emptyset).$$

For example, each set which is completely $[X]^{\leq\omega}$-nonmeasurable is a Bernstein set, each set which is completely $\mathcal{N}$-nonmeasurable on the real line has an inner measure equal to zero and its completion has an inner measure equal to zero, too. Each completely $\mathcal{M}$-nonmeasurable set does not have the Baire property in any nonempty open set.

For the σ-ideal of all null subsets of the Euclidean space $\mathbb{R}^n$, the notion of completely $\mathcal{N}$-nonmeasureability agree with the notion of saturated non-measurable set. In the case of the σ-ideal of all meager subsets of $\mathbb{R}^n$, the notion of the set which is complete $\mathcal{M}$-nonmeasurable is strictly connected to the $(*)$ property. A subset $A \subseteq \mathbb{R}^n$ has $(*)$ property iff for any Baire measurable set $B \subseteq \mathbb{R}^n$ if $B \subseteq A$ or $B \subseteq A^c$ then B is meager set. Saturated non-measurable and also subsets of the Euclidean space which has $(*)$ property are investigated and can be found in some monographs as for example in Marek Kuczma book [Kucz] (see section 3.3. Saturated non-measurable sets).

A family $\mathcal{A}$ of subsets of a set X is point-small iff

$$\{x \in X :\ \bigcup\{A \in \mathcal{A} :\ x \in A\} \notin \mathcal{I}\} \in \mathcal{I}.$$

In [H1] we have obtained the following result.

Theorem 4 ([H1, Thm 3.2])**.** *If $\mathcal{I}$ is a σ-ideal with a Borel base on a Polish space X and $\mathrm{cov}_h(\mathcal{I}) = \mathrm{Cof}(Bor(X)[\mathcal{I}] \setminus \mathcal{I})$, then for each point-small family $\mathcal{A} \subseteq \mathcal{I}$ such that $X \setminus \bigcup \mathcal{A} \in \mathcal{I}$ there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$, such that the union $\bigcup \mathcal{A}'$ is completely $\mathcal{I}$-nonmeasurable.*

The proof of this theorem as proofs of many such claims is based on a transfinite induction. We construct a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ and a set $S \subseteq X$ such that $S \cap \bigcup \mathcal{A}' = \emptyset$, $S \cap B \neq \emptyset$, and $\bigcup \mathcal{A}' \cap B \neq \emptyset$, for each B from some cofinal family $\mathcal{F} \subseteq Bor(X) \setminus \mathcal{I}$. The construction is possible because $cov_h(\mathcal{I}) = \mathrm{Cof}(Bor(X)[\mathcal{I}] \setminus \mathcal{I})$.

It should be noted that for some configuration of cardinal coefficients the theorem about nonmeasurability of unions of sets from a fixed σ-ideal on Polish space occurs in a large generality.

Theorem 5 ([H1, Thm 3.1])**.** *Let $\mathcal{I}$ be a fixed σ-ideal on a Polish space X such that there exists a completely $\mathcal{I}$-nonmeasurable set of size less than $cov_h(\mathcal{I})$. Then for any family $\mathcal{A} \subseteq \mathcal{I}$, such that $X \setminus \bigcup \mathcal{A} \in \mathcal{I}$, there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$, such that the union $\bigcup \mathcal{A}'$ is completely $\mathcal{I}$-nonmeasurable.*

For the σ-ideal of the first category sets on $\mathbb{R}$ it is true that in the generic extension obtained by adding ω_2 independent Cohen reals $\{c_\xi : \xi < \omega_2\}$ (here $cov(\mathcal{M}) = \omega_2 = \mathfrak{c}$) there exists a completely $\mathcal{M}$-nonmeasurable set whose cardinality is strictly smaller than $cov_h(\mathcal{M})$:

$$X = \{c_\xi + r \in \mathbb{R} :\ \xi < \omega_1 \wedge r \in \mathbb{Q}\}.$$

A similar argument works in the measure case, when we add ω_2 independent random reals to the Gödel universe L.

This theorem was applied by Yulia Kuznetsova in [Kuzn], where she considered some problems related to harmonic analysis. She asked whether for every measure zero set A on the real line, there exists a subset $B \subseteq \mathbb{R}$ such that $A + B$ is nonmeasurable. In each model where the above theorem is true for the ideal of the Lebesgue measure zero sets (e.g., in the model obtained by adding ω_2 Solovay reals to the constructible universe) the answer to Kuznetsova's question is positive. As a result, in these models any measurable homomorphism between a locally compact topological group and a topological group is continuous.

Theorem 4 was used in [H1] to prove that in Abelian Polish groups there exist subfamilies of translationally invariant σ-ideals with nonmeasurable unions. Let $(G, +)$ be a fixed Polish Abelian group. An ideal $\mathcal{I} \subseteq \mathscr{P}(G)$ is translationally invariant on G if and only if for any $A \in \mathcal{I}$ and any $g \in G$, we have $g + A = \{g + a \in G : a \in A\} \in \mathcal{I}$. We say that a set $C \subseteq G$ is an $\mathcal{I}$-Gruenhage set, if for any set $B \in Bor(G)[\mathcal{I}] \setminus \mathcal{I}$ and any set $T \in [G]^{<\mathfrak{c}}$ the set $B \setminus (C+T)$ is not empty. Darji and Keleti [DK] proved that if $C \subset \mathbb{R}$ is a compact set of packing dimension $dim_p(C) < 1$, then $\mathbb{R} \neq T + C$ for any $T \in [\mathbb{R}]^{<\mathfrak{c}}$. With this assertion, it is not difficult to show that the classical Cantor set C is an $\mathcal{N}$-Gruenhage set.

Namely, the following theorem was proved in [H1].

Theorem 6 ([H1, Thm 5.2])**.** *If $\mathcal{I}$ is a translationally invariant σ-ideal with a Borel base on a Polish Abelian group $(G,+)$, then for every set $C \subseteq G$, for which $C \cup -C$ is an $\mathcal{I}$-Gruenhage set, there exists $P \subseteq G$, such that $P + C$ is a completely $\mathcal{I}$-nonmeasurable set in G.*

In the proof of this theorem we used the fact that for the symmetric relation

$$R = \{(x,y) \in G : \ x - y \in C \vee y - x \in C\},$$

the family $\mathcal{A} = \{R[x] \in \mathcal{I} : \ x \in G\}$ satisfies the hypothesis of Theorem 4.

It is a natural question whether the conclusion of the above theorem can be strengthened to the condition $P \subseteq C$. The answer is positive in the case of the classical Cantor set. To prove this result we used the ultrafilter method, discussed in the introduction to this section.

Theorem 7 ([H1, Cor 5.10])**.** *If C is the classical Cantor set, then there exists a subset $P \subseteq C$ for which the algebraic sum $P+C$ is nonmeasurable in the sense of Lebesgue and does not have the Baire property.*

Using the fact that every uncountable Borel set has cardinality $\mathfrak{c}$, we obtain the following theorem.

Theorem 8 ([H1, Thm 4.1])**.** *If X is an uncountable Polish space and $\mathcal{A} \subseteq [X]^{\leq\omega}$ is a point-countable family (i.e. $\{A \in \mathcal{A} \ x \in A\} \in [\mathcal{A}]^{\leq\omega}$ for every $x \in X$), such that $\bigcup \mathcal{A} = X$, then there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is a Bernstein set.*

As one can easily see, if $\mathfrak{c}$ is a regular cardinal number, then the countable-point property of $\mathcal{A}$ can be replaced, e.g. $|\{A \in \mathcal{A} : \ x \in A\}| < \mathfrak{c}$ for every $x \in X$. This condition cannot be generalised to $\mathfrak{c}$-point families, for it is well known that if **CH** holds, then there exists a summable family $\mathcal{A} \subseteq [\mathbb{R}]^{\leq\omega}$ covering $\mathbb{R}$, i.e., such that the union of any subfamily is in the σ-field $Bor(\mathbb{R})[[\mathbb{R}]^{\leq\omega}] = Bor(\mathbb{R})$ generated by all Borel sets $Bor(X)$ and the σ-ideal of all countable subsets. Moreover, for each uncountable subfamily $\mathcal{A}' \subseteq \mathcal{A}$ we have $\bigcup \mathcal{A}' = \mathbb{R}$ and, of course, if it is countable, then $\bigcup \mathcal{A}'$ is a countable set.

Theorem 4.4 in the same article [H1] shows a relationship between the summability of a family of closed sets in the Polish space and the Cantor-Bendixon rank. Let $A \subseteq X$ be any subset of a topological space X. By A' we denote the set of all condensation points of A. Using induction on the ordinals ON define A^α in the following way:

$$A^\alpha = \begin{cases} (A^\beta)' & \text{if } \alpha = \beta + 1 \\ \bigcap \{A^\xi : \xi < \alpha\} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Theorem 9 ([H1, Thm 4.4])**.** *Assume that $\mathcal{I}$ is a σ-ideal with a Borel base on a Polish space X. If $\mathcal{A} \subseteq \mathscr{P}(X)$ is an $\mathcal{I}$-summable family of countable closed sets of bounded countable Cantor-Bendixon rank, i.e.*

$$(\exists \alpha < \omega_1)(\forall A \in \mathcal{A}) \ (A^\alpha = \emptyset),$$

then $\bigcup \mathcal{A} \in \mathcal{I}$.

It is an immediate application of this theorem that if $\mathcal{A}$ is a family of closed countable sets with bounded countable Cantor-Bendixon rank and $\bigcup \mathcal{A} \notin \mathcal{I}$, then there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$, such that the union $\bigcup \mathcal{A}'$ is $\mathcal{I}$-nonmeasurable.

In $[H2]$ (jointly with Szymon Żeberski) we investigate the existence of subfamilies of families of sets from a fixed σ-ideal $\mathcal{I}$ whose unions are completely $\mathcal{I}$-nonmeasurable. We have obtained results concerning families of sets for which there exists a parametrisation which is regular in terms of descriptive complexity. We introduce the following notation. Let $F \subseteq X \times Y$ be a fixed relation. For each $x \in X$ and $y \in Y$ let

$$F_x = \{v \in Y : (x, v) \in F\} \text{ and } F^y = \{u \in X : (u, y) \in F\},$$

$\pi_X[F] = \bigcup\{F^y : y \in X\}$ and $\pi_Y[F] = \bigcup\{F_x : x \in X\}$ ($\pi_X[F]$ and $\pi_Y[F]$ denote the projections of F onto X, Y, respectively). If $T \subseteq Y$, then $F^{-1}[T] = \{x \in X : F_x \cap T \neq \emptyset\}$ denotes the preimage of T by F.

Let π be a fixed partition of a Polish space X. The π-saturation of a set $A \subseteq X$ is defined as

$$A^* = \bigcup\{E \in \pi : E \cap A \neq \emptyset\}.$$

A partition π is Borel measurable if the π-saturation of every open set is a Borel set, π is strongly Borel measurable if the π-saturation of any closed set is a Borel set. Every open set in a Polish space is a countable union of closed sets, which implies that each strongly Borel measurable partition is Borel measurable. The concept of strong Borel measurability allows us to find subfamilies with unions which are completely $\mathcal{I}$-nonmeasurable in a fairly wide class of partitions (of a Polish space) into closed sets from a σ-ideal $\mathcal{I}$.

Theorem 10 ([H2, Thm 2.1]). *Let $\mathcal{I}$ be a σ-ideal with a Borel base such that*

$$(\forall B \in Borel(X) \setminus \mathcal{I})(\exists F \in Clo(X)) \ (F \subseteq B \wedge F \notin \mathcal{I}).$$

If $\mathcal{A} \subseteq \mathcal{I}$ is a strongly Borel measurable partition X into closed sets, then there exists $\mathcal{A}' \subseteq \mathcal{A}$ such that the union $\bigcup \mathcal{A}'$ is completely $\mathcal{I}$-nonmeasurable.

Here we present a sketch of the proof. By Theorem 4, it is sufficient to prove that $cov_h(\mathcal{I}) = 2^\omega$. For this purpose we choose a perfect set $F \notin \mathcal{I}$ contained in a fixed Borel set $B \in Bor(X) \setminus \mathcal{I}$. Then $\pi = \{E \cap F : E \in \mathcal{F}\}$ is a strongly Borel measurable partition of F and therefore it is Borel measurable as well. Now we can use the Kuratowski – Ryll-Nardzewski selector theorem. Let S be a Borel selector of the partition π. Then $|S| = \mathfrak{c}$, because if S were a countable set then F would be included in countably many elements of the partition π, thus $F \in \mathcal{I}$ would be an element of the ideal, which contradicts $F \notin \mathcal{I}$.

This claim is transfered immediately to the case of Polish topological groups as follows.

Corollary 1. *Let $(G, +)$ be a Polish Abelian group and let $H < G$ a subgroup which is a perfect set and belongs to an ideal $\mathcal{I}$. If $\mathcal{I}$ is translationally invariant in G, $\mathcal{I}$ has a Borel base, and $\mathcal{I}$ possess $\mathcal{I}$-positive perfect set property, then there exists a set of translations $T \subseteq G$, such that $T + H$ is completely $\mathcal{I}$-nonmeasurable in the group G.*

Theorem 11 ([H2, Thm 2.2]). *Let $\mathcal{I} \supseteq [X]^{\leq\omega}$ be a σ-ideal with a Borel base on a Polish space X and let $f : X \to Y$ be an $\mathcal{I}$-measurable mapping from X into a topological space Y such that for any $y \in Y$ $f^{-1}[\{y\}] \in \mathcal{I}$. Then there exists a subset $T \subseteq Y$ such that $f^{-1}[T]$ is completely $\mathcal{I}$-nonmeasurable.*

Without loss of generality it can be assumed that f is Borel measurable, and then for each Borel set $B \in Bor(X) \setminus \mathcal{I}$, the projection of the set $(B \times Y) \cap f$ onto the space Y is analytic and, hence, it is countable or it has the cardinality $\mathfrak{c}$. If the projection is countable, then B can be covered by countably many members of $\mathcal{I}$, which is impossible. Then $cov_h(\{f^{-1}[\{y\}] : y \in Y\}) = \mathfrak{c}$, and then by Theorem 4 the conclusion of Theorem 11 is proved.

The above result can be extended to a theorem which can be stated in terms of multifunctions.

Theorem 12 ([H2, Thm 2.3]). *If $\mathcal{I}$ is a c.c.c. σ-ideal such as in the previous theorem, and $F : X \to Y$ is an $\mathcal{I}$-measurable multifunction such that $F(x) \in [Y]^{<\omega}$ for every $x \in X$, then there exists a set $T \subseteq Y$ such that $F^{-1}[T]$ is completely $\mathcal{I}$-nonmeasurable.*

The proof of this theorem is based on the Kuratowski – Ryll-Nardzewski selector theorem and the following theorem.

Theorem 13. *Let $\mathcal{I}$ be a c.c.c. σ-ideal with a Borel base on a Polish space X and let $\mathcal{F} \subseteq \mathcal{I}$ be a family of sets such that*

- *$\mathcal{F}$ is point-finite,*
- *$(\forall B \in Bor(X) \setminus \mathcal{I})(B \subseteq [\bigcup \mathcal{F}]_{\mathcal{I}} \longrightarrow |\{F \in \mathcal{F} : F \cap B \neq \emptyset\}| = \mathfrak{c})$.*

Then there exists a subfamily $\mathcal{F}' \subseteq \mathcal{F}$, such that the union $\bigcup \mathcal{F}'$ is completely $\mathcal{I}$-nonmeasurable in the Borel envelope $[\bigcup \mathcal{F}]_{\mathcal{I}}$.

Here, for $A \in \mathscr{P}(X)$, $[A]_{\mathcal{I}} = X \setminus \bigcup \mathcal{A}$, where $\mathcal{A} \subseteq Bor(X) \setminus \mathcal{I}$ is a maximal antichain of Borel $\mathcal{I}$-positive sets which are disjoint from the set A. As $\mathcal{F}$ is point-finite, we can find a subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ which has the same Borel envelope as that of $\mathcal{F}$ (i.e. $[\bigcup \mathcal{F}_0]_{\mathcal{I}} = [\bigcup \mathcal{F}]_{\mathcal{I}}$) such that $cov_h(\mathcal{F}_0) = \mathfrak{c}$. Then, by transfinite induction, we can find a subfamily $\mathcal{F}' \subseteq \mathcal{F}_0$ as in the conclusion of Theorem 13.

Without loss of generality, we can assume that our multifunction F is Borel measurable. So if we choose any $\mathcal{I}$-positive Borel set B, then using the Kuratowski – Ryll-Nardzewski theorem on selectors applied to the restricted function $F \restriction B$ we can find a Borel selector $s \subseteq F \restriction B$, which has size $\mathfrak{c}$, because $\mathcal{F}$ is a point-finite uncountable family. Thus, the second assumption in the auxiliary theorem is also satisfied which allows us to find a subfamily of $\mathcal{F}'$ which satisfies the assertion of Theorem 12.

If we assume that a σ-ideal $\mathcal{I}$ with a Borel base is *c.c.c.*, then $Bor(X)[\mathcal{I}]$ contains all analytic sets, and then, by the Theorem 12, we obtain the following result.

Theorem 14 ([H2, Thm 2.4])**.** *Let X and Y be Polish spaces and let $\mathcal{I}$ be a c.c.c. σ-ideal with a Borel base on X. Let $F \subseteq X \times Y$ be an analytic relation in the product $X \times Y$ such that*

(1) $(\forall y \in Y)\ (F^y \in \mathcal{I})$,
(2) $X \setminus \pi_X(F) \in \mathcal{I}$,
(3) $(\forall x \in X)\ (|F_x| < \omega)$.

Then there exists $T \subseteq Y$ for which $F^{-1}[T]$ is completely $\mathcal{I}$-nonmeasurable.

The above three theorems are provable in **ZFC** theory, but we use additional conditions involving some regularity of a family of sets from $\mathcal{I}$. A natural question is whether the conclusions of these assertions are true in the general case for a point-finite family $\mathcal{A} \subseteq \mathcal{I}$, for which the union is the whole space, maybe, except a set from $\mathcal{I}$. Unfortunately, we do not know the answer, however the nonexistence of a quasi-measurable cardinal which is not greater than $\mathfrak{c}$ gives positive answer, see [$H6$].

We say that an uncountable cardinal κ is **quasi-measurable** if there exists a κ additive ideal $\mathcal{I} \subseteq \mathscr{P}(\kappa)$, which is *c.c.c.* (i.e. each antichain in the algebra $\mathscr{P}(\kappa)/\mathcal{I}$ is at most countable).

The following theorem is the main result of [$H6$], a joint article with Szymon Żeberski.

Theorem 15 ([H6, Thm 3.3])**.** *Assume that there is no quasi-measurable cardinal $\kappa \leq \mathfrak{c}$. If $\mathcal{I}$ is a c.c.c. σ-ideal on a Polish space X and $\mathcal{A} \subseteq \mathcal{I}$ is a point-finite cover of X, then there exists a family of pairwise disjoint families $\{\mathcal{A}_\xi \subseteq \mathcal{A} : \xi < \omega_1\}$ such that each union $\bigcup \mathcal{A}_\xi$ is completely $\mathcal{I}$-nonmeasurable in the space X.*

The proof of this theorem is based on two lemmas in [Zeb] and Theorems 2.1 and 2.2 in [$H6$].

Lemma 1 ([Zeb], Lemma 3.4)**.** *If $\mathcal{I}$ is a c.c.c. σ-ideal on a Polish space X and $\mathcal{A} \subseteq \mathcal{I}$ is a point-finite family, such that $\bigcup \mathcal{A} \notin \mathcal{I}$ and the algebra $\mathscr{P}(\mathcal{A})/\mathcal{I}$ is not c.c.c., then there exists an uncountable collection of pairwise disjoint subfamilies $\{\mathcal{A}_\xi : \xi < \omega_1 \wedge \mathcal{A}_\xi \subseteq \mathcal{A}\}$ with the same Borel envelope that is not in $\mathcal{I}$, i.e. $[\bigcup \mathcal{A}_\xi]_{\mathcal{I}} = [\bigcup \mathcal{A}_\eta]_{\mathcal{I}} \neq 0$ for any $\xi, \eta < \omega_1$.*

Lemma 2 ([Zeb], Lemma 3.5)**.** *If $\mathcal{I}$ is a c.c.c. σ-ideal on a Polish space X and $\mathcal{A}$ is any point-finite covering of X, then the set*

$$\{A \in \mathcal{A} :\ (\exists B \in Bor(X) \setminus \mathcal{I})\ B \subseteq A\}$$

is at most countable.

In [$H6$] we proved the following theorem.

Theorem 16 ([H6, Thm 2.1])**.** *Assume that $\mathcal{A} \subseteq \mathcal{I}$ is a covering of a Polish space X such that for every set $D \in [X]^{<\mathfrak{c}}$ the union $\bigcup_{x\in D} \bigcup\{A \in \mathcal{A} : \ x \in A\}$ does not contain any $\mathcal{I}$-positive Borel set $B \in Bor(X) \setminus \mathcal{I}$. Then the family $\mathcal{I}$ contains $\mathfrak{c}$ many pairwise disjoint subfamilies $\{\mathcal{A}_\xi \subseteq \mathcal{A} : \xi < \mathfrak{c}\}$, such that each union $\bigcup \mathcal{A}_\xi$ is completely $\mathcal{I}$-nonmeasurable in X.*

The above theorem gives a result which is used in the proof of the main result of $[H6]$.

Theorem 17 ([H6, Thm 2.2])**.** *Assume that there is no quasi-measurable $\kappa < \mathfrak{c}$. If $\mathcal{A} \subseteq \mathcal{I}$ is a family of subsets of X Polish, such that for each $x \in X$, $|\{A \in \mathcal{A}\ x \in A\}| < \mathfrak{c}$, and $\bigcup \mathcal{A} \notin \mathcal{I}$, then $\mathscr{P}(\kappa)/\mathcal{I}$ is not ccc.*

We will sketch the proof of Theorem 15. By transfinite induction we are finding a family of pairwise disjoint $\mathcal{I}$-positive Borel sets $\{B_\sigma :\ \sigma < \gamma\}$ and a family $\{\mathcal{A}^\sigma_\xi :\ \xi < \omega_1\}$ such that,

- $(\forall \xi, \eta < \omega_1)\ (\xi \neq \eta \longrightarrow \mathcal{A}^\sigma_\xi \cap \mathcal{A}^\sigma_\eta = \emptyset)$
- $(\forall \xi < \omega_1)\ (B^\sigma \in \ulcorner \bigcup \mathcal{A}^\sigma_\xi \setminus \bigcup_{\rho<\sigma} B_\rho \urcorner_\mathcal{I})$

where for any $Y \subseteq X$, $\ulcorner Y \urcorner_\mathcal{I}$ is the set of $\subseteq_\mathcal{I}$-minimal elements in $\{B \in Bor(X) :\ Y \subseteq_\mathcal{I} B\}$ which is partially ordered by the relation $\subseteq_\mathcal{I}$ (i.e. $u \subseteq_\mathcal{I} v \longleftrightarrow u \setminus v \in \mathcal{I}$), the property *c.c.c.* of $\mathcal{I}$ guarantees that $\ulcorner Y \urcorner_\mathcal{I}$ is not empty. This same property of $\mathcal{I}$ implies that $\gamma < \omega_1$ is a countable ordinal.

For a fixed $\sigma < \gamma$, let $\mathcal{A}^\sigma = \{A \setminus \bigcup_{\rho<\sigma} B_\rho :\ \ A \in \mathcal{A} \setminus \bigcup_{\rho<\sigma} \mathcal{A}^\rho\}$. If $\bigcup \mathcal{A}^\sigma \in \mathcal{I}$, then the process is completed. Otherwise, if $\bigcup \mathcal{A}^\sigma \notin \mathcal{I}$ then by Theorem 17 $\mathscr{P}(\mathcal{A}^\sigma)/\mathcal{I}$ is not *c.c.c.*. Then Lemma 1 allows us to find a family $\{\mathcal{A}^\sigma_\xi :\ \xi < \omega_1\}$ such that for any $\xi, \eta < \omega_1$ $[\bigcup \mathcal{A}^\sigma_\xi]_\mathcal{I} = [\bigcup \mathcal{A}^\sigma_\eta]_\mathcal{I} \neq 0$. Then we can find an $\mathcal{I}$-positive Borel set B^σ, for example any member of $\ulcorner \bigcup \mathcal{A}^\sigma_0 \setminus \bigcup_{\rho<\sigma} B_\rho \urcorner$.

Then, for each $\xi < \omega_1$, the family $\mathcal{A}'_\xi = \bigcup\{\mathcal{A}^\sigma_\xi : \sigma < \gamma\}$ is point-finite and has Borel envelope which is equal to entire space i.e. $[\bigcup \mathcal{A}'_\xi]_\mathcal{I} = [\bigcup_{\sigma<\gamma} B^\sigma]_\mathcal{I} = X$. Moreover, a family $\{\bigcup \mathcal{A}'_\xi :\ \xi < \omega_1\}$ is point-finite and then Lemma 2 ensures that for at most countably many $\xi < \omega_1$ the union $\bigcup \mathcal{A}'_\xi$ contains an $\mathcal{I}$-positive Borel set. Hence there exists $\beta < \omega_1$ such that for every $\xi > \beta$, the union $\bigcup \mathcal{A}'_\xi$ is completely $\mathcal{I}$-nonmeasurable in X.

In every model of **ZFC** in which the additivity a σ-ideal is equal to $\mathfrak{c}$ (for example, it is true for the measure and the category ideals on $\mathbb{R}$ under Martin's axiom) there exists a family $\mathcal{A} \subseteq \mathcal{I}$, which is summable in the following sense:

$$(\forall \mathcal{C} \in \mathscr{P}(\mathcal{A}))\ \big(\bigcup \mathcal{C} \in \mathcal{I} \vee (\bigcup \mathcal{C})^c \in \mathcal{I}\big).$$

This family is the tower described as follows. Let us enumerate a real line $\mathbb{R} = \{x_\xi : \xi < \mathfrak{c}\}$, then for every $\alpha < \mathfrak{c}$, let $A_\alpha = \{x_\xi : \xi < \alpha\}$, then $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$. This family is a point-big, namely the star $\mathcal{A}(x) = \{A \in \mathcal{A}\ x \in A\}$ of any point $x \in \mathbb{R}$ lies in the co-ideal $\mathcal{I}^*$, i.e. its complement is an element of $\mathcal{I}$.

This observation was the motivation to undertake the research of $[H3]$ on measurability of unions of point-big families. The following two theorems are the main results of this paper.

Theorem 18 ([H3, Thm 2.1])**.** *If $\mathcal{I}$ is a σ-ideal with a Borel base on a Polish space X, then for each family $\mathcal{A} \subseteq \mathcal{I}$ satisfying the following conditions:*

(1) $(\forall x \in X)\ (|\mathcal{A}(x)| = \mathfrak{c})$,
(2) $(\forall x, y \in X)(x \neq y \longrightarrow |\mathcal{A}(x) \cap \mathcal{A}(y)| \leq \omega)$,
(3) $cov_h(\mathcal{A}) = \mathfrak{c}$,

there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$, where $\bigcup \mathcal{A}'$ is completely $\mathcal{I}$-nonmeasurable,

Theorem 19 ([H3, Thm 2.2])**.** *If $\mathcal{I}$ is a σ-ideal with a Borel base on a Polish space X, then for each family $\mathcal{A} \subseteq \mathcal{I}$ satisfying the conditions:*

(1) $\bigcup \mathcal{A} = X$,
(2) $(\forall x, y \in X)(x \neq y \longrightarrow |\mathcal{A}(x) \cap \mathcal{A}(y)| \leq \omega)$,
(3) $cov_h(\mathcal{A}) = \mathfrak{c}$,

there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$, where $\bigcup \mathcal{A}'$ is $\mathcal{I}$-nonmeasurable.

Here, $cov_h(\mathcal{A}) = min\{|\mathcal{D}| :\ \mathcal{D} \subseteq \mathcal{A} \wedge (\exists B \in Bor(X) \setminus \mathcal{I})\ B \subseteq \bigcup \mathcal{D}\}$.

In the rest of this paper, I gave definable examples of applications of above theorems. These examples are constructible in any model of **ZFC**. For this purpose, I introduced the concept of the so-called tiny perfect set relative to the fixed family of subsets of a fixed Polish group.

Definition 2 (perfect tiny set)**.** *If $\mathcal{A} \subseteq \mathcal{I}$ is a family of subsets of a fixed Polish group $(G, +)$ and if $\mathcal{I}$ is a translationally invariant σ-ideal with a Borel base on G, then we say that $P \subseteq X$ is a perfect tiny set with respect to the family $\mathcal{A}$, if:*

- $(\forall B \in Bor(X) \setminus \mathcal{I})(\exists s \in G)\ (|(S + P) \cap B| = \mathfrak{c})$,
- $(\forall A \in \mathcal{A})(\forall t \in G)\ (|P \cap (t + A)| \leq \omega)$.

If we consider the family $\mathcal{A}$ of all lines in the Euclidean space $\mathbb{R}^n$ of dimension at least equal to two, then the sphere $S^n \subseteq \mathbb{R}^n$ is an example of the perfect tiny set with respect to $\mathcal{A}$. Similarly, every line $l \subseteq \mathbb{R}^n$ is a tiny perfect set with respect to the family of all n-dimensional spheres in $\mathbb{R}^{n+1}$.

These examples concern families of null subsets of the real plane, where cov_h is equal to $\mathfrak{c}$. The key Lemma 3.2 in $[H3]$ says that every perfect subset of a Polish group which has Haar measure, can be translated into each Borel set of positive measure in such a way that the intersection has size $\mathfrak{c}$.

If we want to prove that $cov_h(\mathcal{A}) = \mathfrak{c}$, it is sufficient to note that, if B is a set of positive Haar measure, and $\mathcal{A}' \in [\mathcal{A}]^{<\mathfrak{c}}$, then there exists $t \in G$ such that $|(t+P) \cap B| = \mathfrak{c}$ and $|(t+P) \cap \bigcup \mathcal{A}'| < \mathfrak{c}$.

From the above observations and Theorems 18 and 19 we obtain the following two conclusions.

Corollary 2 ([H3, Proposition 3.6])**.** *Let $\mathcal{I}$ be any translationally invariant σ-ideal on a Polish group $(G, +)$ with the Borel base and let us assume that $\mathcal{A} \subseteq \mathcal{I}$ is such that:*

- *there is a tiny perfect set with respect to $\mathcal{A}$,*
- $(\forall x \in G)\ (|\mathcal{A}(x)| = \mathfrak{c})$,
- $(\forall x, y \in G)\ (x \neq y \longrightarrow |\mathcal{A}(x) \cap \mathcal{A}(y)| \leq \omega)$.

Then there exists $\mathcal{A}' \subseteq \mathcal{A}$, such that $\bigcup \mathcal{A}'$ is a completely $\mathcal{I}$-nonmeasurable set in G.

Corollary 3 ([H3, Proposition 3.7])**.** *Let $\mathcal{I}$ be any translationally invariant σ-ideal on a Polish group $(G, +)$ with a Borel base and let us assume that $\mathcal{A} \subseteq \mathcal{I}$ and:*

- *there is a tiny perfect set with respect to $\mathcal{A}$,*
- $\bigcup \mathcal{A} = G$,
- $(\forall x, y \in G)\ (x \neq y \longrightarrow |\mathcal{A}(x) \cap \mathcal{A}(y)| \leq \omega)$.

Then there exists $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is a $\mathcal{I}$-nonmeasurable in G.

Using Corollary 3 we can easily obtain the following result.

Corollary 4. *If $n \geq 2$ and $\mathcal{L}$ is any family of lines in $\mathbb{R}^n$ such that $\bigcup \mathcal{L} = \mathbb{R}^n$, then there exists a subfamily $\mathcal{L}' \subseteq \mathcal{L}$ such that $\bigcup \mathcal{L}'$ is a nonmeasurable set with respect to the Lebesgue measure.*

A family of lines in a Euclidean space can be replaced by a family of spheres.

Theorem 20 ([H3, Thm 3.10])**.** *For each family circles of a fixed radius which covers the plane, there exists a subfamily such that its union is nonmeasurable with respect to the Lebesgue measure and there exists a subfamily such that the union does not have the Baire property. Moreover, if we assume that each point on the plane is covered by $\mathfrak{c}$ many circles from our family, then we can find a subfamily, whose union is completely $\mathcal{I}$-nonmeasurable, where $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}\}$.*

Passing from the plane to the n-dimensional Euclidean space, we can prove that:

Theorem 21 ([H3, Thm 3.11])**.** *If $\mathcal{A} \subseteq \{S(x, r) \in \mathscr{P}(\mathbb{R}^n) :\ x \in \mathbb{R}^n \wedge r > 0\}$ is an arbitrary family of $n-1$ spheres in $\mathbb{R}^n$ which satisfies the condition*

$$(\forall x \in \mathbb{R}^n)\ \ (\{y \in \mathbb{R}^n :\ (\exists r > 0)\ (x \in S(y, r) \wedge S(y, r) \in \mathcal{A})\} \text{ has positive measure}).$$

Then there exists $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is completely $\mathcal{N}$-nonmeasurable in $\mathbb{R}^n$.

The articles of Sierpiński [Sier], Cichoń and Jasiński [CJ], Kysiak [Kys1] and the paper written by Ciesielski, Freiling and Fejźić [CFF] were main motivation of the paper [$H5$] written together with Szymon Żeberski.

In [$H5$] we substite the addition operation on the real line by another binary operation defined on a Polish space. One of somewhat technical results which, however, gave nice applications, is the following theorem.

Theorem 22 ([H5, Thm 3.4]). *Let T be an arbitrary set, $\mathcal{I}$ be a σ-ideal with a Borel base on a Polish space X. Let $\lambda < \mathfrak{c}$, or $\lambda = \mathfrak{c}$ and λ be regular. If $(R_\alpha)_{\alpha<\mathfrak{c}} \in (\mathscr{P}(T^2 \times X))^{\mathfrak{c}}$ be a seqence of relations of length $\mathfrak{c}$, such that for each $\alpha < \mathfrak{c}$:*

(1) $\{x : \ |R_\alpha^{-1}(x)| \neq \mathfrak{c}\} \in \mathcal{I}$,
(2) $|R_\alpha \cap S| < \lambda$ *for every S of the form $\Delta, \{a\} \times T \times \{x\}, T \times \{a\} \times \{x\}$, where $a \in T, x \in X$,*
(3) $(\forall B \in Bor(X) \setminus \mathcal{I})(\exists a \in T)\ (|R_\alpha^{-1}(B) \cap \{a\} \times T| = \mathfrak{c})$,
(4) $(\forall (a,b) \in T^2)\ (|R_\alpha(a,b)| < \lambda)$.

Then there exists $A \subseteq T$ such that, for any $\alpha < \mathfrak{c}$, the image $R_\alpha(A^2)$ is a completely $\mathcal{I}$-nonmeasurable in X.

In the proof of Theorem 22 we used the transfinite induction.

Theorem 22 implies two results concerning measure and category.

Corollary 5 ([H5, Cor 3.3]). *There is a subset A of the real line $\mathbb{R}$ such that for every C^1-function $f : \mathbb{R}^2 \to \mathbb{R}$ onto $\mathbb{R}$ the image $f[A \times A]$ is a completely $\mathcal{N}$-nonmeasurable.*

In the proof of this theorem we have used the fact such that for any C^1-function $f : \mathbb{R}^2 \to \mathbb{R}$ onto $\mathbb{R}$ the preimage of any Borel set of positive Lebesgue measure is still of positive (two-dimensional) Lebesgue measure.

Corollary 6 ([H5, Cor 3.4]). *There is a subset A of the real line $\mathbb{R}$ such that for every C^1-function $f : \mathbb{R}^2 \to \mathbb{R}$ onto $\mathbb{R}$ which has non-zero partial derivatives outside a set of the first category, the image $f[A \times A]$ is completely $\mathcal{M}$-nonmeasurable.*

In [CFF], Ciesiselski, Freiling and Fejźić constructed a null perfect set C, such that $C + C$ is an interval and the algebraic sum $A+A$ cannot be a Bernstein set for any $A \subseteq C$. (The key fact implying this theorem states that every point $x \in C + C$ can be represented in finitely many ways.) However, we have obtained a result which gives a set which is completely nonmeasurable with respect to a σ-ideal.

Theorem 23 ([H5, Thm 3.5]). *If T_1, T_2 are arbitrary sets and $\mathcal{I}$ is a σ-ideal with a Borel base on Polish space X, then for every function $f : T_1 \times T_2 \to X$ satisfying the conditions:*

(1) *F is "onto",*
(2) $\{x \in X : \ \omega < |f^{-1}(x)|\} \in \mathcal{I}$,
(3) *for every Borel set $B \in Bor(X) \setminus \mathcal{I}$:*
$$|\{A \in T_1 : \ \ |\{a\} \times T_2 \cap f^{-1}(B)| = \mathfrak{c}\}| = \mathfrak{c},$$

there are $A \subseteq T_1$, $B \subseteq T_2$, for which the image of $f(A \times B)$ is completely $\mathcal{I}$-nonmeasurable. Moreover, if $T_1 = T_2$ then there exists $A \subseteq T_1$ such that $f(A \times A)$ is completely $\mathcal{I}$-nonmeasurable in space X.

An application of the Mycielski gives the following result.

Corollary 7 ([H5, Cor 3.5]). *Let assume that we have three σ-ideals $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ with a Borel base on Polish spaces X_1, X_2, X_3, respectively. If a function $f : X_1 \times X_2 \to X_3$ satisfies the conditions*

- *f is "onto",*
- *$f^{-1}(z)$ is at most countable set for all z's from outside a set from the σ-ideal $\mathcal{I}_3$,*
- *for every Borel set $B \subseteq X_3$ outside $\mathcal{I}_3$, there exists a set $W \in Bor(X_1 \times X_2) \setminus (\mathcal{I}_1 \otimes \mathcal{I}_2)$, such that $W \subseteq f^{-1}(B)$.*

Then there are $A \subseteq X_1$, $B \subseteq X_2$, such that the image $f(A \times B)$ is completely $\mathcal{I}_3$-nonmeasurable in the space X_3.

Results contained in Jacek Cichoń's article [C] were the inspiration for investigating generalised Lusin sets. As already mentioned in the introduction, $\mathbf{MA} + \neg\mathbf{CH}$ does not allow the existence of a Lusin set understood in the classical sense, but in any such model, there are $\mathfrak{c}$-Lusin sets. For a given uncountable cardinal number κ we say that an uncountable subset A of a Polish space X is a κ-Lusin set if $\kappa \leq |A|$ and its trace on each set of the first Baire category has cardinality less than κ. Of course, this set is not a set of the first Baire category and it does not have the Baire property. If the cofinality of κ is uncountable, then the family $[X]^{<\kappa}$ is a proper σ-ideal containing all singletons and it is even a $cof(\kappa)$-complete ideal. A set $A \subseteq X$ is κ-Lusin if and only if $A \notin \mathcal{M}$ and $A \cap Y \in [X]^{<\kappa}$ for each $Y \in \mathcal{M}$. This simple observation leads to the notion of an $(\mathcal{I}, \mathcal{J})$-Lusin set for σ-ideals $\mathcal{I}, \mathcal{J}$ defined on a Polish space X.

Definition 3. *Let $\mathcal{I}, \mathcal{J}$ be σ-ideals on a Polish space X. A set $A \subseteq X$ is called an $(\mathcal{I}, \mathcal{J})$-Lusin set if*

- *$A \notin \mathcal{I}$, and*
- *$(\forall Y \in \mathcal{I})\ (A \cap Y \in \mathcal{J})$.*

Moreover, if κ is a fixed cardinal number, then we say that $A \subseteq X$ is a $(\kappa, \mathcal{I}, \mathcal{J})$-Lusin, if the cardinality of A is equal to κ and A is an $(\mathcal{I}, \mathcal{J})$-Lusin set.

We say that two σ-ideals $\mathcal{I}$ and $\mathcal{J}$ are orthogonal ($\mathcal{I} \perp \mathcal{J}$) in a Polish space X, if there exists a partition $X = I \cup J$, such that $I \in \mathcal{I}$ and $J \in \mathcal{J}$. Of course, Marczewski's decomposition of the real line guarantees that $\mathcal{M} \perp \mathcal{N}$. As we know from [*H*7, Fact 1.1], if $\mathcal{I}, \mathcal{J}$ are orthogonal σ-ideals on X, then there exists an $(\mathcal{I}, \mathcal{J})$-Lusin set. However, if A is an $(\mathcal{I}, \mathcal{J})$-Lusin, then it is not at the same time a $(\mathcal{J}, \mathcal{I})$-Lusin set.

In [*H*7] we (Szymon Żeberski and me) consider the family of sets $\mathcal{A} \subseteq \mathscr{P}(X)$, which are non-equivalent with respect to a fixed family $\mathcal{F} \subseteq X^X$ of functions defined on X.

Definition 4. *Let $\mathcal{F} \subseteq X^X$ be a family of functions. We say that sets $A, B \subseteq X$ are nonequivalent with respect to $\mathcal{F}$ if*

$$(\forall f \in \mathcal{F})\ (A \neq B \longrightarrow \neg(f[A] = B \vee f[B] = A)).$$

In [*H*7], we have proved a theorem which is a generalisation of the well-known Erdös-Sierpiński duality.

Theorem 24 (Erdös-Sierpiński). *Assuming* **CH**, *there exists a bijection $f : \mathbb{R} \to \mathbb{R}$, such that*

$$(\forall A \in \mathscr{P}(\mathbb{R}))\ (A \in \mathcal{M} \longleftrightarrow f[A] \in \mathcal{N}) \wedge (A \in \mathcal{N} \longleftrightarrow f[A] \in \mathcal{M}).$$

Theorem 25 ([H7, Thm 2.1]). *Let $\mathcal{I}, \mathcal{J}$ be arbitrary σ-ideals with a Borel base on a Polish space X. If $\kappa = \mathrm{cov}_h(\mathcal{I}) = \mathrm{cof}(\mathcal{I}) \leq \mathrm{non}(\mathcal{J})$ and $\mathcal{F} \in [X^X]^{\leq\kappa}$ is any family of functions of size not greater than κ, then there exists a family $\mathcal{A}$ of cardinality κ of pairwise nonequivalent $(\kappa, \mathcal{I}, \mathcal{J})$-Lusin sets with respect to $\mathcal{F}$.*

This theorem implies the following corollary, whose hypothesis is, for example, satisfied under **CH**.

Corollary 8 ([H7, Cor 2.3]). *If $\mathcal{I}, \mathcal{J}$ are σ-ideals with a Borel base such that $\mathrm{cov}_h(\mathcal{I}) = \mathrm{non}(\mathcal{J}) = \mathfrak{c}$, then there exists $\mathfrak{c}$-many $(\mathcal{I}, \mathcal{J})$-Lusin sets which are pairwise non-equivalent with respect to the family of all $\mathcal{I}$-measurable functions.*

When $(\mathcal{I}, \mathcal{J}) = (\mathcal{N}, \mathcal{M})$ or $(\mathcal{I}, \mathcal{J}) = (\mathcal{M}, \mathcal{N})$, we obtain the following corollaries.

Corollary 9 ([H7, Cor 2.4]). *For $\mathcal{M}, \mathcal{N}$ we have*

- *Assume that $\mathrm{cov}(\mathcal{N}) = \mathfrak{c}$. There are $\mathfrak{c}$ many different $(\mathfrak{c}, \mathcal{N}, \mathcal{M})$-Lusin sets which are not equivalent to each other with respect to the family of Lebesgue measurable functions.*

- *Assume that* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. *There are* $\mathfrak{c}$ *many different* $(\mathfrak{c}, \mathcal{M}, \mathcal{N})$*-Lusin sets which are not equivalent to each other with respect to the family of Baire measurable functions.*

In the same paper, we investigated the forcing notions, which preserve the property of being an $(\mathcal{I}, \mathcal{J})$-Lusin set for a definable proper ideals $\mathcal{I}, \mathcal{J}$ in a Polish space X. In this paper we considered definable forcing notions $\mathbb{P} = Bor(X) \setminus \mathcal{I}$, which were introduced in the work of Robert Solovay[So1] for the ideals $\mathcal{N}$ and $\mathcal{M}$, and intensively studied by many mathematicians. Most results concerning this topic are contained in the monograph [Zapl]. A forcing notion $\mathbb{P} = Bor(X) \setminus \mathcal{I}$ is definable if each generic filter $G \subseteq \mathbb{P}$ (over a ground model V) is definable from a generic real for which there is a canonical name $\dot{r} \in V^{\mathbb{P}}$ such that $V[G] = V[r]$. Further we consider definable σ-ideals $\mathcal{I}$ with a Borel base for which the Borel codes for sets from $\mathcal{I} \cap Bor(X)$ are absolute between any transitive models of **ZFC** theory $M \subseteq N$. This means that: for every real $x \in \omega^\omega \cap M$

$$M \models \text{"}\#x \in Bor(X) \cap \mathcal{I}\text{"} \longleftrightarrow N \models \text{"}\#x \in Bor(X) \cap \mathcal{I}\text{"}.$$

We say that for a transitive model V of **ZFC** theory a definable forcing notion $\mathbb{P} \in V$ preserves the property of being an $(\mathcal{I}, \mathcal{J})$-Lusin set if for every $(\mathcal{I}, \mathcal{J})$-Lusin set $A \in V$ we have

$$V[G] \models \text{"}A \text{ is a } (\mathcal{I}, \mathcal{J})\text{-Lusin set"},$$

where $G \subseteq \mathbb{P}$ is a generic filter over V.

We say that a σ-ideal $\mathcal{I}$ in a Polish space X has the Fubini property if

$$(\forall A \in Bor(X \times X))\,(\{x \in X : \; A_x \notin \mathcal{I}\} \in \mathcal{I} \longrightarrow \{y \in X : \; A^y \notin \mathcal{I}\} \in \mathcal{I}).$$

We have proved the following theorem for *c.c.c.* forcings, related to the preservation of the property of being a generalized Lusin set.

Theorem 26 ([H7, Thm 3.1])**.** *Let* κ *be an uncountable cardinal number, and* $\mathcal{I}, \mathcal{J}$ *be c.c.c.* σ*-ideals which have the Fubini property. Suppose that* $\mathbb{P}_{\mathcal{I}} = Bor(X) \setminus \mathcal{I}$ *and* $\mathbb{P}_{\mathcal{J}} = Bor(X) \setminus \mathcal{J}$ *are definable forcings. Then the forcing* $\mathbb{P}_{\mathcal{I}}$ *preserves the* $(\kappa, \mathcal{I}, \mathcal{J})$*-Lusin set property.*

In particular, Solovay forcing which adds one random real to the ground model V, preserves all Sierpiński sets lying in V. Similarly, the forcing which add one Cohen real preserves all Lusin sets from the ground model.

An analogous theorem proved for a definable forcing notion which preserves a base of the ideal $\mathcal{I}$.

Theorem 27 ([H7, Thm 3.2])**.** *Let* $(\mathbb{P}, \leq)$ *a be definable forcing notion that preserves a base of a* σ*-ideal* $\mathcal{I}$ *in a Polish space* X*, namely, in any generic extension* $V[G]$

$$\{B \in Bor(X) \cap \mathcal{I} : \; B \textit{ is encoded in } V\}$$

is still a base of $\mathcal{I}$*. Assume that Borel codes for Borel sets from* $\mathcal{I}, \mathcal{J}$ *are absolute between transitive* **ZFC** *models, then the forcing* $(\mathbb{P}, \leq)$ *preserves being a* $(\mathcal{I}, \mathcal{J})$*-Lusin set.*

This theorem implies the following corollaries.

Corollary 10 ([H7, Cor 3.3])**.** *Each forcing* $\mathbb{P}$*, which preserves old reals (i.e.* $(\omega^\omega)^V = (\omega^\omega)^{V^{\mathbb{P}}}$*) and such that codes for Borel sets in the* σ*-ideals* $\mathcal{I}, \mathcal{J}$ *are absolute, preserves being a* $(\mathcal{I}, \mathcal{J})$*-Lusin set.*

Corollary 11 ([H7, Cor 3.5])**.** *Let* λ *be an ordinal number, and* $\mathbb{P}_\lambda = ((P_\alpha, \dot{Q}_\alpha) : \alpha < \lambda)$ *be a countable support iteration of length* λ*, such that for each* $\alpha < \lambda$ *we have* $P_\alpha \Vdash \text{"}\dot{Q} - \sigma\text{-closed"}$ *and coding of Borel sets from ideals* $\mathcal{I}, \mathcal{J}$ *is absolute. Then* $\mathbb{P}_\lambda$ *preserves being a* $(\mathcal{I}, \mathcal{J})$*-Lusin set.*

Results about the preservation of being a $(\mathcal{I}, \mathcal{J})$-Lusin set were based on a method introduced by Martin Goldstern in [Gold]. Let us consider an example of the usage that method. Let Ω be the family of all clopen subsets of the Cantor space 2^ω (which is countable) and consider the space $C^{random} = \{f \in \Omega^\omega : \; (\forall n \in \omega)\lambda(f(n)) < 2^{-n}\}$ (here λ is the Lebesgue measure) and Ω is equipped with the discrete topology. For $n \in \omega$, $f \in C^{random}$, $g \in 2^\omega$, let

$$f \sqsubseteq_n^{random} g \longleftrightarrow (\forall k \geq n)\; g \notin f(k).$$

Let $\sqsubseteq^{random} = \bigcup_{n\in\omega} \sqsubseteq_n^{ramdom}$. Let $f \in C^{random}$. Let

$$A_f = \bigcap_{m\in\omega} \bigcup_{n\geq m} f(n) \in \mathcal{N}.$$

Note that A_f is a set of measure zero. We can prove that for every set $A \in \mathcal{N}$ there exists $f \in C^{random}$ such that $A \subseteq A_f$.

It is well known that, for any $g \in 2^\omega$ and $n \in \omega$, the set $\{f \in C^{random} : f \sqsubseteq_n^{random} g\}$ is closed in C^{random}. We also have $f \sqsubseteq_n^{random} g$ if and only if $g \notin A_f$.

Let $N \prec H_\kappa$ be a countable elementary submodel of H_κ for a large enough κ, such that $P, \sqsubseteq^{random} \in N$. Let P be a forcing notion, and let $\dot{f}_0, \dots \dot{f}_{k-1} \in V^P$ will be names for functions in the C^{random}, i.e. $\Vdash$ "$(\forall i \in k)\ \dot{f}_i \in C^{\hat{random}}$". Let $f_0^*, \dots f_{k-1}^*$ be a sequence of functions. Then the decreasing sequence $((p_n))_{n\in\omega} \in P^\omega$ of P interprets $\{\dot{f}_i : i < k\}$ as $\{f_i^* : i < k\}$ if

$$(\forall i < k)(\forall n \in \omega)\ (p_n \Vdash \text{"}\dot{f}_i \restriction n = f_i^* \restriction n\text{"}).$$

Let $g \in H_\kappa$, then we say that g *covers* N if

$$(\forall f \in N \cap C^{random})\ (f \sqsubseteq^{random} g).$$

Definition 5 (P preserves $\sqsubseteq^{random}$). *Let $(P, \leq)$ and $N \prec H_\kappa$ be as above, where additionally we assume that $(P, \leq)$ is a proper forcing notion. We say that P preserves $\sqsubseteq^{random}$ if for every $p_0 \in P \cap N$, $g \in 2^\omega$ and every sequence $(p_n)_{n\in\omega} \in P^\omega \cap N$ that interprets $\{\dot{f}_i \in V^P : i < k\}$ as $\{f_i^* : i < k\} \in N$, if g covers N with choosen sequence $(n_i)_{i<k}$ such that $f_i^* \sqsubseteq_{n_i} g$ for each $i < k$, then there exists $q \leq p_0$ such that:*

(1) *q is (N, P)-generic,*
(2) *$q \Vdash$ "$(\forall f \in N[G])\ f \sqsubseteq g$",*
(3) *$(\forall i < k)\ (q \Vdash$ "$\dot{f}_i \sqsubseteq_{n_i} g$").*

The main tool of what we want to use are the following two theorems

Theorem 28 ([Gold], Fact 6.11). *If P preserves $\sqsubseteq^{random}$, then $P \Vdash \lambda(V \cap 2^\omega) = 1$.*

Theorem 29 ([Gold], Cor 5.14, Thm 6). *Let $P_\gamma = ((P_\alpha, \dot{Q}_\alpha) : \alpha < \gamma)$ be a countable support iteration of proper forcings which satisfies the condition:*

$$(\forall \alpha < \gamma)\ (\Vdash_\alpha \text{"}\dot{Q} \textit{ preserves } \sqsubseteq^{random}\text{"}),$$

then P_γ preserves $\sqsubseteq^{random}$.

Theorem 28 was used in the proof of the following theorem.

Theorem 30 ([H7, Thm 3.7]). *Assume that forcing notion $\mathbb{P}$ preserves $\sqsubseteq^{random}$, then $\mathbb{P}$ preserves property of being a $(\mathcal{N}, \mathcal{M})$-Lusin set.*

In addition, we have the following

Remark 1. *If $V = L$ and $\mathbb{P}$ is a countable support iteration of forcing notions $((P_\alpha, \dot{Q}_\alpha) : \alpha < \omega_2)$ such that:*

- *if α is an even ordinal, then $\Vdash_\alpha$ "$\dot{Q}_\alpha = \mathcal{R}$",*
- *if α is odd, then $\Vdash_\alpha$ "$\dot{Q}_\alpha = \mathcal{L}$".*

Then $\mathbb{P}$ preserves being $(\mathcal{N}, \mathcal{M})$-Lusin set, $\mathrm{cov}(\mathcal{N}) = \omega_2 = \mathfrak{c}$, and $\mathbb{P}$ adds ω_2 Laver reals. If $A \in V$ is a $(\mathcal{N}, \mathcal{M})$-Lusin set of full Haar outer measure (equal to 1) in 2^ω, then in the generic extension $V[G]$ A is a completely $\mathcal{N}$-nonmeasurable set of cardinality $\omega_1 < \mathfrak{c}$. Here $\mathcal{R}$ is the Solovay forcing notion which adds a random real and $\mathcal{L}$ is the Laver forcing.

The method discovered by Cichoń and Szczepaniak [CS] of constructimg nonmeasurable subsets in Euclidean spaces was an inspiration for [$H4$]. Although the proofs in [$H4$] are very elementary, the results apply to infinite dimensional Banach spaces. The so called Steinhaus property played an important role in the proofs of results in [$H4$]. We say that a σ-ideal $\mathcal{I}$ on an Abelian Polish group has Steinhaus property if and only if for any sets $A, B \in Bor(X)[\mathcal{I}] \setminus \mathcal{I}$ there is a non-empty open set $\emptyset \neq U \subseteq G$, such that $U \subseteq A + B$. An example of such a σ-ideal is the σ-ideal $\mathcal{M}$ of the sets of the first category on any Banach space $(X, \|\cdot\|)$. Let us denote by $B = \{x \in X : \|x\| < 1\}$ the open unit ball in a Banach space X.

In my note [$H4$] I consider only translation invariant ideals. The main results are the following theorems.

Theorem 31 ([H4, Thm 2.2])**.** *If X, Y are Banach spaces and*

(1) *$\mathcal{I}$ is a σ-ideal in Y which has Steinhaus property,*
(2) $(\forall n \in \omega \setminus \{0\})(\forall A \in \mathcal{I})\ (nA = \{n \cdot a : a \in A\} \in \mathcal{I})$,
(3) *$f : X \to Y$ is any isomorphism between the spaces X, Y which is not a homeomorphism,*

then the image of the unit ball $f[B]$ is $\mathcal{I}$-nonmeasurable in the space Y.

Theorem 32 ([H4, Thm 2.4])**.** *If X, Y are Banach spaces and*

(1) *$\mathcal{I}$ is a κ-complete ideal in Y which has Steinhaus property,*
(2) $\min\{|D| : D \in \mathscr{P}(X) \textit{is dense in } X\} < \kappa$,
(3) *$f : X \to Y$ is any isomorphism between the spaces X, Y which is not a homeomorphism,*

then the image of the unit ball $f[B]$ is $\mathcal{I}$-nonmeasurable in the space Y.

When the first space mentioned above is separable then we immediately obtain the following result.

Corollary 12. *If X, Y are Banach spaces, X is separable, $\mathcal{I} \subseteq \mathscr{P}(Y)$ is a σ-ideal on Y which has Steinhaus property, $f : X \to Y$ is an isomorphism which is not a homeomorphism between X and Y, then the image of $f[B]$ unit ball is a $\mathcal{I}$-nonmeasurable set in Y.*

Many of the results I have obtained are related to σ-ideals with Borel base defined on Polish spaces. In [$H8$] I considered problems of nonmeasurability with respect to an ideal without Borel base; for example the Marczewski ideal s_0. Here by the $Perf$ we denote the family of all perfect subsets of a given Polish space X and by $s \subseteq \mathscr{P}(X)$ we denote the family of all Marczewski measurable sets (also called **s-measurable**) in a Polish space X. The class s is defined as follows:

$$(\forall A \in \mathscr{P}(X))\ A \in s \longleftrightarrow (\forall P \in Perf)(\exists Q \in Perf)\ (Q \subseteq P \wedge (Q \subseteq A \vee A \cap Q = \emptyset)).$$

We define the class s_0 of **Marczewski null** sets (or, for short, s_0 **sets**) as follows:

$$(\forall A \in \mathscr{P}(X))\ A \in s_0 \longleftrightarrow (\forall P \in Perf)(\exists Q \in Perf)\ (Q \subseteq P \wedge A \cap Q = \emptyset).$$

It is known that every perfect subset in X is a disjoint union of continuum many perfect subsets. We immediately conclude from this that every Lusin or Sierpiński set is in s_0.

One of the first results obtained in this work is the following:

Proposition 1 ([H8 Prop. 2.2])**.** *If $\mathfrak{c}$ is a regular cardinal number and*

$$\mathcal{A} \subseteq \{A : A \textit{ is a Lusin set}\}$$

is any $\mathfrak{c}$-point family such that the union satisfies $\bigcup \mathcal{A} \notin s_0$, then there exists a subfamily $\mathcal{A}' \subseteq \mathcal{A}$, such that the union $\bigcup \mathcal{A}'$ is not s-measurable.

We obtain an analogous result replacing a family of Lusin sets by a family consisting of Sierpiński sets. Assuming the continuum hypothesis **CH** we can not leave the assumption that $\mathcal{A}$ is $\mathfrak{c}$-point family, [H8, Proposition 2.3].

In the paper [JMS] written by Haim Judah, Arnold Miller and Saharon Shelah it was proved that it is consistent with **ZFC** that $\mathrm{add}(s_0) = \omega_1$ and $\mathrm{cov}(s_0) = \mathfrak{c} = \omega_2$.

As in the case of the existence of nonmeasurable unions of sets from σ-ideals with Borel base, the existence of families of sets from s_0 whose unions are not s_0-nonmeasurable depends on cardinal coefficients.

Theorem 33 ([H8, Thm 2.5])**.** *Assume that* $\mathrm{cov}_h(s_0) = \mathfrak{c}$. *For any* $\mathfrak{c}$*-point family* $\mathcal{A} \subseteq s_0$ *of sets from the Marczewski ideal* s_0*, if* $\bigcup \mathcal{A} \notin s_0$*, then there exists a subfamily* $\mathcal{A}' \subseteq \mathcal{A}$ *such that the union* $\bigcup \mathcal{A}'$ *is not s-measurable.*

In the proof of the above theorem the assumption that $\mathrm{cov}_h(s_0) = \mathfrak{c}$ was essentially used, but we can show that it is consistent with $\mathbf{ZFC} + \neg\mathbf{CH}$ that there exists a partition $\mathcal{A}$ of size ω_1 of the real line $\mathbb{R}$ such that one can find a subfamily $\mathcal{B} \subseteq \mathcal{A}$ for which the union $\bigcup \mathcal{B}$ is not an s-measurable set (Theorem 2.3 in [H8]).

We have the following result concerning the situation $\mathrm{cov}_h(s_0) < \mathfrak{c}$:

Theorem 34 ([H8, Thm 2.6])**.** *It is relatively consistent with* **ZFC** *that* $\mathrm{cov}_h(s_0) < \mathfrak{c}$ *and there is a partition* $\mathcal{A} \subseteq s_0$ *of the cardinality* ω_1 *of the real line* $\mathbb{R}$*, for which there is a subfamily* $\mathcal{B} \subseteq \mathcal{A}$ *whose union is not s-measurable.*

To construct an appropriate model I have considered the $\aleph_{\omega_1}$ finite support iteration of Cohen forcing which adds one Cohen real.

Using a transfinite induction, we have shown the following theorem.

Theorem 35 ([H8, Thm 3.1])**.** *If* $cov(s_0) = \mathfrak{c}$*, and* $\mathfrak{c}$ *is regular, then for each family* $\mathcal{A} \subseteq s_0$*, which satisfies the following conditions*

- $\mathcal{A}$ *is a large-point family, i.e.* $\{x \in X : \ |\{A \in \mathcal{A} : \ x \in A\}| < \mathfrak{c}\} \in s_0$,
- $\{(x, y) \in X^2 : \ |\{A \in \mathcal{A} : x, y \in A\}| = \mathfrak{c}\} \in s_0 \times s_0$.

there is a subfamily $\mathcal{A}' \subseteq \mathcal{A}$*, such that* $\bigcup \mathcal{A}'$ *is a Bernstein set.*

Marczewski measurability is closely connected with perfect sets P which are defined from so called perfect trees S by taking the following operation $[S] = \{x \in 2^\omega : \ (\forall n \in \omega) s \restriction n \in S\}$. All mentioned above trees forms Sacks forcing $\mathbb{S}$ with the inclusion as the order. Laver's and Miller's ideals as well as the measurability notion can be defined in a similar way as it was done for Marczewski ideal. Another example of a family of trees is the family of so called **complete Laver trees** sp: $T \in sp$ if $T \subseteq \bigcup_{n\in\omega} \omega^n$ is a tree, and for any $t \in T$

$$\{n \in \omega : \ t^\frown n \in T\} \in [\omega]^\omega.$$

In the same work I have investigated relationships between nonmeasurability with respect to a tree forcing notion and the existence of maximal almost disjoint families of functions on the Baire space. We say that $\mathcal{A} \subseteq \omega^\omega$ is a maximal almost disjoint family of functions in the Baire space (shortly m.a.d. family) if and only if for any different elements $a, b \in \mathcal{A}$ the common part is at most finite and that $\mathcal{A}$ is maximal with respect to inclusion, i.e. for any $x \in \omega^\omega$ there is an $a \in \mathcal{A}$ such that $x \cap a$ is infinite.

In connection with these concepts, assuming **CH**, I have received the following result.

Theorem 36 ([H8, Thm 4.1])**.** *Under* **CH** *there are* $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$ *m.a.d families, such that* $\mathcal{A}$ *is not l-measurable and* $\mathcal{B}$ *is not sp-measurable in the Baire space.*

On the other hand, I obtained the following result under the negation of the continuum hypothesis.

Theorem 37 ([H8, Thm 4.2])**.** *It is relatively consistent with* $\mathbf{ZFC} + \neg\mathbf{CH}$ *that there exists a m.a.d* $\mathcal{A}$ *which is not sp-measurable in the Baire space.*

This result was obtained by using a finite support iteration of posets $(Q_T, \leq)$ of length κ ($\omega_1 < \kappa$), where $T \subseteq \omega^{<\omega}$ is an sp -tree in which all branches form a family of almost disjoint functions in the Baire space.

The forcing notion $(Q_T, \leq)$ is defined as follows: $p = (x_p, s_p^g, s_p^b, \mathcal{F}_p) \in Q_T$ iff

- $x_p \in \omega^{<\omega}$ and
- s_p^g, s_p^b are nonempty finite subtrees of T (finite subsets of T closed under restriction),
- $\mathcal{F}_p \in [\omega^\omega]^{<\omega}$,

For $t \in \omega^{<\omega}$ and a nonempty finite tree $\tau \subseteq \omega^{<\omega}$ let $t \restriction \tau = \bigcup\{s \in \tau : s \subseteq t\}$ be the maximal initial subsequence of t that belongs to τ.

The order is defined as follows: for every $p = (x_p, s_p^g, s_p^b, \mathcal{F}_p) \in Q_T$ and $q = (x_q, s_q^g, s_q^b, \mathcal{F}_q) \in Q_T$ we have $p \leq q$ iff

(1) $x_q \subset x_p \wedge s_q^g \subseteq s_p^g \wedge s_q^b \subseteq s_p^b \wedge \mathcal{F}_q \subseteq \mathcal{F}_p$,
(2) $(\forall t \in s_p^g)\ x_p \cap t \subseteq (t \restriction s_q^g) \cup x_q$,
(3) $(\forall h \in \mathcal{F}_q)(x_p \cap h = x_q \cap h)$,
(4) $(\forall h \in \mathcal{F}_q)(\forall t \in s_p^b)\ t \cap h = (t \restriction s_q^b) \cap h$,
(5) $(\forall h \in \mathcal{F}_q)(\forall t \in s_p^g)\ t \cap h = (t \restriction s_q^g) \cap h$.

First I proved that this forcing notion is σ-centered (so is c.c.c.). Now, let $G \subseteq Q_T$ be a generic filter over V, and in the generic extension $V[G]$ set

$$x_G = \bigcup\{x_p : \ p \in G\},$$

$$S_G^g = \bigcup\{s_p^g : p \in G\} \text{ and } S_G^b = \bigcup\{s_p^b : p \in G\},$$

I proved the following claims.

Claim 1. $(\forall S \in SPT(T) \cap V)$ ($S_G^g \cap S \in SPT(T)$ *and* $S_G^b \cap S \in SPT(T)$),

Claim 2. *For every* $S \in SPT(T) \cap V$ *the set*

$$\{z \in [S_G^b \cap S] : \ |x_G \cap z| = |x_G \setminus z| = \omega\}$$

is comeager in $[S_G^b \cap S]$.

Claim 3. *If* $\mathcal{F} \subseteq \omega^\omega \cap V$ *is an almost disjoint family, then the families* $\{x_G\} \cup [S_G^g] \cup \mathcal{F}$ *and* $[S_G^b] \cup \mathcal{F}$ *are also almost disjoint.*

Next, in the step β of the iteration $((P_\alpha : \alpha \leq \kappa), (\dot{Q}_\beta : \beta < \kappa))$ where we have $\Vdash_{P_\beta} \ \dot{Q}_\beta = \hat{Q}_T$ show up the generic objects $x_\beta = i_{G_{\beta+1}}(\dot{x}_{H_\beta})$, $S_\beta^g = i_{G_{\beta+1}}(\dot{S}_{H_\beta}^g)$, $S_\beta^b = i_{g_{\beta+1}}(\dot{S}_{H_\beta}^b)$(here $G \subseteq P_\kappa$ is a generic filter over the ground model $V \models GCH$), where H_β is $i_{G_{\beta+1}}(\dot{Q}_T)$ generic filter over $V[G_\beta]$, such that $G_{\beta+1} = G_\beta * H_\beta$.

Next, we define $\mathcal{A}_0 = \emptyset$, $\mathcal{A}_{\beta+1} = \mathcal{A}_\beta \cup \{x_\beta\} \cup [S_\beta^g]$, where in the limit ordinal case, we take $\mathcal{A}_\beta = \bigcup_{\xi<\beta} \mathcal{A}_\xi$. Finally by taking any *m.a.d.* family containing $\mathcal{A} = \bigcup_{\beta<\kappa} \mathcal{A}$ we can obtain the desired set satisfying the conclusion of our theorem.

The last result of this work is the relative consistency of the **ZFC** theory and the inequality $cov(s_0) < \mathfrak{a}$ (here $\mathfrak{a}$ is the smallest size of m.a.d. family in the Baire space).

My other scientific achievement

List of my publications in my other research achievement:

[P1] J. Kraszewski, R. Rałowski, P. Szczepaniak and Sz. Żeberski, Bernstein sets and kappa coverings, Mathematical Logic Quarterly, 56 (2) 2010, pp. 216-224.

[P2] M. Bienias, Sz. Głąb, R. Rałowski, Sz. Żeberski, Two point sets with additional properties, Czechoslovak Mathematical Journal, vol 63, no 4(2013), pp. 1019-1037.

[P3] R. Rałowski, P. Szczepaniak and Sz. Żeberski, A generalization of Steinhaus theorem and some nonmeasurable sets, Real Analysis Exchange, vol. 35, nr 1 (2009/2010), pp. 1-9.

[P4] T. Banakh, M. Morayne, R. Rałowski, Sz. Żeberski, Topologically invariant σ-ideals on the Hilbert cube, Israel Journal of Mathematics, vol. 209, (2015), pp. 715-743.

[P5] T. Banakh, M. Morayne, R. Rałowski, Sz. Żeberski, Topologically invariant σ-ideals on the Euclidean spaces, Fundamenta Mathematicae, vol. 231, (2015), pp. 101-112.

[P6] T. Banakh, R. Rałowski, Sz. Żeberski, Classifying invariant σ-ideals with analytic base on good Cantor measure spaces, `accepted to` Proceedings of American Mathematical Society.

[P7] M. Burnecki, R. Rałowski, Topologies on the group of invertible transformations, Banach Center Publications, 95 (2011), pp. 273-280.

[P8] R. Gielerak and R. Rałowski, Statistical mechanics of Class of Anyonic Systems. The Rigorous Approach, J. of Nonlinear Math. Phys. Vol 11 (2004), 85–91

[P9] R. Gielerak and R. Rałowski, Convergence of virial expansions for some anyonic-like systems, Proceedings of the Seminar on Stability Problems for Stochastic Models, Part I (Nałeczów, 1999). J. Math. Sci. (New York) 105 (2001), no. 6, 2555–2556

[P10] R. Rałowski, On the kernel of the Hermitian Form and Partition Function, Proceedings of 6th Int. School on Theoretical Physics, Symmetry and Structural Properties of Condensed Matter. Word Scientific, 2001.

[P11] R. Gielerak and R. Rałowski, Wick Algebras Approach to Physics of 2D Systems, Proceedings of 5th Int. School on Theoretical Physics, Symmetry and Structural Properties of Condensed Matter. Word Scientific, 1999.

[P12] R. Rałowski, On Wick algebras with additional twisted commutation relations, J. Phys. A 30 (1997),no. 9, 3235–3247.

[P13] W. Marcinek R. Rałowski, On Wick algebras with braid relations, J. Math. Phys. 36(1995), no. 6, 2803–2812.

[P14] M. Kozłowski and R. Rałowski, The dielectric response with respect to the weight distribution of relaxation times, Journ. of Math. Chem. vol. 46, nr 4 (2009), pp. 1087-1102,

[P15] R. Rałowski and M. Kozłowski, The Havriliak-Negami Dielectric Response in Time Domain, Polish J. Chem., Vol 79 (2005), 1353–1356.

[P16] M. Kozłowski, R. Rałowski, H. Kołodziej, An Application of the Burr Function to the Description of Dielectric Relaxation Data in Frequency Domain, IEEE Trans. DEI. Vol 10 (2003), 256-259.

The papers that include the remainder of my research can be divided into three parts. The first part consists of purely mathematical papers; the second includes work in the field of mathematical physics and the last one consists of my articles in the field of physical chemistry.

In $[P1]$ we have considered sets A in Abelian Polish groups, such that each set of fixed cardinality κ can covered by a single translation of A. These sets are called κ-covering sets. Similarly, we say that a set $A \subseteq G$ is a $< \kappa$-covering, if any subset of the group G of size less than κ, can be covered by a single translation of A. The inspiration for this work were the results obtained by Muthuvel [Mu] concerning κ-covering sets for κ being a finite cardinal and by Nowik [Now1, Now2] who considered ω or $< \omega$-covering sets with low descriptive complexity. In these articles Nowik gave examples of a partition of the Cantor space 2^ω into continuum many Borel sets which are also ω-covering sets, and of a partition into two sets, such that the first set N is a G_δ-set of measure zero and the second set

M is an F_σ meager set , such that every two disjoint countable sets can be inserted using a single translation into the sets M and N, respectively.

The first result of our work is about the existence of a partition of the real line $\mathbb{R}$ into two Bernstein sets, such that each of them is not a 2-covering. On the other hand, we have found a partition of $\mathbb{R}$ into continuum many Bernstein sets which are $< cof(\mathfrak{c})$-covering. Moreover, if $\mathcal{I}$ is a σ-ideal on the real line $\mathbb{R}$ which has Steinhaus property, and if we assume that $\mathrm{non}(\mathcal{I}) < \mathfrak{c}$, then there exists a completely $\mathcal{I}$-nonmeasurable set which is also a $< \mathfrak{c}$-covering set.

The concept of a κ-covering set became the starting point for some variations of this idea, namely, S-covering and I-covering.

We say that a family $\mathcal{A}$ is a κ-S-covering if

- $\mathcal{A}$ is a family of pairwise disjoint subsets of the real line $\mathbb{R}$,
- $|\mathcal{A}| = \kappa$,
- $(\forall F \in [\mathbb{R}]^\kappa)(\exists t \in \mathbb{R})(\forall A \in \mathcal{A})(|(t+F) \cap A| = 1 \wedge t + F \subseteq \bigcup \mathcal{A})$.

Ostatnie wyniki związane z diagramem Cichonia powstały dzięki nowym metodom

Our results apply to families $\mathcal{A}$ whose elements are completely nonmeasurable sets with respect to some σ-ideals on $\mathbb{R}$. The following theorem is an example of such application.

Theorem 38 ([P1, Thm 3.3]). *Let κ be a fixed cardinal number such that $2 < \kappa < \mathfrak{c}$. If $2^\kappa \leq \mathfrak{c}$, then there exists a partition of $\mathbb{R}$ onto κ many Bernstein sets $\{B_\xi : \xi < \kappa\}$ such that*

- *for any $\xi < \kappa$, B_ξ is not a 2-covering set but*
- *$\{B_\xi :\ \xi < \kappa\}$ is κ - S-covering.*

The above theorem shows that S-covering and κ-covering are completely different notions.

It is known that, under **MA**, if $\omega \leq \kappa < \mathfrak{c}$ then $2^\kappa = \mathfrak{c}$, which ensures consistency of the conclusion of the above theorem with **ZFC** theory.

In the next theorem we consider a slightly more general situation.

Theorem 39 ([P1, Thm 3.5]). *Let κ be a cardinal number that satisfies the equality $2^\kappa = \mathfrak{c}$. Let $(G, +)$ is uncountable Abelian Polish group with a metric d. In addition, let $\mathcal{I} \subseteq \mathscr{P}(G)$ be a fixed σ-ideal on G such that*

- $(\forall B \in Bor(G) \setminus \mathcal{I})(\forall \mathcal{D} \in [\mathcal{I}]^{<\mathfrak{c}})\ (|B \setminus \bigcup \mathcal{D}| = \mathfrak{c})$,
- *there exists $a \in rng(d) \setminus \{0\}$, such that $(\forall x \in G)$ $(\{Y \in G :\ d(x,y) = a\} \in \mathcal{I})$.*

Then there exists a family of pairwise disjoint sets $\{B_\xi : \xi < \kappa\}$ with the following properties:

1. *B_ξ is completely $\mathcal{I}$-nonmeasurable in G for any $\xi < \kappa$,*
2. *B_ξ is not a 2-covering set for any $\xi < \kappa$,*
3. *$\{B_\xi : \xi < \kappa\}$ is κ - S-covering.*

Translation in the definition of a κ-covering set, can be replaced, for example, by any isometry of the real plane. Then we deal with the concept of a set, which is κ - I-covering. Namely, every subset of the plane $A \subseteq \mathbb{R}^2$ is κ - I-covering if the following condition is fulfilled

$$(\forall B \in [\mathbb{R}^2]^\kappa)(\forall \varphi \in \mathbb{R}^{2^{\mathbb{R}^2}})\ (\varphi \text{ is an isometry and } \varphi[B] \subseteq A).$$

The following two theorems show the difference between 2 and 3 - I coverings.

Theorem 40 ([P1, Thm 4.3]). *Each Bernstein set in $\mathbb{R}^2$ is a 2 - I-covering set.*

Theorem 41 ([P1, Thm 4.4]). *There exists a Bernstein set in $\mathbb{R}^2$, which is not a 3 - I-covering set.*

The conclusuion of the Theorem 40 can not be extended to arbitrary complete $\mathcal{I}$-nonmeasurable sets.

Theorem 42 ([P1, Thm 4.5]). *If $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}\}$ then there exists a completely $\mathcal{I}$-nonmeasurable set in $\mathbb{R}^2$, which is not a 2 - I-covering set.*

In [$P2$], written together with Marek Bienias, Szymon Głąb and Szymon Żeberski, we investigated sets which were introduced by Stefan Mazurkiewicz. In the literature, these sets are known as Mazurkiewicz sets but also as two-point sets. We say that a subset of the plane $A \subseteq \mathbb{R}^2$ is a Mazurkiewicz set if every line has exactly two common points with A. It is known that these sets are quite complex, which was showed by Larman in [Lar] namely, they can not be F_σ but in the constructible universe L, these sets can be coanalytic, which was proved by Miller in [Mi].

The first result in this article is the existence of the Mazurkiewicz set, which is a Hamel basis of the real plane $\mathbb{R}^2$ over the field of rational numbers $\mathbb{Q}$ and is also a completely $\mathcal{I}$-nonmeasurable set with respect to any σ-ideal with a Borel base containing all singletons. Then we have proved the existence of a Mazurkiewicz set, which is in the Marczewski ideal s_0. In addition, if a σ-ideal $\mathcal{I}$ has the property that for every Borel set B not in $\mathcal{I}$ there exist $\mathfrak{c}$ many parallel lines, such that each of them has the intersection with the set B of the cardinality equal to continuum, then, firstly, there exists a Mazurkiewicz set in the ideal s_0, which is a Hamel base and is also a completely $\mathcal{I}$-nonmeasurable set, and, secondly, there exists a Mazurkiewicz set, which is a Hamel base and is completely $\mathcal{I}$-nonmeasurable and is also an s-nonmeasurable set.

The concept of Mazurkiewicz set can be naturally generalized to a κ point set. For a cardinal number $2 \leq \kappa \leq \mathfrak{c}$, it is a subset of the plane which meets every straight line in exactly κ points. We have proved that for any positive integer $n \geq 2$, every n-point set can be decomposed into n pairwise disjoint bijections on $\mathbb{R}$.

We have found a relationship between Bernstein sets on the line $\mathbb{R}$ and Mazurkiewicz sets. Namely, for any Bernstein set $B \subseteq \mathbb{R}$ there is a Mazurkiewicz set $A \subseteq \mathbb{R}^2$, which is both a set of measure zero and of the first category in $\mathbb{R}^2$ with the property that $f^{-1}[(0,1)]$ is equal to B for any function $f \subseteq A$

It is easy to see that each Mazurkiewicz set is neither a Bernstein nor a Lusin, nor a Sierpiński set. For this reason, we have introduced a notion of a partial Mazurkiewicz set, i.e. such a set, which has at most two common elements with every line. We have obtained the result that says that under **CH**, there is a Lusin set, which is a partial Mazurkiewicz set. The analogous result holds for Sierpiński's set.

In addition, in [$P1$] we have shown the existence of an $\aleph_0$-point set, which is not a 2 - I-covering set. We showed that there is an $\aleph_0$-point set, which is an $\aleph_0$-covering set.

We showed that, when we add ω_2 independent Cohen reals to the constructible universe L, then we obtain a model of **ZFC** theory, in which $\omega_1 < \mathfrak{c} = \omega_2$ and there is an $\aleph_1$-point set, which is at the same time an $\aleph_1$-covering one.

For a fixed natural number n greater than one, we have found examples of n-point sets, one of which is not a 2 - I-covering and the other is n-covering.

In the same work we investigated the combinatorial properties of Mazurkiewicz sets as, in some sense, families of almost disjoint sets. Let h be a fixed definable Borel bijection between the real line $\mathbb{R}$ and the Ramsey space $[\omega]^\omega$ of all infinite subsets of ω. Let π_1, π_2 be the orthogonal projections of the plane $\mathbb{R}^2$ onto the first and the second axis, respectively. Then it is relatively consistent with the **ZFC** theory that $\neg$**CH** and there is a partial Mazurkiewicz set $A \subseteq \mathbb{R}^2$, for which the image $h[\pi_1[A] \cup \pi_2[A]]$ is a maximal almost disjoint family in ω of cardinality ω_1.

The final result of the paper [$P2$] says that in the model obtained by adding ω_2 independent Cohen reals to the constructible universe L, there is a partial Mazurkiewicz set $C \subseteq \mathbb{R}^2$ of size ω_2, which is a Lusin set and it has the following property:

$$(\exists A \in \mathcal{N})(\forall D \in [C]^{\omega_1})\ (A + D = \mathbb{R}^2).$$

The analogous result where instead of a Lusin set one obtains a Sierpiński set can be obtained by adding ω_2 Solovay reals to L.

The Steinhaus theorem which says that for any two sets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ with positive Lebesgue measure, the algebraic sum $A + B$ has a nonempty interior was the inspiration for another work, [$P3$], written jointly with Przemysław Szczepaniak and Szymon Żeberski.

The first result of this article is the following generalization of the above mentioned Steinhaus Theorem.

Theorem 43 ([P3, Thm 2.1])**.** *Let $\mathcal{I}$ be $\mathcal{N}$ or $\mathcal{M}$. Let a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of the class C^1 satisfy*

$$\{(x, y) \in \mathbb{R}^2 : \frac{\partial f}{\partial x}(x, y) = 0 \vee \frac{\partial f}{\partial y}(x, y) = 0\} \in \mathcal{I}.$$

Let $A, B \in Bor(\mathbb{R}) \setminus \mathcal{I}$. Then the image $f[A \times B]$ contains a non-empty open interval.

This theorem in its basic version, which is the Steinhaus theorem, was main tool for the proof of the Cichoń-Szczepaniak Theorem [CS] on the unit ball in a Euclidean space.

Theorem 44 (Cichoń-Szczepaniak)**.** *If m, n are two different positive integers and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a given isomorphism over the field of rational numbers $\mathbb{Q}$. Then for any set $A \subseteq \mathbb{R}^n$ such that A, A^c have non-empty interior in $\mathbb{R}^n$, the inner m-dimensional Lebesgue measure of the image $f[A]$ is equal to zero, and the outer m-dimensional Lebesgue measure of $f[A]$ is full.*

Application of Theorem 44 gives a nonmeasurable sets with some algebraic properties. Examples are provided by the following theorems.

Theorem 45 ([P3, Thm 3.2])**.** *There exists a completely $\mathcal{N}$-nonmeasurable set $A \subsetneq \mathbb{R}$ such that $A + A = A$ and $A - A = \mathbb{R}$.*

Theorem 46 ([P3, Thm 3.3])**.** *There is a partition $\mathcal{A} = \{A_n : n \in \omega\}$ of the real line $\mathbb{R}$ onto completely $\mathcal{N}$-nonmeasurable sets, such that for every $n \in \omega$ we have $A_n + A_n = A_n$.*

Theorem 47 ([P3, Thm 3.5])**.** *There is a partition $\mathcal{A} = \{A_n : n \in \omega\}$ of the real line $\mathbb{R}$ onto completely $\mathcal{N}$-nonmeasurable sets, such that*

$$(\forall m, n \in \omega)\ (m \neq n \longrightarrow A_m + A_n = \mathbb{R} \setminus \{0\}).$$

Theorem 48 ([P3, Thm 3.7])**.** *There is a set $A \subseteq \mathbb{R}$, such that $A, A + A, A + A + A, \ldots$ are completely $\mathcal{N}$-nonmeasurable and $\bigcup_{n \in \omega} \underbrace{A + \ldots + A}_{n} = \mathbb{R}$.*

Theorem 49 ([P3, Thm 3.9])**.** *There is a set $A \subseteq \mathbb{R}$, such that*

$$A \subsetneq A + A \subsetneq A + A + A \subsetneq \ldots$$

are completely $\mathcal{N}$-nonmeasurable and $\bigcup_{n \in \omega} \underbrace{A + \ldots + A}_{n}$ is also completely $\mathcal{N}$-nonmeasurable set on $\mathbb{R}$.

Theorem 50 ([P3, Thm 3.11])**.** *There is a set $A \subseteq \mathbb{R}$, such that*

$$A \supsetneq A + A \supsetneq A + A + A \supsetneq \ldots$$

are completely $\mathcal{N}$-nonmeasurable sets on $\mathbb{R}$.

We gave the multiplicative couterparts (which are applications of the results obtained above).

Theorem 51 ([P3, Cor 3.2])**.** *There is a set $A \subseteq \mathbb{R}$ completely $\mathcal{N}$-nonmeasurable, such that $A \cdot A = A$.*

Theorem 52 ([P3, Cor 3.3])**.** *There is a set $A \subseteq \mathbb{R}$, such that*

$$A \subsetneq A \cdot A \subsetneq A \cdot A \cdot A \subsetneq \ldots$$

are completely $\mathcal{N}$-nonmeasurable and $\bigcup_{n \in \omega} \underbrace{A \cdot \ldots \cdot A}_{n}$ is also completely $\mathcal{N}$-nonmeasurable on $\mathbb{R}$.

The question whether the minimum size of a family of Cantor sets which is necessary to cover the Hilbert cube $\mathbb{I}^\omega$ is the same as the minimum size of a family of Cantor sets whose union covers the unit interval $\mathbb{I} = [0,1]$ was open for quite a while and was attracting attention of some mathematicians (for instance was posed on the UCL webpage of Marianna Csörney). In $[P4]$, written jointly with Taras Banakh, Michał Morayne and Szymon Żeberski, we gave the affirmative answer. In this article, we have considered topologically invariant ideals on the Hilbert cube $\mathbb{I}^\omega$ equipped with the product topology. Treating ideals $\mathcal{I} \subseteq [\mathbb{I}^\omega]^{\leq\omega}$ and $\mathscr{P}(\mathbb{I}^\omega)$ as *trivial* ones on the Hilbert cube, we showed that the ideal of meager sets $\mathcal{M}$ on $\mathbb{I}^\omega$ is a maximal ideal among all topologically invariant σ-ideals with the base which has Baire property and that σ-ideal $\mathcal{M}$ is the greatest among non-trivial topologically invariant σ-ideals on $\mathbb{I}^\omega$ with base generated by the σ-compact sets in $\mathbb{I}^\omega$. On the other hand, we have considered the σ-ideal generated by the so-called minimal Cantor sets on the Hilbert cube $\mathbb{I}^\omega$, that is these sets C which are homeomorphic to the Cantor set and such that for any perfect set $P \subseteq \mathbb{I}^\omega$ there is a homeomorphism $h \in Homeo(\mathbb{I}^\omega)$ for which $h[C] \subseteq P$. This ideal is denoted by $\sigma\mathcal{C}_0$. It is contained by every nontrivial topologically invariant σ-ideal with the analytic base on $\mathbb{I}^\omega$. A Cantor set is minimal if and only if it is a $\mathcal{Z}_\omega$-set. A closed set $A \subseteq \mathbb{I}^\omega$ is a $\mathcal{Z}_\omega$-set if and only if the set

$$\{f \in \mathcal{C}(\mathbb{I}^\omega, \mathbb{I}^\omega) :\ f[\mathbb{I}^\omega] \cap A = \emptyset\}$$

is a dense set in the space of all continuous functions $\mathcal{C}(\mathbb{I}^\omega, \mathbb{I}^\omega)$ equipped with the compact-open topology.

The combinatorial characterization of $\operatorname{cov}(\mathcal{M}), \operatorname{non}(\mathcal{M})$ given by Tomasz Bartoszyński (eg [Bart], [BartJud]) implies the following inequalities

$$\operatorname{cov}(\sigma\mathcal{C}_0) = \operatorname{cov}(\mathcal{M}) \leq \operatorname{cov}(\mathcal{I}) \leq \operatorname{cov}(\sigma\mathcal{C}_0),\ \operatorname{non}(\mathcal{M}) = \operatorname{non}(\sigma\mathcal{C}_0) \leq \operatorname{non}(\mathcal{I}) \leq \operatorname{non}(\mathcal{M}).$$

In the study of cardinal coefficients as $\operatorname{add}(\cdot)$, or $\operatorname{cof}(\cdot)$, a major role is played by the fact that the space $Homeo(\mathbb{I}^\omega)$ of all homeomorphisms of the Hilbert cube $\mathbb{I}^\omega$ forms a Polish space with the compact-open topology metrizable by

$$\tilde{d}(f,g) = \sup_{x\in\mathbb{I}^\omega} d(f(x), g(x)) + \sup_{x\in\mathbb{I}^\omega} d(f^{-1}(x), g^{-1}(x)).$$

The $\mathcal{Z}$-set unknotting theorem guarantees that every two minimal Cantor sets $A, B \subseteq \mathbb{I}^\omega$ are *ambiently* homeomorphic which means that there is a homeomorphism $h \in Homeo(\mathbb{I}^\omega)$ such that $f[A] = B$. It follows that for a fixed minimal Cantor set A and any dense $G \in G_\delta$ contained in $\mathbb{I}^\omega$ the set

$$\{h \in Homeo(\mathbb{I}^\omega) :\ h[A] \subseteq G\}$$

is a dense G_δ in the space $Homeo(\mathbb{I}^\omega)$.

For given any two σ-ideals $\mathcal{I}, \mathcal{J}$ on a Polish space, we can define the relative cardinal coefficients as follows

$$\operatorname{add}(\mathcal{I}, \mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{J}\},$$
$$\operatorname{cof}(\mathcal{I}, \mathcal{J}) = \min\{|\mathcal{B}| :\ \mathcal{B} \subseteq \mathcal{J} \wedge (\forall A \in \mathcal{I})(\exists B \in \mathcal{B})\ A \subseteq B\}.$$

One of the main result of this article is characterizing the cardinal coefficients for σ-ideals $\mathcal{M}$ and $\sigma\mathcal{C}_0$ which can be written as:

- $\operatorname{non}(\sigma\mathcal{C}_0) = \operatorname{non}(\mathcal{M})$,
- $\operatorname{cov}(\sigma\mathcal{C}_0) = \operatorname{cov}(\mathcal{M})$,
- $\operatorname{add}(\sigma\mathcal{C}_0) = \operatorname{add}(\mathcal{M}) = \operatorname{add}(\sigma\mathcal{C}_0, \mathcal{M})$,
- $\operatorname{cof}(\sigma\mathcal{C}_0) = \operatorname{cof}(\mathcal{M}) = \operatorname{cof}(\sigma\mathcal{C}\mathcal{M})$.

We also studied topologically invariant and proper σ-ideals on the Hilbert cube, which are not included in the σ-ideal $\mathcal{M}$ of meager sets. It turns out that the σ-ideal $\sigma\mathcal{G}_0$ generated by the so-called *tame* - G_δ sets is the smallest topologically invariant σ-ideal on $\mathbb{I}^\omega$ among all topologically invariant σ-ideals with the Baire property on the Hilbert cube which are not included in the σ-ideal $\mathcal{M}$. We have

$$\omega_1 \leq \operatorname{add}(\sigma\mathcal{G}_0) \leq \operatorname{cov}(\sigma\mathcal{G}_0) \leq \operatorname{add}(\mathcal{M}) \leq \operatorname{cof}(\mathcal{M}) \leq \operatorname{non}(\sigma\mathcal{G}_0) \leq \operatorname{cof}(\sigma\mathcal{G}_0) \leq \mathfrak{c}.$$

To sum up, for any σ-ideals $\mathcal{I}, \mathcal{J}$ on $\mathbb{I}^\omega$ with analytic base such that $\mathcal{I} \subseteq \mathcal{M}$ and $\mathcal{J} \not\subseteq \mathcal{M}$ we have the following topological variant of Cichoń's diagram:

$$\begin{tikzcd}
 & & & & \mathrm{non}(\sigma\mathcal{G}_0) \arrow[r] & \mathrm{non}(\mathcal{J}) \arrow[r] & \mathrm{cof}(\mathcal{J}) \arrow[r] & \mathfrak{c} \\
 & & \mathrm{non}(\mathcal{I}) \arrow[r, equal] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cof}(\mathcal{I}) \arrow[urr, bend right] & & \\
 & & \mathrm{add}(\mathcal{I}) \arrow[r] \arrow[u] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r, equal] \arrow[u] & \mathrm{cov}(\mathcal{I}) \arrow[u] & & \\
\omega_1 \arrow[r] \arrow[urr, bend left] & \mathrm{add}(\mathcal{J}) \arrow[r] & \mathrm{cov}(\mathcal{J}) \arrow[r] & \mathrm{cov}(\sigma\mathcal{G}_0). \arrow[u] & & & &
\end{tikzcd}$$

The article $[P5]$, written jointly with Banakh, Morayne and Żeberski, examined the cardinal coefficients for topologically nontrivial invariant σ -ideals on Euclidean spaces. Recall, by a nontrivial σ-ideal we understand a σ-ideal containing uncountable sets and not equal to the family of all subsets of the space considered. As previously, first we determine the greatest with respect to inclusion non-trivial σ-ideal that is topologically invariant having a base with Baire property. This is the ideal of all meager sets $\mathcal{M}$ on $\mathbb{R}^n$ (Theorem 2.1 in $[P5]$).

We have also found the least with respect to inclusion topologically invariant σ-ideal with analytic base. This ideal, denoted by $\sigma\mathcal{C}_0$, is generated by the family of so-called of *tame*-Cantor sets (Theorem 2.2 in $[P5]$). A tame-Cantor set is a set $h[C]$, where $h : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism of $\mathbb{R}^n$ and $C \subseteq \mathbb{R} \times \{0\}^{n-1}$ is a Cantor set (on a copy of $\mathbb{R}$). Of course, all tame-Cantor sets are ambiently homeomorphic to each other. One of equivalent definitions of a tame-Cantor set is that a closed subset of $\mathbb{R}^n$ is a tame-Cantor set, if for any $\epsilon > 0$, there is a finite family $\mathcal{F}$ of homeomorphic copies of n-dimensional cube $[0,1]^n$, each of them of diameter smaller than ϵ, such that the desired set is contained in interior of $\bigcup \mathcal{F}$. The main result of $[P5]$ concerning cardinal coefficients is the following theorem.

Theorem 53 ([P5], Thm 2.4)**.** *For $\sigma\mathcal{C}$ on $\mathbb{R}^n$ the following equalities hold*

- $non(\sigma\mathcal{C}) = non(\mathcal{M})$,
- $cov(\sigma\mathcal{C}) = cov(\mathcal{M}))$,
- $add(\sigma\mathcal{C}) = add(\sigma\mathcal{C}, \mathcal{M}) = add(\mathcal{M})$,
- $cof(\sigma\mathcal{C}) = cof(\sigma\mathcal{C}, \mathcal{M}) = cof(\mathcal{M})$.

Then for nontrivial topological invariant σ -ideals $\sigma\mathcal{C} \subseteq \mathcal{I} \subseteq \mathcal{M}$ with analytic base in $\mathbb{R}^n$, we have the following relations between cardinal coefficients.

Corollary 13 ([P5], Corollary 2.3)**.** *For any nontrivial topologically invariant σ-ideal $\mathcal{I}$ with analytic base in $\mathbb{R}^n$ we have*

- $non(\mathcal{I}) = non(\mathcal{M})$,
- $cov(\mathcal{I}) = cov(\mathcal{M})$,
- $add(\mathcal{I}) \leq add(\mathcal{M})$,
- $cof(\mathcal{M}) \leq \mathrm{cof}(\mathcal{I})$.

Here the inequalities can be sharp as one can see in Example 2.6 of $[P5]$; for the non-trivial topologically invariant σ -ideal $\mathcal{I}$ on $\mathbb{R}^2$ generated by the segment $[0,1] \times \{0\}$, $add(\mathcal{I}) = \omega_1$, while $cof(\mathcal{I}) = \mathfrak{c}$.

The above corollary can be represented by the following diagram

$$\begin{tikzcd} & \mathrm{non}(\mathcal{I}) \arrow[r, equal] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{I}) \arrow[r] & \mathfrak{c} \\ \omega_1 \arrow[r] & \mathrm{add}(\mathcal{I}) \arrow[r] \arrow[u] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r, equal] \arrow[u] & \mathrm{cov}(\mathcal{I}) \arrow[u] & \end{tikzcd}$$

From the above corollary we have obtained relations between cardinal coefficients for the σ-ideals $\sigma\mathcal{D}_k$ generated by the closed subsets of $\mathbb{R}^n$ of topological dimension $k < n$.

Theorem 54 ([P5], Thm 2.7)**.** *For any nonnegative integer $k < n$ the σ-ideal $\sigma\mathcal{D}_k$ generated by the closed subsets of $\mathbb{R}^n$ of dimension equal at most k we have*

- $non(\sigma\mathcal{D}_k) = non(\mathcal{M})$,
- $add(\sigma\mathcal{D}_k) = add(\mathcal{M})$,
- $cov(\sigma\mathcal{D}_k) = cov(\mathcal{M})$,
- $cof(\sigma\mathcal{D}_k) = cof(\mathcal{M})$.

In [*P*6], an article written jointly with with Banakh and Żeberski, we consider the classification of all σ-ideals with Borel base which are invariant with respect to a good Borel measure defined on the Cantor space.

An ordered pair (X, λ) is called a Cantor measure space, if X is homeomorphic to the Cantor space 2^ω and $\lambda : Bor(X) \to [0, \infty)$ is a σ -additive Borel function, vanishing on all singletons of the space X. We say that two measure-spaces $(X, \lambda), (Y, \mu)$ are isomorphic if there exists a homeomorphism h between the spaces X, Y such that for any Borel set $B \subseteq X$, $\lambda(B) = \mu(h[B])$.

We say that a mapping $h : X \to X$ is invariant with respect to λ if for every Borel set $B \subseteq X$, $\lambda(h[B]) = \lambda(b)$. Furthermore, we say that a σ -ideal $\mathcal{I}$ defined on (X, λ) is invariant with respect to λ if for any set $A \in \mathcal{I}$ and measure-preserving homeomorphism $h : X \to X$ we have $h[A] \in \mathcal{I}$.

Let us note that there is a measure λ on the Cantor set X such that the group $\mathcal{H}_\lambda(X)$ of all homeomorphisms preserving measure λ consists of one element only, $\mathcal{H}_\lambda(X) = \{id_X\}$. Then every σ-ideal on X with Borel base is invariant with respect to the measure λ. Then there are $2^{\mathfrak{c}}$ many λ-invariant σ-ideals with Borel base on X.

A Cantor measure space (X, λ) is called good if for any non-empty open set $U \subseteq X$, $\lambda(U) > 0$ and λ is *homogeneous* in the following sense: for any two clopen sets $U, V \subseteq X$ such that $\lambda(U) < \lambda(V)$, there is a clopen subset $U' \subseteq V$ such that $\lambda(U) = \lambda(U')$.

Akin in [Akin] proved that every two good Cantor measure spaces $(X, \lambda), (Y, \mu)$ are isomorphic if and only if the spectra $\lambda(Clopen(X))$ and $\mu(Clopen(Y))$ are equal, where

$$\lambda(Clopen(X)) = \{\lambda(U) : \ U \subseteq X \text{ jest clopen w } X\}.$$

The main result is contained in the following theorem.

Theorem 55 (P6, Thm. 1.1)**.** *Every nontrivial invariant σ -ideal with analytic base on a good Cantor measure space (X, λ) is one of the following σ-ideals*

$$\mathcal{E},\ \mathcal{M} \cap \mathcal{N},\ \mathcal{M},\ \mathcal{N}.$$

Here $\mathcal{E}$ is the σ -ideal generated by the collection of all closed zero measure sets with respect to λ.
We have the following diagram

$$\begin{tikzcd} & & & & \mathcal{M} \arrow[dr] & \\ {[X]^{\leq 0}} \arrow[r] & {[X]^{\leq \omega}} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{M} \cap \mathcal{N} \arrow[ur] \arrow[dr] & & P(X) \\ & & & & \mathcal{N} \arrow[ur] & \end{tikzcd}$$

In the proof of the above theorem, the homogeneity of λ plays an important role. The following lemmas, which can also be considered as separate results, were used in the proof of this theorem.

Lemma 3 ([P6], Lemma 2.4)**.** *Every measure-preserving homeomorphism $h : A \to B$ between two closed nowhere dense sets $A, B \subseteq X$ of a good Cantor measure space (X, λ) can be extended to a measure preserving homeomorphism $f : X \to X$ of the whole space X.*

Lemma 4 ([P6], Lemma 2.5)**.** *Let us assume that $(X, \lambda), (Y, \mu)$ are two Cantor measure spaces, such that $\lambda(X) < \mu(Y)$. If $G_X \subseteq X$, $G_Y \subseteq Y$ are G_δ null sets with respect to the measures λ and μ, respectively, and G_Y is dense in Y, then there is a measure-preserving embedding $f : X \to Y$ such that $f[G_X] \subseteq G_Y$.*

Lemma 5 ([P6], Lemma 2.7)**.** *If $A \subseteq X$ is a closed subset of a good Cantor measure space (X, λ) and $\lambda(A) > 0$, then for every positive real $\epsilon > 0$, there is a finite set of measure (λ) preserving homeomorphisms $h_1, \ldots, h_n$ of X such that*

$$\lambda(\bigcup_{i=1}^{n} h_i[[A]) > \lambda(X) - \epsilon.$$

Lemma 6 ([P6], Lemma 2.10)**.** *For any two F_σ meager subsets A, B of a good Cantor measure space (X, λ), such that $\lambda(A) = \lambda(B) = \lambda(X)$, there is a measure-preserving homeomorphism $h : X \to X$, such that $h[A] = B$.*

Lemma 7 ([P6], Lemma 2.11)**.** *If $A \subseteq X$ is an analytic subset of a Cantor measure space (X, λ) which is not a member of the σ-ideal $\mathcal{E}$, then A contains a G_δ set $G \subseteq A$ such that $\lambda(G) = 0$ and $\lambda \restriction \overline{G}$ is a strictly positive measure (i.e. taking a positive value on each non-empty open set contained in $\overline{G}$).*

Together with Maciej Burnecki, we have considered problems related to the so-called coarse-topologies on the group G of all $\mathcal{N}$-invariant transformations on the unit segment $\mathbb{I} = [0, 1]$ preserving the σ-ideal of null sets with respect to the Lebesgue measure. These topologies are defined using a fixed Orlicz function $\varphi : \mathbb{R} \to [0, \infty]$ which satisfies the so-called Δ' condition with a positive constant $c > 0$.

The first result in $[P7]$ is a theorem which says that if the Orlicz function satisfies Δ' condition with a constant $c > 0$ and if $h : [0, \infty) \to [0, \infty)$ is a Borel measurable function for which there is a positive constant $\lambda > 0$ such that for every $x \in [0, \infty)$, $h(x) \leq \varphi^{-1}(\lambda x)$, then for any $\tau \in G$ and $f \in L^0(m)$, the transformation

$$T_\tau^{(h)} = (f \circ \tau^{-1})(h \circ \omega_\tau)$$

is a bounded linear operator on the Orlicz space $L^\varphi(m)$ and the inequality

$$|T_\tau^{(h)}\|_\varphi \leq \max\{1, c\lambda\}$$

holds. Here m is the Lebesgue measure on $\mathbb{I}$, ω_r is the Radon-Nikodym derivative of the measure $m \circ \tau$ for $\tau \in G$ and $L^0(m)$ is the set of all m-measurable functions on the unit interval $\mathbb{I}$.

The Δ' condition ensures that the Orlicz space $L^\varphi(m)$ is separable and the result mentioned above guarantees that the set

$$G_h = \{T_\tau^{(h)} : \ \tau \in G\}$$

is bounded in the operator space $\mathcal{L}(L^\varphi(m))$ with respect to the strong operator topology generated by the base of the form:

$$V(P, \epsilon, x_1, \ldots, x_n) = \{Q \in \mathcal{L}(L^\varphi(m)) : \ (\forall i \in \{1, \ldots, n\}) \ \ |Q(x_i) - P(x_i) \mid < \epsilon\}.$$

Moreover, the separability of the Orlicz space $L^\varphi(m)$ implies that the topology restricted to a bounded G_δ set $W \subseteq \mathcal{L}(L^\varphi(m))$ is metrizable as follows

$$(\forall P, Q \in W) \ \left(d(P, Q) = \sum_{n \in \omega} \frac{\|P(f_n) - Q(f_n)\|}{2^n \|f_n\|}\right),$$

where $\{f_n : n \in \omega\}$ is a fixed, countable dense set in the Orlicz space $L^\varphi(m)$. G_h is the image of the mapping T^h defined by

$$G \ni \tau \mapsto T^{(h)}(\tau) = T^{(h)}_\tau \in \mathcal{L}(L^\varphi(m)).$$

By the $\Theta_{\varphi,h}$ we denote the topology on the group G induced by the topology on G_h by mapping $T^{(h)}$.

The main result of this paper says that all topologies $\Theta_{\varphi,h}$ are equal whenever φ satisfies the Δ' condition, h is any Borel measurable function defined on the interval $[0,\infty)$ such that $h(0) = 0$ and the following two conditions are satisfied:

- $(\exists\lambda > 0)(\forall x, y \in [0,\infty))\ (|\varphi(h(x)) - \varphi(h(y))| \leq \lambda|x - y|)$,
- $(\exists\eta > 0)(\forall x, y \in [0,\infty))\ (|\varphi^{-1}(x) - \varphi^{-1}(y)| \leq \eta|h(x) - h(y)|)$.

These topologies were introduced and investigated on the L^p spaces by Choksie and Kakutani, see [CK], and then some results were transfered to Orlicz spaces by Burnecki [Burn].

My first scientific paper was in the domain of theoretical physics. In the quantum mechanics, there are two types of elementary indistinguishable particles, namely, those that have rational spin (angular quantum momentum) which are called *fermions* (protons, electrons, quarks) and those that have integer spin called *bosons* such as photons, gluons responsible for the transmission of the strong interactions, and the two bosons W and Z which transfer the weak interactions. Pauli's exclusion principle prohibits the appearance of two fermions in a one quantum state, while bosons can exist in the same quantum state in any quantity. These phenomena of interactions of many particles, can be described in the so-called second quantization formalism, where the basic operators are defined on a separable Hilbert space. There are the so-called creation a^+ and annihilation a operators. The creation and annihilation operators for bosons satisfy the following relations: for any $i, j \in n$ for $n \in \omega$

$$a_i a_j^+ - a_j^+ a_i = \delta_{ij} 1 \ \wedge\ a_i a_j - a_j a_i = 0 \ \wedge\ a_i^+ a_j^+ - a_j^+ a_i^+ = 0 \qquad \textbf{CCR} \text{ relations}$$

while for fermions we have

$$a_i^+ a_j^+ + a_j a_i = \delta_{ij} 1 \ \wedge\ a_i a_j + a_j a_i = 0 \ \wedge\ a_i^+ a_j^+ + a_j^+ a_i^+ = 0. \qquad \textbf{ACR} \text{ relations}$$

Within the formalism of the second quantization John Bardeen, Leon Cooper and Robert Shrieffer built a theory to explain superconductivity in metals at temperatures near absolute zero. The basic mechanism is the appearance at low temperatures of a quantum state in the metal particles in which two electrons with opposite spins bind to form a boson which is called a Cooper pair. Cooper pairs can be present in the same quantum state in any quantity and then can take the form of a superconducting gas which does not interact with the cristal lattice of a metal. Moreover, the magnetic field does not penetrate to the interior of a metal; such a phenomenon is called the Meissner effect. The BCS theory can predict the phase transition into the superconductivity state at much lower temperatures than the transition temperature of nitrogen from the gas phase into the liquid state. However, the phenomenon of the transition to the superconducting state of certain substances having a structure far from the crystal symmetry has been observed experimentally. There is a chance that this phenomenon can be explained by the quasi-particle theory. Such particles are called *anyones* and were described in a joint paper of Leinass and Myrhaim [LM] and in Wilczek's article [Wil]. The above-mentioned quasi-particles are the basis for explaining the quantum Hall effect. In contrast to bosons or fermions the phase function of many particles may change in an arbitrary manner and these particles form the intermediate system between fermions and bosons. In connection with this phenomenon, in [$P12$], [$P13$] I investigated the existence and properties of operator algebras defined on the quotient spaces of a tensor product of a one particle Hilbert space. These are a generalization of the algebra generated by the operators of creation and annihilation of bosons, or fermions:

$$a_i a_j^+ - \sum_{k,l} c_{ij}^{kl} a_k^+ a_l = \delta_{ij} 1, \quad a_i a_j - \sum_{kl} \tilde{b}_{ij}^{kl} a_k a_l = 0 \tag{1}$$

As a result of these considerations I obtained a necessary condition for the existence of such representations. Namely, the matrices $B = (b_{ij}^{kl})_{i,j,k,l \in \{1,\ldots,n\}}$ and $C = (c_{ij}^{kl})_{i,j,k,l \in \{1,\ldots n\}}$ for some $n \in \mathbb{N}$ are related in the following way:

(2) $$(1-B)(1+\tilde{C}) = 1 \text{ where } \tilde{c}^{kl}_{ij} = c^{ik}_{jl}.$$

Let E be a linear space over the field of complex numbers $\mathbb{C}$ with a fixed basis $\{e_i\}_{i\in I}$ where $I \subseteq \mathbb{N}$. Let $TE = \bigoplus_{n\in\omega} E^{\otimes n}$ denote the tensor algebra. Let $C : E\otimes E \to E\otimes E$ be a fixed linear operator and $g : E\otimes E \to \mathbb{C}$ be the linear functional such that $g(e_i \otimes e_j) = \delta_{ij}$ for any i, j. For $1 \leq i < m$ let $C^{(i)}_m : E^{\otimes m+1} \otimes TE \to E^{\otimes m+1} \otimes TE$ be defined as follows:

$$C^{(i)}_m(x_1 \otimes \ldots \otimes x_i \otimes x_{i+1} \otimes \ldots \otimes x_m \otimes x) = x_1 \otimes \ldots \otimes C(x_i \otimes x_{i+1}) \otimes \ldots \otimes x_m \otimes x.$$

The operator $h_m : E^{\otimes m+1} \to E^{\otimes m-1}$ is defined as follows:

$$h_m = \sum_{k=1}^{m} (1_{k-1} \otimes g \otimes 1_{mk}) C^{(k-1)}_m \ldots C^{(0)}_m.$$

If (2) is fulfilled and if we assume that

- $\tilde{B}^{(2)} C^{(1)} C^{(2)} = C^{(1)} C^{(2)} \tilde{B}^{(1)}$ and
- $[h^{(1)} h^k C^{(k-1)} \ldots C^{(1)} + \ldots + h^{(k-2)} C^{(k-3)} \ldots C^{(1)} h^{(2)}](1 - \tilde{B}^{(1)}) = 0$ for $k \in \mathbb{N} \setminus \{0, 1, 2\}$,

then the operator representation of particles fulfilling equation (1) exists.

If we assume that the operator C has the operator norm on a Hilbert space $E \otimes E$ not greater than 1, then the operator of position can be written as follows:

$$E \ni f \mapsto \phi(f) = (a(f) + a^+(f))/\sqrt{2}$$

and is essentially selfadjoint on the Fock space $\mathcal{F}$, which is a quotient tensor algebra with a scalar product. For any $x, y \in E^{\otimes n+1}$ this scalar product is defined as follows:

$$< x, y >_C = < x, P_{n+1} y >_0 \wedge P_{n+1} = (1 \otimes P_N) R_{n+1} \wedge R_{n+1} = 1 + C^{(1)}_{n+1} + \ldots + C^{(1)}_m \ldots C^{(n)}_m$$

where

$$< u_1 \otimes \ldots \otimes u_n, v_1 \otimes \ldots \otimes u_m >_0 = \delta_{n,m} \prod_{i=1}^{n} < u_i, v_i >, \text{for the product } < \cdot, \cdot >: E^2 \to \mathbb{C}.$$

for any fixed vectors $u_1, \ldots u_n, v_1, \ldots v_m \in E$. In the Fock space $\mathcal{F}$, the creation operators are conjugated to the anihillation operators and vice versa.

Examples of such algebras are of course algebras generated by bosons or fermions. Moreover, for any complex number $q_i \in \mathbb{C}$, such that $|q_i| = 1$ for any $i \leq dimE$ the matrix elements in the case of the so-called colored bosons are expressed as follows:

$$b^{kl}_{ij} = \tilde{b}^{kl}_{ij} = q_j \bar{q}_i \delta_{il} \delta_{jk} \quad \wedge \quad a^{kl}_{ij} = q_i \bar{q}_j \delta_{il} \delta_{jk}$$

and for the colored fermionic statistics:

$$b^{kl}_{ij} = \tilde{b}^{kl}_{ij} = -q_j \bar{q}_i \delta_{il} \delta_{jk} \quad \wedge \quad a^{kl}_{ij} = -q_i \bar{q}_j \delta_{il} \delta_{jk}.$$

The operator algebra determined by the matrices

$$\tilde{b}^{kl}_{ij} = 1 \wedge c_{ij} = q^{2\delta_{ij}}$$

is generated by the creation and annihilation operators related to each other in the following way:

$$b_i b^+_j - q^{2\delta_{ij}} b^+_j b_i = \delta_{ij} 1 \wedge b_i b_j - b_j b_i = 0 = b^+_i b^+_j - b^+_j b^+_i.$$

Using appropriate transformations on the above operators b_i, b^+_j, proposed by Chaichian, Gross and Presnajder [CGP], we obtain the deformed algebra $SU_q(n)$ derived from cans and Woronowicz [PW] generated by the operators A_i, A^+_j of relations defined as:

$$A_i A_j = q A_j A_i \quad \wedge \quad A^+_j A^+_i = q A^+_i A^+_j \quad \text{for } i < j,$$

$$A_i A^+_j = q A^+_j A_i \quad \text{for } i \neq j,$$

$$A_iA_i^+ - q^2A_i^+A_i = 1-(1-q^2)\sum_{k>i}A_k^+A_k.$$

In $[P10]$ I determined the kernel of a twisted scalar product defined on the tensor product TE of the n-dimensional complex space E. This product is definable from the operator $T : E\otimes E \to E\otimes E$ given by the following formula

$$(\forall i,j\in\{1,\dots,n\})(q_{i,j}\in\mathbb{C})\ T(e_i\otimes e_j) = q_{i,j}e_j\otimes e_i \wedge |q_{i,j}| = 1\wedge q_{i,j} = \overline{q_{j,i}}.$$

Then for $x, y \in TE$ the product is given by the formula $\langle x,y\rangle_T = \langle x,Py\rangle_0$ where for $x_1,\dots,x_n,y_1,\dots,y_m \in E$:

$$\langle x_1\otimes\ldots\otimes x_n, y_1\otimes\ldots\otimes y_m\rangle_0 = \delta_{n,m}\prod_{i=1}^{n}\langle x_i,y_i\rangle.$$

Here the projection $P=\bigoplus_{n\in\omega}P_n$ is acting on TE and for $n\in\omega$

$$P_n = (1+T_1^n)\ldots(1+T_{n-1}^n+\ldots+T_{n-1}^n\cdots T_1^n \in End(E^{\otimes n})).$$

If $dimE\geq 2$, then $ker\langle\cdot,\cdot\rangle_T = \{x\in TE:\ \langle x,x\rangle_T\} = \bigoplus_{n\in\omega}Z_n$, where

$$Z_n = \{x\in E^{\otimes n}:\ x=\sum_{\bar{i}}(\sum_{\sigma\in G_{\bar{i}}}\alpha_{\sigma,\bar{i}}F(\sigma))e_{\bar{i}} \longrightarrow (\forall \bar{i}=(i_1,\dots,i_n))\sum_{\sigma\in G_{\bar{i}}}\alpha_{\sigma,\bar{i}} = 0\},$$

$G_{\bar{i}} = Imf_{\bar{i}}$, $\bar{i}=(i_1,\dots,i_n)$, $f_{\bar{i}}((j_1,\dots,j_n)) = \sigma\in S_n$ and the permutations σ fulfill the following equality $\sigma(j_1,\dots,j_n) = \bar{i} = (i_1,\dots,i_n)$. In addition, in the same paper I found the statistical sum for a great canonical ensemble of the mixed bosonic and fermionic particle system described by the Manin algebra specified by the twist quantum operator $T\in End(E\otimes E)$ acting on the square tensor of D-dimensional space E over the field of complex numbers $\mathbb{C}$ with fixed orthonormal basis $\{e_i : i\in\{1,\dots,k\}\}\cup\{f_j :\ j\in\{1,\dots,D-k\}\}$

$$T(e_i\otimes e_i) = e_i\otimes e_i \wedge T(f_j\otimes f_j) = -f_j\otimes f_j,$$
$$i\neq j\longrightarrow T(e_i\otimes e_j) = q_{i,j}e_j\otimes e_i\wedge T(f_i\otimes f_j) = q_{i+k,j+k}f_j\otimes f_i,$$
$$T(e_i\otimes f_j) = q_{i,j+k}f_j\otimes e_i,$$

where the complex numbers $q_{i,j}\in\mathbb{C}$ satisfy the relation

$$q_{i,j}\cdot\overline{q_{i,j}} = 1\wedge q_{i,j} = \overline{q_{j,i}}.$$

Then for the Hermitian operator h defined on the above mentioned basis of the space E, $h(e_i) = \epsilon_i e_i$, $h(f_j) = \eta_j f_j$, for the fixed real numbers $\epsilon_i\eta_j\in\mathbb{R}$ corresponding to the energy of bosonic and fermionic particles, the statistical sums on the n-particle space $E^{\otimes n}$ are as follows

$$tr(P_n e^{d\Gamma_n(h)}) = \prod_{i=1}^{k}(1-e^{\epsilon_i})^{-1}\prod_{j=1}^{D-k}(1+e^{\eta_j}),$$

$d\Gamma_n(h) = h\otimes 1\otimes\ldots\otimes 1 + 1\otimes h\otimes 1\otimes\ldots\otimes 1+\ldots 1\otimes\ldots\otimes 1\otimes h.$

In $[P8], [P9]$ written jointly with with Roman Gielerak, we considered Leinaas-Myrheim systems of quasi-particles with given commutation relations of creation and annihilation operators

$$\begin{cases} a_r(x)a_r(x)^+ - e^{r(x,y)}a_r(x)^+a_r(x) & = \delta(x-y)1\\ a_r(x)a_r(x) - e^{r(x,y)}a_r(x)a_r(x) & = 0\\ a_r(x)^+a_r(x)^+ - e^{r(x,y)}a_r^+(x)a_r^+(x) & = 0,\end{cases}$$

($r : \mathbb{R}^{2d}\to\mathbb{R}$ and $y(x,y)+y(y,x) = 0$ for any $x,y\in\mathbb{R}^d$). Here h is a given one particle hamiltonian $h = h_V = -\Delta^\sigma+\mu$, where $\sigma\in C^3(\partial V)$ is a boundary condition of the class C^3 for the set $V\subseteq\mathbb{R}^d$ and here Δ^σ is the Laplace operator, while μ is the chemical potential and $\beta > 0$ is the so-called inverse temperature. We showed the existence of the thermodynamic limit of the free density energy

$$\lim_{V\nearrow\mathbb{R}^d}\frac{\ln Z_V^r(\beta,\mu)}{|V|}$$

in a finite volume of the grand canonical Gibbs ensemble:

$$Z_V^r(\beta,\mu) \equiv Tr_{\Gamma_r(h_V)}\Gamma_r(e^{-\beta h(\mu)}).$$

Here $\Gamma_r(h_V)$ is a Fock module as the resulting quotient algebra of the tensor algebra $T(H)$ by the two-sided ideal generated by the above-defined commutation relations of the creation a_r^+ and the annihilation operators a_r (here $H = H_V = L^2(V)$ denotes the one particle Hilbert space of all square integrable functions on V on which acts the hamiltonian h_V). From the one particle hamiltonian we require the existence of the trace $tr_H(e^{-\beta h_V(\mu)})$ defined on the one particle Hilbert space H. I should mention here that the above result is the starting point for the study of phase transitions of the Leinaas-Myrheim quasi-particles.

The last three works from the list, $[P14]$, $[P15]$ and $[P16]$, concern physical chemistry. Together with Mirosław Kozłowski and Hubert Kołodziej we studied the behavior of the dielectric response in the time and frequency domains for chemical compounds possessing a certain crystal structure.

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