Summary of professional accomplishments
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1. Education and scientific degrees:

1. Jagiellonian University, M. Sc., 1992 (mathematics)

2. Silesian University, Ph. D., 1998 (mathematics),

Title of Ph. D. Thesis:

Markov semigroups and their application to study of stability of partial differential equations

2. University appointments:

1. 1992–1998 Assistant Professor, Institute of Mathematics, Silesian University,

2. 1998– 2011 Adjunct (Assoc. Professor), Institute of Mathematics, Silesian University,

3. 01.10.2011 — Senior lecturer, Institute of Mathematics, Silesian University

Other appointments:

4. 01.10.1999 - 30.09.2000 Adjunct (Assoc. Professor), Institute of Mathematics, Polish Academy of Sciences,

5. 15.02.2003 - 14.08.2003 Adjunct (Assoc. Professor), Institute of Mathematics, Polish Academy of Sciences,

3. Basis of Habilitation:

The habilitation thesis

Stochastic semigroups, their properties and applications to structured models of population dynamics

contains five published papers.

3.1. List of publications

[KP1] K. Pichór, R. Rudnicki, Continuous Markov semigroups and stability of transport equations, *J. Math. Anal. Appl.* **249** (2000), 668–685,

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DOI:10.1006/jmaa.2000.6968

[KP2] R. Rudnicki, K. Pichór, Markov semigroups and stability of the cell maturity distribution, *J. Biol. Systems* **8** (2000), 69–94.
DOI: 10.1142/S0218339000000067

[KP3] K. Pichór, Asymptotic stability and sweeping of substochastic semigroups, *Ann. Polon. Math.* **103** (2012), 123-134,
DOI:10.4064/ap103-2-2

[KP4] J. Banasiak, K. Pichór, R. Rudnicki, Asynchronous exponential growth of a general structured population model, *Acta Appl. Math.* **119** (2012), 149-166,
DOI: 10.1007/s10440-011-9666-y

[KP5] K. Pichór, Asymptotic behaviour of a structured population model, *Mathematical and Computer Modelling* **57** (2013), 1240–1249,
DOI: 10.1016/j.mcm.2012.10.027

3.2. Description of results

3.2.1. Introduction

My scientific activity is connected with three fields of mathematics: differential equations, probability theory and biomathematics. I have studied problems concerning the asymptotic properties of the solutions of generalized Fokker-Planck equations and transport equations. These equations generate continuous stochastic (Markov) semigroups on $L^1(X)$, it means that the problem of asymptotic stability of the solutions of partial differential equations is equivalent to analogous properties of the stochastic semigroups generated by these equations. I have elaborated new and effective criteria of asymptotic stability of stochastic and substochastic semigroups. These semigroups play an important role in such diverse areas as astrophysics – fluctuations in the brightness of the Milky-Way [7], in the theory of stochastic processes (diffusion processes [26] and jump processes [33, 36]), in the theory of dynamical systems and in population dynamics [9, 27]. In particular, they are applied in the description of structured populations models and fragmentation processes and are used in investigations of distributions of genes in the genome [35], birth and death processes and branching processes [1]. My habilitation thesis concerns applications of stochastic semigroups mainly to structured population models, but some theoretical results are also illustrated by examples of diffusion processes, piecewise deterministic processes and birth-death processes.

3.2.2. Inspiration to my studies

Theory of stochastic operators and semigroups, called also Markov operators and semigroups appeared in different fields of mathematics: Markov chains, ergodic theory, theory of diffusion and transport equations. Results concerning this subject matter which had been obtained until the middle of the Eighties of the last century were basically collected in books of S.R. Foguel [11], E. Nummelin [31] and A. Lasota, M.C. Mackey [23]. The first two books contains mainly results from the theory of Harris operators. The book of Lasota and Mackey [23] is an excellent survey of many results concerning their applications to dynamical systems and partial differential equations (often with some integral perturbation). Inspiration to my studies were some results of A. Lasota, in particular, theorems concerning locally expending maps, the lower function method and asymptotic periodicity. However, they were by no means exhaustive, and in the Nineties it turned out that using techniques of Harris operators led to new results concerning long-time asymptotics, in particular to so called "Foguel alternative". Since these results are essential to understand my thesis, I will present them with necessary definitions in the next section.

3.2.3. Asymptotic properties of stochastic semigroups

In this Section we study asymptotic properties of stochastic semigroups: asymptotic stability, sweeping and the Foguel alternative.

Asymptotic stability

Let the triple (X, Σ, m) be a σ-finite measure space. Denote by D the subset of the space $L^1 = L^1(X, \Sigma, m)$ which contains all densities

$$D = \{f \in L^1 : \ f \geq 0, \ \|f\| = 1\}.$$

A continuous semigroup $\{P(t)\}_{t\geq 0}$ of linear operators on L^1 is said to be a *stochastic or Markov semigroup* if $P(t)(D) \subset D$. If P is an operator, then a semigroup $\{P^n\}_{n\in\mathbb{N}}$, $\mathbb{N} = \{0, 1, 2, \dots\}$, is said to be a discrete time semigroup. A density f_* is called *invariant* if $P(t)f_* = f_*$ for each $t > 0$. The stochastic semigroup $\{P(t)\}_{t\geq 0}$ is called *asymptotically stable* if there is an invariant density f_* such that

$$\lim_{t\to\infty} \|P(t)f - f_*\| = 0 \quad \text{for} \quad f \in D.$$

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A stochastic semigroup $\{P(t)\}_{t\ge 0}$ is called *partially integral* if there exist $t_0 > 0$ and a measurable non-negative function $q(x,y)$ such that

$$\int_X \int_X q(x,y)\, m(dx)\, m(dy) > 0 \tag{1}$$

and

$$P(t_0)f(x) \ge \int_X q(x,y) f(y)\, m(dy) \quad \text{for every } f \in D. \tag{2}$$

If in formula (2) we have equals sign, then the semigroup is called *integral semigroup*. We have the following

Theorem 1 ([32]). *Let $\{P(t)\}_{t\ge 0}$ be a partially integral stochastic semigroup. Assume that the semigroup $\{P(t)\}_{t\ge 0}$ has an invariant density f_* and has no other periodic points in the set of densities. If $f_* > 0$ a.e. then the semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable.*

Now we formulate corollaries which are useful in applications. We say that a stochastic semigroup $\{P(t)\}_{t\ge 0}$ *spreads supports* if for every set $A \in \Sigma$ and for every $f \in D$ we have

$$\lim_{t\to\infty} m(\operatorname{supp} P(t)f \cap A) = m(A)$$

and *overlaps supports* if for every $f, g \in D$ there exists $t > 0$ such that

$$m(\operatorname{supp} P(t)f \cap \operatorname{supp} P(t)g) > 0.$$

Corollary 1 ([32]). *A partially integral stochastic semigroup which spreads supports and has an invariant density is asymptotically stable.*

Corollary 2 ([32]). *A partially integral stochastic semigroup which overlaps supports and has an invariant density $f_* > 0$ a.e. is asymptotically stable.*

These corollaries generalize some earlier results [4, 28, 33, 34] for integral stochastic semigroups. Another proof of Corollary 2 is given in [5].

Sweeping

A stochastic semigroup $\{P(t)\}_{t\ge 0}$ is called *sweeping* with respect to a set $A \in \Sigma$ if for every $f \in D$

$$\lim_{t\to\infty} \int_A P(t)f(x)\, m(dx) = 0. \tag{3}$$

The notion of sweeping was introduced by Komorowski and Tyrcha [22]. Now we formulate the following condition.

(KT): There exists a measurable function f_* such that: $P(t)f_* \le f_*$ for $t \ge 0$, $f_* \notin L^1$, $0 < f_* < \infty$ a.e. and $\int_A f_*\, dm < \infty$.

Theorem 2 ([22]). *Let $\{P(t)\}_{t\geq 0}$ be an integral stochastic semigroup which has no invariant density. Assume that the semigroup $\{P(t)\}_{t\geq 0}$ and a set $A \in \Sigma$ satisfy condition (KT). Then the semigroup $\{P(t)\}_{t\geq 0}$ is sweeping with respect to A.*

In paper [32] it was shown that Theorem 2 holds for a wider class of operators than integral ones.

Theorem 3 ([32]). *Let $\{P(t)\}_{t\geq 0}$ be a stochastic semigroup which overlaps supports. Assume that the semigroup $\{P(t)\}_{t\geq 0}$ and a set $A \in \Sigma$ satisfy condition (KT). Then the semigroup $\{P(t)\}_{t\geq 0}$ is sweeping with respect to A.*

The main difficulty in applying Theorems 2 and 3 is to prove that a stochastic semigroup satisfies condition (KT), in particular that it has a subinvariant function f_*. Now we formulate a criterion for sweeping which will be useful in applications.

Theorem 4 ([32]). *Let X be a metric space and Σ be the σ-algebra of Borel sets. We assume that a stochastic semigroup $\{P(t)\}_{t\geq 0}$ has the following properties:*

(a) *for every $f \in D$ we have $\int_0^\infty P(t)f\,dt > 0$ a.e. or $\sum_{n=0}^\infty P^n f > 0$ a.e. if $\{P(t)\}_{t\geq 0}$ is a discrete time semigroup,*

(b) *for every $y_0 \in X$ there exist $\varepsilon > 0$ and a measurable function $\eta \geq 0$ such that $\int \eta\,dm > 0$ and*

$$q(x,y) \geq \eta(x)\mathbf{1}_{B(y_0,\varepsilon)}(y),$$

where q is a function satisfying (1) and (2). If the semigroup $\{P(t)\}_{t\geq 0}$ has no invariant density then it is sweeping with respect to compact sets.

Foguel alternative

We say that a stochastic semigroup $\{P(t)\}_{t\geq 0}$ satisfies the *Foguel alternative* if it is asymptotically stable or sweeping from a sufficiently large family of sets. For example this family can be all compact sets.

From Corollary 1 and Theorem 4 it follows

Theorem 5. *Let X be a metric space and Σ be the σ-algebra of Borel sets. Let $\{P(t)\}_{t\geq 0}$ be a stochastic semigroup. We assume that there exist $t > 0$ and a continuous function $q : X \times X \to (0,\infty)$ such that*

$$P(t)f(x) \geq \int_X q(x,y)f(y)\,m(dy) \quad \text{for } f \in D. \tag{4}$$

Then this semigroup is asymptotically stable or is sweeping with respect to compact sets.

Using Theorem 5 one can check that the Foguel alternative holds for multi-state diffusion processes [26, 32], [KP7] diffusion with jumps [KP8] and transport equations [KP9].

3.2.2. Results concerning asymptotic properties

In this Section we investigate continuous stochastic semigroups.

In [KP1] we strengthen Theorem 1 in the case of continuous time stochastic semigroups. We give new sufficient conditions for asymptotic stability of partially integral continuous stochastic semigroups.

Theorem 6 (KP1 Theorem 2). *Let (X, Σ, μ) be a σ-finite measure space and let $\{P(t)\}_{t\geq 0}$ be a partially integral stochastic semigroup. Assume that the semigroup $\{P(t)\}_{t\geq 0}$ has the only one invariant density f_*. If $f_* > 0$ a.e. then the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable.*

The assumption that the semigroup $\{P(t)\}_{t\geq 0}$ is continuous in the statement of Theorem 6 is essential. The proof of Theorem 6 is based on the theory of Harris operators given in [11, 18].

Remark 1. *In applications we often replace the assumption that the invariant density is unique by the following one. We assume that there does not exist a set $E \in \Sigma$ such that $m(E) > 0$, $m(X \setminus E) > 0$ and $P(t)E = E$ for all $t > 0$. Here $P(t)$ is the operator acting on the σ-algebra Σ given by: if $f \geq 0$, $\operatorname{supp} f = A$ and $\operatorname{supp} Pf = B$ a.e., then $PA = B$.*

If X is a compact space then from Theorem 4 and Theorem 6 it follows

Corollary 3 (KP12 Corollary 1). *Let X be a compact metric space and Σ be the σ-algebra of Borel sets. Let $\{P(t)\}_{t\geq 0}$ be a stochastic semigroup which satisfies conditions:*
(a) for every $f \in D$ we have $\int_0^\infty P(t)f\,dt > 0$ a.e.,
(b) for every $y_0 \in X$ there exist $\varepsilon > 0$, $t > 0$, and a measurable function $\eta \geq 0$ such that $\int \eta\,dm > 0$ and

$$P(t)f(x) \geq \eta(x) \int_{B(y_0,\varepsilon)} f(y)\,m(dy)$$

for $x \in X$, where $B(y_0, \varepsilon)$ is the open ball with center y_0 and radius ε, Then the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable.

From Theorems 4 and 6 it also follows the Foguel alternative:

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Corollary 4. *Let X be a metric space and Σ be the σ-algebra of Borel sets. Let $\{P(t)\}_{t\geq 0}$ be an integral stochastic semigroup with a continuous and positive kernel $k(t,x,y)$ for $t>0$. If the semigroup $\{P(t)\}_{t\geq 0}$ has an invariant density, then it is asymptotically stable, and if $\{P(t)\}_{t\geq 0}$ has no invariant density, then it is sweeping with respect to compact sets.*

We now give an application of Theorem 6 to jump process. Let $X=\mathbb{R}^d$, $\Sigma=\mathcal{B}(X)$ be the σ-algebra of Borel subsets of X and μ be the Lebesgue measure on X. Consider the following equation

$$\frac{\partial u}{\partial t}=Au-\lambda u+\lambda Pu, \tag{5}$$

where

$$Au=-\sum_{i=1}^{d}\frac{\partial(b_i u)}{\partial x_i},$$

$\lambda>0$ and P is a stochastic operator corresponding to the iterated function system $(S_1(x),\dots,S_N(x),p_1(x),\dots,p_N(x))$. Assume that $S_i: X\to X$, for $i=1,\dots,N$, is a sequence of continuously differentiable transformations, such that $\det S_i'(x)\neq 0$ for almost every x and that $p_i: X\to[0,1]$, $i=1,\dots,N$, is a sequence of continuous functions such that $\sum_{i=1}^{N}p_i(x)=1$ for each $x\in X$. Equation (5) has the following interpretation. We consider particles which move along the solutions of the equation $x'=b(x)$. At any time interval $[t,t+\Delta t]$ a particle with the probability $p_i(x)\Delta t+o(\Delta t)$ jumps from the point x to $S_i(x)$. For each $\bar{x}\in X$ denote by $\pi_t\bar{x}$ the solution $x(t)$ of the equation $x'(t)=b(x(t))$ with the initial condition $x(0)=\bar{x}$. Assume that $\pi_t(X)\subset X$ for all $t\geq 0$. Then equation (5) generates a stochastic semigroup $\{P(t)\}_{t\geq 0}$ on the space $L^1(X,\mathcal{B}(X),\mu)$. Note that $P(t)E\subset E$ for all $t\geq 0$ and some measurable set E if and only if $S_i(E)\subset E$ for all $i=1,\dots,N$ and $\pi_t(E)\subset E$ for all $t\geq 0$. We have the following

Theorem 7 (KP1 Proposition 1.). *Assume that the semigroup $\{P(t)\}_{t\geq 0}$ has a non-zero invariant function and has no non-trivial invariant sets. Let $(i_1,\dots,i_d)$ be a given sequence of integers from the set $\{1,\dots,N\}$. Let $x_0\in X$ be a given point and let $x_j=S_{i_j}(x_{j-1})$ for $j=1,\dots,d$. Set*

$$v_j=S'_{i_d}(x_{d-1})\dots S'_{i_j}(x_{j-1})b(x_{j-1})-b(x_d)$$

for $j=1,\dots,d$. Assume that $p_{i_j}(x_{j-1})>0$ for all $j=1,\dots,d$ and suppose that the vectors $v_1,\dots,v_d$ are linearly independent. Then the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable.

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If the stochastic semigroup $\{P(t)\}_{t\geq 0}$ has a non-zero invariant function $f \in L^1$, then the function $f_* = f^+/\|f^+\|$, assuming that $\|f^+\| > 0$, is an invariant density for $\{P(t)\}_{t\geq 0}$. We prove that the semigroup $\{P(t)\}_{t\geq 0}$ is partially integral. Then from Theorem 6 and Remark 1 the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable.

In [KP1] we consider also randomly controlled dynamical system (see [KP1, Section 3.2] for detail). We have k dynamical systems $\pi_t^i(x)$ corresponding to the equations $x' = b(x,i)$, $i = 1,\ldots,k$ and we exchange their randomly. Denote by $\{P(t)\}_{t\geq 0}$ the semigroup corresponding to this system. Let $(i_1,\ldots,i_{d+1})$ be a sequence of integers from the set $\Gamma = \{1,\ldots,k\}$. For $x \in X$ and $t > 0$ we define the function $\psi_{x,t}$ on the set $\Delta_t = \{\tau = (\tau_1,\ldots,\tau_d) : \tau_i > 0, \tau_1 + \cdots + \tau_d \leq t\}$ by

$$\psi_{x,t}(\tau_1,\ldots,\tau_d) = \pi^{i_{d+1}}_{t-\tau_1-\tau_2-\cdots-\tau_d} \circ \pi^{i_d}_{\tau_d} \circ \cdots \circ \pi^{i_2}_{\tau_2} \circ \pi^{i_1}_{\tau_1}(x).$$

We have the following Theorem concerning asymptotic stability of the semigroup $\{P(t)\}_{t\geq 0}$.

Theorem 8 (KP1 Proposition 2). *Assume that the semigroup $\{P(t)\}_{t\geq 0}$ has a non-zero invariant function and has no non-trivial invariant sets. Suppose that for some $x_0 \in X$, $t_0 > 0$ and $\tau^0 \in \Delta_{t_0}$ we have*

$$\det\left[\frac{d\psi_{x_0,t_0}(\tau^0)}{d\tau}\right] \neq 0. \tag{6}$$

Then the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable.

Note that a measurable set $E \subset X \times \Gamma$ is invariant with respect to the semigroup $\{P(t)\}_{t\geq 0}$ if and only if E is of the form $E = E_0 \times \Gamma$ and $\pi_t^i(E_0) = E_0$ for $t \geq 0$ and $i = 1,\ldots,k$. The proof of Theorem 8 is similar to the proof of Theorem 7.

In the paper [KP3] we investigate *substochastic semigroups*, i.e. semigroups $\{P(t)\}_{t\geq 0}$ of linear and positive operators on L^1 satisfying $\|P(t)\| \leq 1$ for $t \geq 0$. We found new sufficient conditions for asymptotic stability and sweeping of partially integral continuous substochastic semigroups. Our criteria generalize results of [KP1] and [32]. In particular, earlier results concerning asymptotic stability of integral stochastic semigroups which spreads or overlaps supports given in [4, 5, 28] follow from our main theorem.

Theorem 9 (KP3, Main Theorem 3.1). *Let X be a metric space and $\Sigma = \mathcal{B}(X)$. Let $\{P(t)\}_{t\geq 0}$ be a substochastic semigroup on $L^1(X)$ which has the only one invariant density f_* and let $S = \operatorname{supp} f_*$. Assume that $\{P(t)\}_{t\geq 0}$ is*

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a partially integral semigroup with the kernel $k(t,x,y)$ such that

$$\int_S \int_S k(t_0,x,y)\, m(dx)\, m(dy) > 0$$

for some $t_0 > 0$. Moreover, we assume that for some $t_1 > 0$
(a) *there does not exist a nonempty measurable set $B \subsetneq X \setminus S$ such that $P^*(t_1)\mathbf{1}_B \geq \mathbf{1}_B$ and*
(b) *for every $y_0 \in X \setminus S$ there exist $\varepsilon > 0$ and a measurable function $\eta \geq 0$ such that $\int_{X\setminus S} \eta\, dm > 0$ and*

$$k(t_1,x,y) \geq \eta(x) \tag{7}$$

for $x \in X$ and $y \in B(y_0,\varepsilon)$, where $B(y_0,\varepsilon)$ is the open ball with center y_0 and radius ε.
Then for every $f \in D$ there exists a constant $c(f)$ such that

$$\lim_{t\to\infty} \mathbf{1}_S P(t) f = c(f) f_*$$

and for every compact set $F \in \Sigma$ and $f \in D$ we have

$$\lim_{t\to\infty} \int_{F\cap X\setminus S} P(t)f(x)\, m(dx) = 0.$$

The proof of this theorem is based on the results concerning properties of Harris operators [11, 18]. Note that, if a substochastic semigroup $\{P(t)\}_{t\geq 0}$ on $L^1(X)$ has the only one invariant density f_* and $\operatorname{supp} f_* = X$ then $\{P(t)\}_{t\geq 0}$ is a stochastic semigroup [KP3 Lemma4.2] and from Theorem 9 it follows that

$$\lim_{t\to\infty} P(t) f = f_*$$

for each density f.

We now give an application of Theorem 9 to a birth-death process. A general birth-death process on $\mathbb{N} = \{0,1,\ldots\}$ is described by the following system of equations

$$x_i'(t) = -a_i x_i(t) + b_{i-1} x_{i-1}(t) + d_{i+1} x_{i+1}(t) \tag{8}$$

for $i \geq 0$, where $b_{-1} = d_0 = 0$, $b_i \geq 0$, $d_{i+1} \geq 0$ for $i \geq 0$, $a_0 = b_0$, $a_i = b_i + d_i$ for $i \geq 1$. Let us assume that the system (8) generates a stochastic semigroup $\{P(t)\}_{t\geq 0}$ and that $d_i > 0$ for $i > 0$.

We now suppose that there exists $n > 0$ such that $b_n = 0$, $b_i > 0$ for $i \neq n$. We claim that Theorem 9 applies. Thus, for each $\bar{x} \in l^1$ there exists a constant

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$c(\bar{x})$ such that the solution of (8) with the initial condition $x(0) = \bar{x}$ satisfies

$$\lim_{t\to\infty} x_i(t) = c(\bar{x})x_i^* \quad \text{for } i \le n,$$

$$\lim_{t\to\infty} x_i(t) = 0 \qquad\quad \text{for } i > n,$$

where $x^* = (x_i^*)$ is an invariant density with $\operatorname{supp} x^* = \{0, 1, \dots, n\}$.

3.2.5. Results concerning applications to structured population models.

Size structured model and age structured model

In the paper [KP2] we investigate a model of size-structured cell populations. We assume that a cell is characterized by its size (maturity) x which ranges from $x = a$ to $x = 1$. The cell size grows according to equation $x'(t) = g(x(t))$. Cells can die or divide with rates $\mu(x)$ and $b(x)$. We assume that the cells cannot divide before they have reached a minimal maturation $a_0 \in (a, 1)$. We denote by $\mathcal{P}(x, A)$ the probability for a daughter cell born from a mother cell of size x to have a size in the set A. The function $u(x, t)$ describing the distribution of the size satisfies the following equation

$$\frac{\partial u}{\partial t} = -\frac{\partial (gu)}{\partial x} - (\mu + b)u + 2P(bu), \tag{9}$$

where $P : L^1[a, 1] \to L^1[a, 1]$ is a Markov operator such that $P^*\mathbf{1}_B(x) = \mathcal{P}(x, B)$.

A simple version of size-structured model was introduced for the first time probably by Bell and Anderson [6] and was studied and generalized in many papers (see e.g. [9, 12, 30, 38]). Our model includes models of a population of organisms reproducing by binary fission with equal [9, 12, 14, 27] and unequal division [2, 13, 16, 19] and both types of binary fission models [21].

We assume that a new born daughter cell has the size which is randomly distributed in the interval $(a, x - h]$, where h is a positive constant and $x \ge a_0$ is the size of a mother cell, that is,

$$\mathcal{P}(x, [a, x - h]) = 1 \qquad \text{for all} \quad x \in [a, 1].$$

Since the cells have to divide before they reach the maximal size $x = 1$, we assume that

$$\int_a^1 b(x)\,dx = \infty. \tag{10}$$

We prove that equation (9) generates a continuous semigroup $\{T(t)\}_{t\ge 0}$ of linear operators on $L^1[a, 1]$ [KP2 Theorem1]. The main result of the paper is the theorem on asynchronous exponential growth of the population (AEG) [KP2 Theorem 3]. In order to formulate the theorem on (AEG) we need auxiliary conditions. We additionally assume that $\mathcal{P}$ satisfies one of the following conditions: (I): there exists a measurable function $q : [a, 1] \times [a_0, 1] \to [0, \infty)$ such that $\int_a^1 \int_{a_0}^1 b(x)q(y, x)\, dx\, dy > 0$ and $\mathcal{P}(x, A) \geq \int_A q(y, x)\, dy$ for $x \in [a_0, 1]$ and $A \in \Sigma$, or (I'): there exist $x_0 \in (a, 1)$, $\varepsilon > 0$ and function $r\colon (x_0 - \varepsilon, x_0 + \varepsilon) \to [a, 1]$, such that $r'(x_0) \neq 0$, $b(x_0) > 0$, $g(r(x_0)) \neq r'(x_0)g(x_0)$ and

$$\mathcal{P}(x, \{r(x)\}) \geq \varepsilon \quad \text{for } x \in (x_0 - \varepsilon, x_0 + \varepsilon).$$

Note that in the model with equal division we have $\mathcal{P}\left(x, \{\frac{x}{2}\}\right) = 1$. In this case $r(x) = x/2$. If we assume that there is $x_1 \in (a, 1)$ such that $b(2x_1) > 0$, $2g(x_1) \neq g(2x_1)$ then condition (I') holds. If we consider the model with unequal division, i.e. $\mathcal{P}(x, A) = \int_A q(y, x)\, dy$ then condition (I) holds. The process of reproduction can be neither equal nor unequal division [21]. In this case condition (I) also holds.

The proof of Theorem [KP2 Theorem 3] goes as follows. Equation (9) can be written as an evolution equation $u'(t) = Au$. First we show that A is an infinitesimal generator of a continuous semigroup $\{T(t)\}_{t\geq 0}$ of linear operators on $L^1[a, 1]$. Then we prove that there exist $\lambda \in \mathbb{R}$ and continuous and positive functions v and w such that $Av = \lambda v$ and $A^*w = \lambda w$. From this it follows that the semigroup $\{P(t)\}_{t\geq 0}$ given by $P(t) = e^{-\lambda t}T(t)$ is a Markov semigroup on the space $L^1(X, \Sigma, m)$, where m is a Borel measure on the interval $[a, 1]$ given by $m(B) = \int_B w(x)\, dx$ [KP2 Theorem 2]. Moreover, for some $c > 0$ the function $f_* = cv$ is an invariant density with respect to $\{P(t)\}_{t\geq 0}$. Finally, from Theorem 1 we conclude that this semigroup is asymptotically stable. Since the Lebesgue measure and the measure m are equivalent we obtain that $e^{-\lambda t}u(\cdot, t)$ converges to $f_*\Phi(u)$ in $L^1(a, 1)$, where $\Phi(\psi) := \int_a^1 \psi(x)w(x)\, dx$.

In [KP4] we consider a structured cell population model described by a first order partial differential equation perturbed by a general birth operator. We investigate different processes usually treated separately. Our scheme describes in a unified way a wide class of structured population models and also includes the classical age structured Sharpe-Lotka-McKendrick model [29, 37]. We start with the Kolmogorov's backward equation

$$\frac{\partial v}{\partial t} = g(x)\frac{\partial v}{\partial x} - \mu(x)v(t, x) + \int_a^1 v(t, y)\mathcal{P}(x, dy),$$

where $\mu(x)$ is the rate of loss of individuals with parameter x by death or by division and $\mathcal{P}(x, A)\Delta t$ is the probability that an individual with the parameter

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x has a descendant in the set A. If $u(t,x)$ is the distribution of x then u satisfies the following Fokker-Planck equation

$$\frac{\partial u}{\partial t} + \frac{\partial (g(x)u)}{\partial x} = -\mu(x)u(t,x) + Pu(t,x), \tag{11}$$

with the boundary condition

$$g(a)u(t,a) = \int_a^1 b^a(x)u(t,x)\,dx, \tag{12}$$

where $b^a(m)$ describes the rate at which the individuals with the parameter m produce individuals with the minimal parameter a, like in the age-structured population model. We prove that the system (11) — (12) with the initial condition $u(0,x) = u_0(x)$ generates a strongly continuous positive semigroup $\{T(t)\}_{t\geq 0}$, on $L^1[a,1]$ [KP4 Theorem 1]. In the proof of [KP4 Theorem 1] we apply Desch's theorem, see e.g. [8] or [3, Theorem 5.13]. Using the theory of positive stochastic semigroups we establish new criteria for an asynchronous exponential growth (AEG) of solution to the system (11) — (12) [KP4 Theorem 2]. The proof of Theorem [KP4 Theorem 2] goes as follows. We replace a continuous semigroup $\{T(t)\}_{t\geq 0}$ of linear operators with a stochastic semigroup $\{P(t)\}_{t\geq 0}$ and using Theorem 6 we show that the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable.

In the paper [KP5] we study a general maturity-structured population model which is also described by the system (11) — (12). Note that models of cellular replication studied in [KP2], [KP4] are based on assumption that the death or birth rate is unbounded at $m = 1$. In [KP5] we assume that $g(1) = 0$. This condition guarantees that the maturity variable cannot exceed 1. An advantage of our approach [KP5], is that both the birth and death rates can be bounded. Using the Phillips perturbation theorem [10] we show that the system generates a C_0 semigroups $\{T(t)\}_{t\geq 0}$. The main result of the paper is the theorem on asynchronous exponential growth (AEG) of the population [KP5 Theorem 1]. The idea of the proof is similar to that of [KP4 Theorem 2]. The most difficult part is to check that the eigenvector of the adjoint equation is bounded and separated from zero.

4. Other results

In the paper [KP6] we consider a stochastically perturbed discrete time dynamical system of the form $x_{n+1} = S(x_n)\xi_n$. We found sufficient conditions for the weak convergence of the distributions of x_n to a stationary measure. The proof is based on a theorem of A. Lasota and J.A. Yorke [24] concerning Markov operators on measures.

In [KP7] we study semigroups generated by parabolic systems describing the evolution of densities of two-state diffusion processes. In the paper we formulate new sufficient conditions for asymptotic stability of abstract stochastic semigroups [KP7 Theorem 2] and apply them to the system of partial differential equations [KP7 Theorem 1]. The proofs are based on the Foguel alternative. In order to exclude sweeping we introduce a notion called a Hasminskiĭ function. Consider a stochastic semigroup $\{P(t)\}_{t\geq 0}$ and let A be the infinitesimal generator of $\{P(t)\}_{t\geq 0}$. Let $\mathcal{R} = (I - A)^{-1}$ be the resolvent operator at point 1. A measurable function $V : X \to [0, \infty)$ is called a *Hasminskiĭ function* for the stochastic semigroup $\{P(t)\}_{t\geq 0}$ and a set $Z \in \Sigma$ if there exist $M > 0$ and $\varepsilon > 0$ such that

$$(13)\quad \int_X V(x)\mathcal{R}f(x)\,dm(x) \leq \int_X (V(x) - \varepsilon)f(x)\,dm(x) + \int_Z M\mathcal{R}f(x)\,dm(x).$$

Theorem 10. *Let $\{P(t)\}$ be a stochastic semigroup generated by the equation*

$$\frac{\partial u}{\partial t} = Au.$$

Assume that there exists a Hasminskiĭ function for the semigroup $\{P(t)\}_{t\geq 0}$ and a set Z. Then the semigroup $\{P(t)\}$ is not sweeping with respect to the set Z.

In application we take V such that the function A^*V is "well defined" and it satisfies the following condition $A^*V(x) \leq -c < 0$ for $x \notin Z$. Then we check that V satisfies inequality (13). The function V was called a Hasminskiĭ function because he showed [15] that the semigroup generated by the Fokker-Planck equation has an invariant density if there exists a positive function V such that $A^*V(x) \leq -c < 0$ if $\|x\| \geq r$. We apply the method of Hasminskiĭ function to semigroups generated by two-state diffusion processes. This process is described by the following system of equations

$$(14)\quad \begin{cases} \dfrac{\partial u_1}{\partial t} = -pu_1 + qu_2 + A_1u_1 \\[2mm] \dfrac{\partial u_2}{\partial t} = pu_1 - qu_2 + A_2u_2. \end{cases}$$

A semigroup generated by this system satisfies the Foguel alternative. In order to prove asymptotic stability it is sufficient to construct a proper Hasminskiĭ function. One can check that if there exist non-negative C^2-functions V_1 and V_2 such that

$$-p(x)V_1(x) + p(x)V_2(x) + A_1^*V_1(x) \leq -\varepsilon,$$
$$q(x)V_1(x) - q(x)V_2(x) + A_2^*V_2(x) \leq -\varepsilon$$

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for $\|x\| \geq r$, then the corresponding Markov semigroup is asymptotically stable [KP7 Theorem 1]. In this case inequality (13) was proved by using some generalization of the maximum principle. The paper contains some interesting examples and remarks. One might expect that if both equations $\frac{\partial u}{\partial t} = A_1 u$ and $\frac{\partial u}{\partial t} = A_2 u$ are asymptotically stable then system (14) is also asymptotically stable. But it is not true (see [KP7 Remark 2]). It is interesting that system (14) can be asymptotically stable when both equations $\frac{\partial u}{\partial t} = A_1 u$ and $\frac{\partial u}{\partial t} = A_2 u$ are not stable [KP7 Remark3].

In [KP8] we study diffusion with jumps. Consider the following equation

$$\frac{\partial u}{\partial t} = Au - \lambda u + \lambda P u, \tag{15}$$

where $\lambda > 0$,

$$Au = \sum_{i,j=1}^{d} \frac{\partial^2 (a_{ij} u)}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \frac{\partial (b_i u)}{\partial x_i} \tag{16}$$

and P is a stochastic operator corresponding to the iterated function system

$$(S_1(x), \ldots, S_N(x), p_1(x), \ldots, p_N(x)).$$

We assume that for each j we have

$$\lim_{\|x\| \to \infty} \|S_j(x)\| = \infty.$$

Assume that

$$\lim_{\|x\| \to \infty} 2\langle x, b(x) \rangle + \lambda \sum_{j=1}^{n} p_j(x) \left(\|S_j(x)\|^2 - \|x\|^2 \right) = -\infty,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$. Then a stochastic semigroup $\{P(t)\}_{t \geq 0}$ generated by equation (15) is asymptotically stable [KP8 Theorem 3]. Our criterion generalizes the results of [17] and [33]. The proof is based on the method of Hasminskiĭ function. Inequality (13) was proved by using some generalization of the maximum principle.

In [KP9] we investigate a semigroup generated by transport equation. Consider a partial differential equation with an integral perturbation

$$\frac{\partial u}{\partial t} + \lambda u = - \sum_{i=1}^{d} \frac{\partial (b_i u)}{\partial x_i} + \lambda \int k(x, y) u(y, t)\, dy. \tag{17}$$

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If $k(x, y)$ is a continuous and strictly positive function and there exists a C^1-function $V : X \to [0, \infty)$ such that

$$\sum_{i=1}^{d} b_i \frac{\partial V}{\partial x_i} - \lambda V(x) + \lambda \int k(y, x)V(y)\, dy \le -c < 0$$

for $\|x\| \ge r$, $r > 0$, then a stochastic semigroup $\{P(t)\}_{t\ge0}$ generated by equation (17) is asymptotically stable [KP9 Theorem 1]. Equation (17) can be written as an evolution equation $\frac{\partial u}{\partial t} = Au$. The proof of inequality (13) is different and based on an approximation of V by a sequence of elements from the domain of the operator A^*.

In the paper [KP10] we present results in the theory of stochastic operators and semigroups The main subject of the paper are stochastic semigroups generated by partial differential equations (transport equations). Equations of this type appear in the theory of stochastic processes (diffusion processes and jump processes), in the theory of dynamical systems and in population dynamics.

In [KP11] we consider a system of stochastic equations which models the population dynamics of a prey-predator type. We analyse long-time behaviour of densities of the distributions of the solutions. We prove that the densities can coverge in L^1 to an invariant density or can converge weakly to a singular measure.

The paper [KP12] is devoted to a stochastic process used in modelling gene expression in eucaryotes [25]. Similar model was also investigated in [20]. We show that its distributions satisfy a Fokker-Planck-type system of partial differential equations. Then, we construct a stochastic semigroup corresponding to this system. The main result of the paper is asymptotic stability of the involved semigroup in the set of densities [KP12 Theorem3]. The strategy of the proof is as follows. First, it is shown that the transition function of the related stochastic process has a kernel (integral) part. Then we find a set E on which the density of the kernel part of the transition function is positive. Next we show that the set E is an "attractor". Since the attractor E is a compact set, from Corollary 5 it follows that the semigroup is asymptotically stable.

In [KP13] we present some recent results concerning the generation and the long-time behaviour of stochastic semigroups and illustrate them by some biological applications. The general results are applied to biological models described by piecewise deterministic Markov processes: birth-death processes, the evolution of the genome, genes expression and physiologically structured models.

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Other publications by Katarzyna Pichór

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