Summary of scientific achievements

1. Name and surname Katarzyna Kuhlmann (previously Osiak)

2. Academic degrees

- Master of Science in Mathematics, degree granted in July 2001 at University of Silesia, Katowice, Poland. Title of Master's thesis: *Representations of groups of small order.*
  Advisor: dr hab. Andrzej Sładek, prof. UŚ
- Doctor's Degree in Mathematical Sciences, degree granted in September 2005 at University of Silesia, Katowice, Poland. Title of Ph.D. thesis: *Spaces of higher level orderings.*
  Advisor: dr hab. Andrzej Sładek, prof. UŚ

3. Employment history

- from October 2016: University of Szczecin, Poland, assistant professor
- 2013–1014: University of Saskatchewan, Saskatoon, Canada, sessional lecturer
- 2008–2009: Ben Gurion University of the Negev, Beer Sheva, Israel, postdoctoral fellow
- 2008 (6 weeks): University of Saskatchewan, Saskatoon, Canada, postdoctoral fellow
- 2003–2016: University of Silesia, Katowice, Poland, assistant professor
- 1995–1999: Institue of Occupational Medicine and Environmental Health, Sosnowiec, Poland, technician
- 1994–1995: Institute of Meteorology and Water Management, Katowice, Poland, technician

4. Indication of scientific achievement

Title of habilitation thesis:

Spaces of $\mathbb{R}$-places

Papers constituting the habilitation thesis:

[1] K. Osiak, *The Boolean space of* $\mathbb{R}$*-places*, Rocky Mountain J. Math. **40** (2010), no. 6, 2003–2011

[2] I. Efrat, K. Osiak, *Topological spaces as spaces of* $\mathbb{R}$*-places*, J. Pure Appl. Algebra **215** (2011), no. 5, 839–846

[3] F.-V. Kuhlmann, M. Machura, K. Osiak, *Metrizability of spaces of* $\mathbb{R}$*-places of function fields of transcendence degree 1 over real closed fields*, Comm. Algebra **39** (2011), no. 9, 3166–3177

[4] M. Machura, M. Marshall, K. Osiak, *Metrizability of the space of* $\mathbb{R}$*-places of a real function field*, Math. Z. **266** (2010), no. 1, 237–242

[5] F.-V. Kuhlmann, K. Kuhlmann, *Embedding theorems for spaces of* $\mathbb{R}$*-places of rational function fields and their products*, Fund. Math. **218** (2012), no. 2, 121–149

[6] K. Kuhlmann, *The structure of spaces of* $\mathbb{R}$*-places of rational function fields over real closed fields*, Rocky Mountain J. Math. **46** (2016), no. 2, 533–557

[7] P. Koprowski, K. Kuhlmann, *Places, cuts and orderings of function fields*, J. Algebra **468** (2016), 253–274.

A) Introduction and motivation of research

Real algebra has its beginning at the end of the 19^{th} century, when D. Hilbert formulated his famous 17^{th} problem asking whether every polynomial $f \in \mathbb{R}[x_1, ..., x_n]$ which takes only non-negative values is a sum of squares of real rational functions. An affirmative answer to this question was given by E. Artin and O. Schreier in 1927 in the paper [AS] which provided the foundation for the theory of ordered fields.

Let K be an ordered field, i.e., a field with a total order relation $<$ which is compatible with addition and with multiplication by positive elements. The *positive cone* P of $<$ is the set of positive elements with respect to $<$. It is a subgroup of the multiplicative group $\dot{K}$ of K with $[\dot{K} : P] = 2$, and is additively closed. The subgroups of $\dot{K}$ with these properties are precisely the positive cones of total order relations on K which are compatible with the operations. We call them *orderings* of K.

One of the main theorems of Artin-Schreier theory says that a field K admits an ordering if and only if it is a formally real field, i.e., -1 is not a sum of squares in K. Formally real fields which do not admit proper formally real algebraic field extensions are called *real closed fields*. A real closed field K has only one ordering $P = \dot{K}^2$.

Motivated by Artin-Schreier theory, R. Baer and W. Krull revealed the relation between orderings and valuations (see [B1], [B2] and [K]). Let Γ be a linearly ordered set and ∞ an element larger than all elements in Γ. A *valuation* v of an additive group K is map $v : K \to \Gamma \cup \{\infty\}$ with the properties: $v(a) = \infty \iff a = 0$ and $v(a - b) \geqslant \min\{v(a), v(b)\}$ (note that we use Krull's notation here). If additionally Γ is an ordered abelian group, K is a field and v restricted to $\dot{K}$ is a group homomorphism, then we obtain a *field valuation* of K. Then the set $A_v = \{a \in K : v(a) \geqslant 0\}$ is the *valuation ring* of v with unique maximal ideal $I_v = \{a \in K : v(a) > 0\}$. The field $Kv = A_v/I_v$ is its *residue field*, and by vK we denote its *value group* $v(\dot{K})$. The ring homomorphism $A_v \mapsto Kv$ can be extended to a map $\xi_v : K \to Kv \cup \{\infty\}$ by sending all elements in $K \setminus A_v$ to ∞; then ξ_v is called the *place associated to* v.

A valuation v of K is called *real* if Kv is a formally real field. The corresponding place is called a *real place*. We say that an ordering P is *compatible* with the valuation v if A_v is convex with respect to P. In this case the image of $P \cap A_v$ under the residue map $A_v \to Kv$ is an ordering on Kv. In particular, Kv is formally real. The famous Baer-Krull Theorem states that if v is a real valuation then every ordering $\bar{P}$ on Kv can be "lifted" to K, i.e., there is an ordering P of K compatible with v which induces $\bar{P}$ on Kv. Moreover, the number of orderings of K which are compatible with v and induce the same ordering on Kv is equal to the cardinality of the group $\mathrm{Hom}(vK/2vK, \{-1, 1\})$. The set of all valuation rings of valuations compatible with an ordering P forms a poset having as its minimal element the convex hull $A(P)$ of the rational numbers in K with respect to P. An ordering P is called *archimedean* if $A(P) = K$. The valuation having $A(P)$ as its valuation ring is called the *natural valuation of the ordering* P. The ordering induced on the residue field of the natural valuation is archimedean, so there is a unique embedding of the residue field in the real numbers respecting the ordering. Composing the residue map with this embedding gives a place $K \to \mathbb{R} \cup \{\infty\}$ called an $\mathbb{R}$*-place*.

Denote by $X(K)$ the set of all orderings of a field K and by $M(K)$ the set of all $\mathbb{R}$-places of K. We have observed that every ordering $P \in X(K)$ uniquely determines an $\mathbb{R}$-place. By the Baer-Krull Theorem, the so obtained map

$$\lambda : X(K) \to M(K)$$

is surjective. D. K. Harrison (unpublished result) and J. Leicht with F. Lorenz [LL] observed that there is a bijection between orderings of K and prime ideals of the Witt ring $W(K)$ (which consists of the equivalence classes of anisotropic quadratic forms over K). In this way, $X(K)$ obtains a topology induced by the Zariski topology on the prime spectrum of $W(K)$. The subbasis for this topology can be chosen to be the family of *Harrison sets*

$$H(a) = \{P \in X(K) \colon a \in P\},\ a \in K.$$

With this topology, $X(K)$ becomes a boolean space (compact, Hausdorff and totally disconnected). In 1975, T. Craven [C] proved that every boolean space X can be realized as a space of orderings of some field K.

By the surjectivity of λ one can consider the quotient topology on the space $M(K)$. With this topology, the space $M(K)$ is compact. D. W. Dubois [D] proved that it is also Hausdorff. He used the properties of the *real holomorphy ring* $\mathcal{H}(K)$ of the field K, which is defined to be the intersection of all real valuation rings of K. It plays an important role in real algebra and real algebraic geometry. The elements of $\mathcal{H}(K)$ separate points in $M(K)$, i.e., for any two distinct $\mathbb{R}$-places ξ_1 and ξ_2 there is $a \in \mathcal{H}(K)$ such that $\xi_1(a) > 0$ and $\xi_2(a) < 0$. The subbasis for the topology on $M(K)$ can be chosen to be the family of sets

$$U(a) = \{\xi \in M(K) \colon \xi(a) > 0\},\ a \in \mathcal{H}(K).$$

Let $L|K$ be a field extension. If P is an ordering of L, then $P \cap K$ is an ordering of K. We call P *an extension* of $P \cap K$ and the map $\mathrm{res} : X(L) \to X(K)$, $\mathrm{res}(P) = P \cap K$, the *restriction map*. It was observed by M. Knebusch [Kn] that the restriction of $\lambda_L(P)$ to K coincides with $\lambda_K(P \cap K)$. More precisely, we have the following commuting diagram of *continuous* maps:

$$\begin{array}{ccc} X(L) & \xrightarrow{\lambda_L} & M(L) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\ X(K) & \xrightarrow{\lambda_K} & M(K) \end{array}$$

In view of Craven's result about the realizability of boolean spaces as spaces of orderings it is natural to ask which compact Hausdorff spaces can be realized as spaces of $\mathbb{R}$-places. This problem, described as difficult and attractive, was posted by E. Becker, D. Gondard and M. Marshall in the two papers [BG] and [GM]. A full answer to this problem is not known. Partial answers are obtained in this habilitation thesis. But let us present now some facts which were known already before.

It is easy to observe that if K is a totally archimedean field (i.e., all orderings of K are archimedean) then the map λ is a homeomorphism, so $M(K)$ is boolean. In particular, all finite spaces can be realized as spaces of $\mathbb{R}$-places, since there are totally archimedean fields with any given finite number of orderings (see [E]).

In 1971, R. Brown [Br] proved that if F is an algebraic function field of transcendence degree 1 over a totally archimedean field K which has finitely many orderings, then $M(F)$ is a disjoint union of finitely many simple closed curves.

M. Knebusch in [Kn1] and [Kn2] studied algebraic curves over real closed fields. Let X be a smooth irreducible complete algebraic curve over a real closed field K and let F be the field of rational functions on X defined over K. Then F is a finitely generated field extension of K of transcendence degree 1. Let γ be the set of real points of X, that is, points which determine K-rational places (i.e., places with values in $K \cup \{\infty\}$ that are trivial on K). The composition of a K-rational place of F with the unique $\mathbb{R}$-place of K yields an $\mathbb{R}$-place of F. If K is an archimedean field, then the $\mathbb{R}$-place of K is an embedding of K in $\mathbb{R}$ and the points of γ correspond exactly to the $\mathbb{R}$-places of F. If K is non-archimedean, then the points of γ correspond only to a subset of $M(F)$. A result of A. Prestel (see [P], Theorem 9.9]) implies that this subset is dense in $M(F)$.

In higher dimensions the situation is much more complicated. The reason is the fact that a function field F of transcendence degree at least 2 admits an abundance of smooth projective models. The relation between $M(F)$ and the various models was described by H.-W. Schülting in [Sch]. He proved that if F is a function field over a real closed field K, then the space $M(F)$ is homeomorphic to the inverse limit of its smooth models. Let V_r be the set of real points of a smooth complete $\mathbb{R}$-variety V with formally real function field F equipped with the euclidean topology. L. Bröcker proved (unpublished result) that the number of connected components of $M(F)$ is equal to the number of semialgebraic connected components of V_r and therefore the latter is a birational invariant of non-singular complete $\mathbb{R}$-varieties.

Bröcker's result is not true for function fields over a non-archimedean real closed field K. A counterexample was given by H.-W. Schülting in [Sch] for a function field of transcendence degree 2 over K, but it is not difficult to give such a counterexample even for a function field of a curve. Schülting's example answered Brown's question posted in [Br].

For any formally real field K the connected components of $M(K)$ were also studied by J. Harman [H] and E. Becker [Be2]. In both cases Becker's theory of higher level orderings was used. J. Harman proved that for every field K with connected space $M(K)$, the space of $\mathbb{R}$-places of rational function fields over K is connected as well. In a very recent paper R. Brown and J. Merzel ([BM]) proved that the space $M(\mathbb{R}(x,y))$ of $\mathbb{R}$-places of the rational function field in two variables is not only connected (which follows from Harman's result), but also path-connected.

Becker's approach in [Be2] involved the units $\mathbb{E}(K)$ of the real holomorphy ring of K. The connected components of $M(K)$ can be separated by elements of $\mathbb{E}(K)$, i.e., for every component π of $M(K)$ there is $a \in \mathbb{E}(K)$ such that $\pi \subset U(a)$ and $M(K) \setminus \pi \subset U(-a)$. Let $\mathbb{E}^+(K)$ be the set of totally positive units of $\mathcal{H}(K)$ (i.e., units which are positive with respect to every ordering of K). Both $\mathbb{E}(K)$ and $\mathbb{E}^+(K)$ are subgroups of the multiplicative group $\dot{K}$. Becker proved in [Be2] that the number of connected components of $M(K)$ is equal to $\log_2[\mathbb{E}(K) : \mathbb{E}^+(K)]$. This number can also be expressed by using sums of 2^n-th powers in K (see [BG]).

The real holomorphy ring of K gives a lot of information about $M(K)$. An element $a \in \mathcal{H}(K)$ determines a continuous real-valued function on $M(K)$ given by $\xi \mapsto \xi(a)$ for $\xi \in M(K)$. Set $S^n(\mathcal{H}(K)) = \{(a_0,...,a_n) \in \mathcal{H}(K)^{n+1}\colon\ a_0^2 + ... + a_n^2 = 1\}$. Every $a = (a_0,...,a_n) \in S^n(\mathcal{H}(K))$ determines a continuous function $\hat{a} : M(K) \to S^n$, with S^n the n-dimensional sphere, where $\hat{a}(\xi) = (\xi(a_0),...,\xi(a_n))$. So we have a map from $S^n(\mathcal{H}(K))$ into the set $C(M(K),S^n)$ of continuous functions on $M(K)$ with values in S^n. Becker in his (yet unpublished) book [Be3] proved that the density of the image of $S^n(\mathcal{H}(K))$ in $C(M(K),S^n)$ is equivalent to the algebraic property that every element of $\mathbb{E}^+(K)$ which is a sum of n squares can be written as a sum of squares of totally positive units. To determine whether the density holds, we need more information about the topological properties of the space $M(K)$.

In some cases the properties of $M(K)$ can be deduced from the properties of the space of orderings $X(K)$, as we will see in the next section. For that we will need the following notion. A *cut* in an ordered set X is a pair (D,E) such that $D \cup E = X$ and $D < E$, meaning that $d < e$ for every $d \in D$ and $e \in E$. In this case the set D is called a *lower cut set* and E is called an *upper cut set*. The cuts $(\emptyset, X)$ and $(X, \emptyset)$ are called *improper*, all others are called *Dedekind cuts*. If D has a last element or E has a first element, then the cut (D,E) is called *principal*. Principal cuts are denoted by a^- or a^+, depending on whether a is the minimal element of the upper cut set or the maximal element of the lower cut set. By $\mathcal{C}(X)$ we denote the set of all cuts in X. R. Gilmer [G] showed that for any real closed field K the orderings of the rational function field $K(x)$ correspond bijectively to the cuts in K.

Since $\mathbb{R}$ is complete, it is cut complete, that is, every Dedekind cut in $\mathbb{R}$ is principal. For each $a \in \mathbb{R}$ the orderings on $\mathbb{R}(x)$ corresponding to the two principal cuts a^- and a^+ have the same natural valuation ring with residue field $\mathbb{R}$. Hence they induce the same $\mathbb{R}$-place ξ_a of $\mathbb{R}(x)$. Thus we have a bijection between the elements $a \in \mathbb{R}$ and the places ξ_a. The two improper cuts also induce the same $\mathbb{R}$-place ξ_∞. Therefore we can identify the $\mathbb{R}$-places of $\mathbb{R}(x)$ with the elements of the set $\mathbb{R} \cup \{\infty\}$, that is, the circle.

The situation is more complicated if we consider a non-archimedean real closed field K. To understand how the map $\lambda : X(K) \to M(K)$ works in this case we will need the notion of an ultrametric. Let X be an arbitrary set and Γ be a totally ordered set and ∞ an element larger than all elements in Γ. A map $u : X \times X \to \Gamma \cup \{\infty\}$ is called an *ultrametric on* X if for every $x, y, z \in X$, (i) $u(x, y) = \infty \Leftrightarrow x = y$, (ii) $u(x, y) = u(y, x)$, (iii) $u(x, y) \geqslant \min\{u(x, z), u(z, y)\}$. Note that a valuation v on an abelian group or field K determines an ultrametric on K by $u(a, b) = v(a - b)$, for $a, b \in K$. Having an ultrametric u on X we define *ultrametric balls* in the natural way. Let S be an upper cut set in Γ. An *ultrametric ball* centered in $x \in X$ with radius S is

$$B_S(x) = \{y \in X \colon u(x, y) \in S \cup \{\infty\}\}.$$

Note that for each $x \in X$, both $X = B_\Gamma(x)$ and the singleton $\{x\} = B_\emptyset(x)$ are ultrametric balls. Further, if $s \in \Gamma$, then we will write $B_{s^-}(x)$ instead of $B_S(x)$ when $S = \{t \mid t \geq s\}$ is the upper cut set of the cut s^-, and $B_{s^+}(x)$ when $S = \{t \mid t > s\}$ is the upper cut set of the cut s^+.

In the case of an ultrametric determined by the natural valuation of an ordered group K, the ultrametric balls of K are the cosets of convex subgroups of K.

Ultrametric balls have two important properties:

- every point x in an ultrametric ball B is its center, i.e., if $y \in B_S(x)$ then $B_S(x) = B_S(y)$,
- if the ultrametric balls B_1 and B_2 are not disjoint, then one of them is contained in the other.

B) Description of the main results of the habilitation thesis

A general goal of my work was to obtain more classes of spaces which can be realized as spaces of $\mathbb{R}$-places and analyze the properties of such spaces.

Paper [1]

In this paper we consider the realizability of boolean spaces as spaces of $\mathbb{R}$-places. Since every finite space is realized, we consider only infinite boolean spaces. Every boolean space is a closed subset of some Cantor cube $D_\mathfrak{m}$ of weight $\mathfrak{m}$. The first important result of the paper is the following theorem.

Theorem 1 [1, Theorem 3.2] *For every infinite cardinal number* $\mathfrak{m}$*, the Cantor cube* $D_\mathfrak{m}$ *of weight* $\mathfrak{m}$ *is homeomorphic to the space* $M(K)$ *for some formally real field* K*.*

The field K is constructed as follows. Take a real closed field R of cardinality $\mathfrak{m}$ and let $R(x)$ be the rational function field over R. Define

$$K = R(x)\left(\left\{\sqrt{\frac{x-a}{x}} \,:\; a \in R\right\}\right).$$

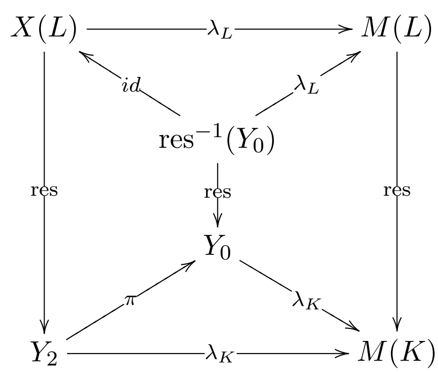
It was shown in [10] that the space of orderings of K is the disjoint union of two Harrison sets: $H(x)$ and $H(-x)$, each of which is homeomorphic to $D_\mathfrak{m}$. The first set contains only the extensions of the ordering P_∞ which corresponds to the improper cut $(R, \emptyset)$ of R, and the second contains only the extensions of the ordering $P_{-\infty}$ which corresponds to the improper cut $(\emptyset, R)$ of R. Moreover, all elements $\frac{x-a}{x}$ are units in the valuation ring $A(P_\infty) = A(P_{-\infty})$. To finish the proof of the theorem, we use the following lemma.

Lemma 2 [1, Lemma 3.1] *Let P be an ordering of the field F and let $L = F(\{\sqrt{a} : a \in \mathcal{A}\})$, where $\mathcal{A} \subset \{a \in F : 0 < \lambda_F(P)(a) < \infty\}$. Then λ_L is injective on the set* $\mathrm{res}^{-1}(P)$.

T. Craven proved in [Cr] that every algebraic extension K of $R(x)$ satisfies the Strong Approximation Property (meaning that any two disjoint closed subsets of $X(K)$ can be separated by Harrison sets). It is equivalent to the property that the Harrison subbasis is a basis for the topology on $X(K)$. Therefore every closed subset Y of $X(K)$ can be written as $Y = \bigcap_{\alpha \in \mathcal{A}} H(\alpha)$ for some $\mathcal{A} \subset K$. Craven proved that for the field $L = K(\{\sqrt[2^n]{\alpha} : \alpha \in \mathcal{A}, n \in \mathbb{N}\})$ the restriction map $\mathrm{res} : X(L) \longrightarrow X(K)$ is a homeomorphism onto Y. To obtain a homeomorphism for $\mathbb{R}$-places, the set $\mathcal{A}$ has to be properly chosen.

Proposition 3 [1, Proposition 4.2] *Let K be a formally real field. Suppose that Y_1 is a closed subset of $X(K)$ such that $\lambda_K|_{Y_1}$ is a bijection onto $M(K)$ and that Y_2 is a closed subset of $X(K)$ such that $Y_2 = \bigcap_{\alpha \in \mathcal{A}} H(\alpha)$, where $\mathcal{A} \subset \mathbb{E}(K)$. Set $Y_0 = Y_1 \cap Y_2$. Then for the field $L = K(\{\sqrt[2^n]{\alpha} : \alpha \in \mathcal{A}, n \in \mathbb{N}\})$ the map λ_L restricted to* $\mathrm{res}^{-1}(Y_0)$ *is a homeomorphism onto $M(L)$.*

The choice of $\mathcal{A}$ as a subset of $\mathbb{E}(K)$ yields that Y_0 is nonempty and allows us to construct a well-defined map $\pi : Y_2 \to Y_0$ which assigns to every $P \in Y_2$ a unique $Q \in Y_0$ such that $\lambda_K(P) = \lambda_K(Q)$. We have the following commuting diagram of continuous maps



where the left and centre vertical maps are bijective, and the map λ_K is injective on Y_0. These facts together with the commutativity of the diagram imply the bijectivity of the restriction of λ_L to $\mathrm{res}^{-1}(Y_0)$.

Let K be the field constructed in the proof of Theorem 1. The Harrison set $H(x) \subset X(K)$ is homeomorphic to the Cantor cube $D_{\mathfrak{m}}$. We use the Separation Criterion [L, Proposition 9.13] to prove that every closed subset $Y_0 \subset H(x)$ can be written in the form $Y_0 = \bigcap_{\alpha \in \mathcal{A}} H(\alpha)$ with $\mathcal{A} \subset \mathbb{E}(K)$. Using Proposition 3 for $Y_1 = H(x)$ and $Y_2 = Y_0$ we obtain the main theorem of [**1**]:

Theorem 4 [1, Theorem 4.4] *Every boolean space is realized as the space of $\mathbb{R}$-places of some field L.*

Paper [2]

In this paper we investigate which classes of compact Hausdorff topological spaces can be realized as spaces of $\mathbb{R}$-places. We prove that the family of topological spaces which are realizable in this way is closed under three topological operations:

- finite disjoint unions;
- closed subsets;

- direct products with boolean spaces.

We use the language of localities introduced by I. Efrat, but here we will describe each of the constructions above using the classical language of orderings and valuations. First we show how to enlarge fields without changing their spaces of $\mathbb{R}$-places.

Proposition 5 [2, Proposition 4.1] *For every field K and cardinal number α there exists a field F extending K with $\operatorname{trdeg} F|K = \alpha$ and such that $\operatorname{res}\colon M(F) \to M(K)$ is a homeomorphism.*

The field constructed for the proof of this proposition is the relative algebraic closure of $K(\mathbb{Z}^\alpha)$ in the power series field $K((\mathbb{Z}^\alpha))$.

Take a finite set $M(F_1), ..., M(F_n)$ of spaces of $\mathbb{R}$-places. By Proposition 5 we can assume that the fields $F_1, ..., F_n$ have the same transcendence degree over $\mathbb{Q}$. By fixing transcendence bases, we may then assume that the fields $F_1, ..., F_n$ are algebraic extensions of $\mathbb{Q}(T)$ for some set T of algebraically independent elements. For every $i = 1, ..., n$, the formal power series field $F_i((x+i))$ with its canonical discrete valuation v_i is henselian with residue field F_i. We define K_i to be the relative algebraic closure of $F_i(x)$ in $F_i((x+i))$. From [2, Corollary 3.8] we obtain:

$$M(K_i) \;\cong\; M(F_i)\,. \tag{1}$$

The extension $(F_i(x), v_i) \subset (K_i, v_i) \subset (F_i((x+i)), v_i)$ of valued fields is *immediate* (meaning that all valuations have the same value group and residue field). Consider the field $F = \bigcap_{i=1}^n K_i$ with the restricted valuations v_i, $i = 1, ..., n$. We denote the set of orderings of F compatible with the valuation v_i by $X(F, v_i)$ and the set of the corresponding $\mathbb{R}$-places by $M(F, v_i)$.

Proposition 6 [2, Proposition 4.2] *Let $v_1, ..., v_n$ be distinct valuations of rank 1 on a field F. For each $1 \le i \le n$ let (K_i, v_i) be an immediate henselian extension of (F, v_i), and assume that $F = \bigcap_{i=1}^n K_i$. Then:*

(a) $X(F) = \dot{\bigcup}_{i=1}^n X(F, v_i)$;

(b) $M(F) = \dot{\bigcup}_{i=1}^n M(F, v_i)$;

(c) $\operatorname{res}\colon \dot{\bigcup}_{i=1}^n X(K_i) \to X(F)$ *is a homeomorphism;*

(d) $\operatorname{res}\colon \dot{\bigcup}_{i=1}^n M(K_i) \to M(F)$ *is a homeomorphism.*

Using this proposition together with (1), we obtain the first main theorem of [2]:

Theorem 7 [2, Theorem 4.3] *Let $F_1, ..., F_n$ be formally real fields. There exists a field F such that*

$$M(F) \;\cong\; \dot{\bigcup_{i=1}^{n}} M(F_i)\,.$$

The second important result of [2] is the following theorem.

Theorem 8 [2, Theorem 5.4] *Let Y be a closed subset of $M(K)$. Then there exists an algebraic extension F of K such that*

(a) $\operatorname{res}\colon X(F) \to X(K)$ *maps $X(F)$ bijectively onto $\lambda_K^{-1}(Y)$;*

(b) $\operatorname{res}\colon M(F) \to M(K)$ *maps $M(F)$ bijectively onto Y.*

The construction of F is as follows. Take any closed subset Y of $M(K)$. By the Separation Criterion, we can choose for every P such that $\lambda_K(P) \notin Y$ an element $a_P \in \mathcal{H}(K)$ such that $\lambda_K^{-1}(Y) \subset H(a_P)$, $P \notin H(a_P)$, and a_P is a unit under the natural valuation of every ordering in $\lambda_K^{-1}(Y) \cup \{P\}$. From this we deduce that $\lambda_K^{-1}(Y) = \bigcap_{P \notin \lambda_K^{-1}(Y)} H(a_P)$. We define F to be the compositum of all fields $K(\{\sqrt[2^n]{a_P} : n \in \mathbb{N}\})$. The fact that $\mathrm{res}\colon X(F) \to X(K)$ is a bijection onto $\lambda_K^{-1}(Y)$ follows again from Craven's result in [C]. This implies that the image of $\mathrm{res}\colon M(F) \to M(K)$ is Y. To show injectivity of the restriction map we use the following lemma.

Lemma 9 [2, Lemma 5.2] *Let $a \in \mathcal{H}_K$ and let $F_a = K(\{\sqrt[2^n]{a} : n \in \mathbb{N}\})$. Then the map* $\mathrm{res}\colon M(F_a) \to M(K)$ *is injective on the subbasic set $U(a)$.*

From this we obtain that the restriction $\mathrm{res}\colon M(F_{a_1,\dots,a_k}) \to M(K)$, where $F_{a_1,\dots,a_k}$ is the compositum of the fields F_{a_i} for $a_1, \dots, a_k \in \mathcal{H}(K)$, is injective on $U(a_1) \cap \ldots \cap U(a_k)$. Since F is a direct limit of such fields, the injectivity of the map $\mathrm{res}\colon M(F) \to M(K)$ follows from the next lemma.

Lemma 10 [2, Lemma 3.6] *Let F_i, $i \in I$, be a directed system of fields with respect to inclusions, and let $F = \varinjlim F_i$. Then $\varprojlim\colon M(F) \to \varprojlim M(F_i)$ is a homeomorphism.*

The obvious corollary to Theorem 8 is:

Corollary 11 [2, Corollary 5.5] *If a topological space M is realizable as a space of $\mathbb{R}$-places, then so is every closed subset of M.*

The last construction, i.e., the realization of the product of a realizable space with a boolean space, is a combination of the two constructions described before. Having a realizable space $M = M(K)$ we can use the union construction to create a field K_n having as its space of $\mathbb{R}$-places the disjoint union of 2^n copies of M. Then we use transfinite induction to prove the following proposition:

Proposition 12 [2, Proposition 6.1] *Let K be a field and let α be a set. There is a field extension $K_\alpha|K$ and a homeomorphism $\tau_\alpha\colon M(K_\alpha) \xrightarrow{\sim} \{0,1\}^\alpha \times M(K)$ such that the following triangle commutes:*

$$\begin{array}{ccc} M(K_\alpha) & \xrightarrow[\sim]{\tau_\alpha} & \{0,1\}^\alpha \times M(K) \\ & \searrow^{\mathrm{res}} & \downarrow{\scriptstyle \mathrm{proj}} \\ & & M(K). \end{array}$$

From Corollary 11 we obtain:

Corollary 13 [2, Corollary 6.2] *Let K be a field and X a boolean space. There exists a field extension F of K such that $M(F)$ is homeomorphic to $X \times M(K)$.*

This result is a generalization of Theorem 4 (which we obtain when we take K to be a field with a unique $\mathbb{R}$-place, for example any real closed field).

Paper [3]

In this paper we study the space of $\mathbb{R}$-places of the rational function field $R(x)$ over an arbitrary (possibly non-archimedean) real closed field R. The main theorem of the paper states:

Theorem 14 [3, Theorem 4.7] *Let R be a real closed field. Then $M(R(x))$ is metrizable if and only if R contains a countable dense subfield.*

As we mentioned in the Introduction, the orderings of $R(x)$ correspond bijectively to the cuts in R. The set $C(R)$ of cuts in R is a linearly ordered set, so we can consider the order topology on it. Theorem 2.1 of [**3**] states that the bijection of $X(R(x))$ with $C(R)$ is in fact a homeomorphism.

The next step is to determine which orderings (hence also cuts) determine the same $\mathbb{R}$-place. For that we use the ultrametric u in R induced by the natural valuation v of R. The value group vR is a divisible ordered abelian group which is nontrivial if R is non-archimedean. Every ultrametric ball B determines two cuts in R: B^- with the lower cut set $\{a \in R\colon a < B\}$ and B^+ with the upper cut set $\{a \in R\colon a > B\}$. The cuts B^- and B^+ are called *ball cuts.*

Theorem 15 [3, Theorem 2.2] *Take a real closed field R and two distinct orderings P_1, P_2 of $R(x)$. Then $\lambda(P_1) = \lambda(P_2)$ if and only if the corresponding cuts are ball cuts of the some ultrametric ball B in R.*

Analyzing cuts in real closed fields and using Theorem 15, we obtain:

Theorem 16 [3, Theorem 3.2] *Let $R' \subset R$ be an extension of real closed fields. Then R' is dense in R if and only if* $\mathrm{res} : M(R(x)) \to M(R'(x))$ *is a homeomorphism.*

By Urysohn's Metrization Theorem, a compact Hausdorff space is metrizable if and only if it is second-countable. Any second-countable space is separable. The cellularity of a topological space M is defined as

$$\sup\{|\mathcal{F}| : \mathcal{F} \text{ is a family of pairwise disjoint open subsets of } M\}.$$

The cellularity is not bigger than the density of M. Hence if the cellularity is uncountable, then the density is uncountable, which implies that the space is not separable and consequently not metrizable.

Recall that a subbasis for the space $M(K)$ can be chosen as a family of sets indexed by elements of the real holomorphy ring of K. If K is countable, then this subbasis (and consequently, also a basis) of $M(K)$ is countable, so $M(K)$ is second-countable and we obtain:

Corollary 17 [3, Corollary 4.1] *If K is a countable field, then $M(K)$ is metrizable.*

From this corollary and Theorem 16 follows the "if" part of Theorem 14. To prove the "only if" part, we use the following proposition.

Proposition 18 [3, Proposition 4.3] *Suppose that vR and Rv are countable and $M(R(x))$ is metrizable. Then R contains a countable dense subfield.*

We see that if R does not contain a countable dense subfield but vR and Rv are countable, then $M(R(x))$ cannot be metrizable. We have to show the latter also in the case where vR or Rv is uncountable. To illustrate the flavour of the proof, we consider the case where Rv is uncountable. We can see Rv as a subfield of R. For any $a \in R$ and $s \in vR$ we define

$$U_{a,s}\colon = \{\xi \in M(R(x))\colon v_\xi(x-a) > s\},$$

where v_ξ is the valuation on $R(x)$ corresponding to ξ. In [**3, Lemma 4.4**] we show that the set $U_{a,s}$ is nonempty and open in $M(R(x))$. Now we take $b \in R$ such that $t = v(b) > s$. Then the sets $U_{a+kb,t}$, where $k \in Rv$, are pairwise disjoint open subsets of $U_{a,s}$, showing that the cellularity of $M(R(x))$ is uncountable.

Using similar constructions for the remaining two cases of vR being uncountable and of both vR and Rv being countable, we prove:

Theorem 19 **[3, Theorem 4.5]** *Let R be a real closed field that does not admit a countable dense subfield. Pick some $a \in R$ and $s \in vR$. Then $U_{a,s}$ contains uncountably many pairwise disjoint open sets. In particular, $M(R(x))$ has uncountable cellularity and is not metrizable.*

This finishes the proof of Theorem 14. We also give an example which shows that countability of vR and Rv is not sufficient for the metrizability of $M(R(x))$.

Example 20 **[3, Example 4.8]** *Take a countable, archimedean real closed field k and a countable, nontrivial, divisible ordered abelian group Γ. The field $R = k((\Gamma))$ is real closed, the value group of its natural valuation v is Γ and the residue field is k. The space $M(R(x))$ has uncountable cellularity, so it is not metrizable.*

For function fields of transcendence degree 1 over real closed fields, we obtain an implication in one direction:

Theorem 21 **[3, Theorem 4.9]** *Take a real closed field R that does not admit a countable dense subfield. Further, take a formally real function field F of transcendence degree 1 over R. Then $M(F)$ is not metrizable.*

Paper [4]

For a function field F of transcendence degree higher than 1 over a real closed field R, the structure of its space of $\mathbb{R}$-places is even more complicated, even if we consider function fields over the reals. The main theorem of paper [4] states:

Theorem 22 **[4, Theorem 1.1]** *For any uncountable real closed field R, the space of $\mathbb{R}$-places of the rational function field $R(x, y)$ is not metrizable.*

I will give a sketch of the proof. The set $Y = H(x) \cap \bigcap_{r \in \dot{R}^2} H(r - x)$ is closed in $X(R(x, y))$. For arbitrary $r \in R$, the set $U_r = Y \cap \bigcup_{a \in \mathbb{N}} [H(ax - (y - r)) \cap H(ax + (y - r))]$ is open in Y. Moreover, the sets U_r are nonempty, pairwise disjoint and *full*, meaning that $\lambda^{-1}(\lambda(U_r)) = U_r$. Then the sets $V_r = \lambda(U_r)$ are nonempty, open and pairwise disjoint subsets of $N = \lambda(Y)$, which shows that the cellularity of N is not smaller then $|R|$. Thus N as well as $M(R(x, y)$ are not metrizable.

The proof of Theorem 22 requires only that $x, y \in F$, $R \subseteq F$ and $U_r \neq \emptyset$ for uncountably many $r \in R$. Using this observation, we can carry over the argument to prove some generalizations.

Theorem 23 **[4, Theorem 3.1]** *Suppose that $R(x, y) \subseteq F \subseteq R'((x, y))$, where R is an uncountable real closed field, R' is a real closed extension of R, and $R'((x, y))$ is the formal power series field in two variables over R'. Then $M(F)$ is not metrizable.*

An immediate consequence of this theorem is:

Corollary 24 **[4, Corollary 3.2]** *For any uncountable real closed field R, the space of $\mathbb{R}$-places of the formal power series field $R((x, y))$ is not metrizable.*

Suppose that F is a function field over R of transcendence degree $d \geqslant 2$. Viewing F as the function field of a real algebraic variety V over R and passing to the completion of the coordinate ring of V at some fixed real regular point, we obtain that $F \subseteq R((x_1, \ldots, x_d))$ for some elements $x_1, \ldots, x_d$ in the coordinate ring. Applying Theorem 23 with $x = x_1, y = x_2$ and R' the real closure of $R((x_3, \ldots, x_d))$ with respect to some fixed ordering, we obtain:

Corollary 25 **[4, Corollary 3.3]** *Suppose that R is an uncountable real closed field and F is a finitely generated formally real field extension of R of transcendence degree ≥ 2. Then $M(F)$ is not metrizable.*

For an archimedean real closed field R we obtain:

Corollary 26 **[4, Corollary 3.4]** *Suppose that R is an archimedean real closed field and F is a finitely generated formally real field extension of R. Then $M(F)$ is metrizable if and only if either R is countable or* $\operatorname{trdeg} F|R \leq 1$.

Take R to be a proper real closed extension of $\mathbb{R}$. Then R is non-archimedean, so there is a positive infinitesimal element y. Using a similar argument as in the proof of Theorem 22, we obtain:

Theorem 27 **[4, Theorem 3.5]** *If R is a proper real closed extension of $\mathbb{R}$, then the space of $\mathbb{R}$-places of the rational function field $R(x)$ is not metrizable.*

Paper [5]

An open question is whether some 2-dimensional euclidean topological space (for instance the torus) can be realized as a space of $\mathbb{R}$-places. Our hope was to obtain such a space as a closed subset of some realizable space. The natural candidate for that could be the space of $\mathbb{R}$-places of the rational function field $\mathbb{R}(x, y)$. In the paper [**5**], we obtain some negative results.

At the beginning we consider possible embeddings of $M(R(x))$ in $M(F(x))$ for some formally real field extension F of a real closed field R.

Theorem 28 **[5, Theorem 1.2]** *Take a real closed field R and a formally real extension field F of R. A continuous embedding ι of $M(R(x))$ in $M(F(x))$ compatible with restriction exists if and only if vR is a convex subgroup of vF, for the natural valuation v of some ordering of F. In particular, such an embedding always exists when R is archimedean ordered. If F is real closed, then there is at most one such embedding.*

There is a surprising consequence of this theorem. If R is a non-archimedean real closed field and F is an elementary extension (e.g., ultrapower) of R of high enough saturation, then vR will not be a convex subgroup of vF and there will be no such embedding ι.

For the proof of Theorem 28 we consider an extension $F|R$ of ordered fields (for now we do not assume R to be real closed), and analyze the relation between cuts in R and cuts in F. If (D', E') is a cut in F, then $(D' \cap R, E' \cap R)$ is a cut in R, which we call the *restriction* of (D', E'). Let (D, E) be a cut in R. We say that the element $a \in F$ *fills* (D, E) if $D < a < E$ holds in F. Two cuts in R are called *equivalent* if they are determined by the same ultrametric ball in R.

Usually several cuts in F restrict to the same cut in R. This means that in general there are several order preserving embeddings of $\mathcal{C}(R)$ in $\mathcal{C}(F)$ which are compatible with restriction. The question arises whether there are such embeddings that are also continuous with respect to the order topology and in addition compatible with the equivalence of cuts.

Proposition 29 **[5, Proposition 4.7]** *Take any extension $F|R$ of ordered fields. If there is at least one non-ball cut in R that is filled in F, then there exists no embedding of $\mathcal{C}(R)$ in $\mathcal{C}(F)$ that is continuous with respect to the order topology and compatible with restriction.*

To prove our main result we consider another topology on the sets of cuts. We say that an interval in $\mathcal{C}(K)$ is *full* if it is closed under the equivalence of cuts. We call the topology generated by the full sets the *full topology*.

Proposition 30 **[5, Prop. 4.8 and Prop. 4.9]** *There is an embedding $\tilde{\iota} : \mathcal{C}(R) \to \mathcal{C}(F)$ which is continuous with respect to the full topology and compatible with restriction if and only if vR is a convex subgroup of vF.*

The embedding $\tilde{\iota}$ we construct for the proof of this proposition is also compatible with the equivalence of cuts.

Now assume that R and F are both real closed. Then we have homeomorphisms $\chi_R : \mathcal{C}(R) \to X(R(x))$ and $\chi_F : \mathcal{C}(F) \to X(F(x))$ between the spaces of cuts and the spaces of orderings of the respective rational function fields. Assume that vR is convex in vF. Then we can define an embedding $\iota : M(R(x)) \longrightarrow M(F(x))$ by $\iota(\xi) := \lambda \circ \chi_F(\tilde{\iota}(C))$, where C is the cut in R such that $\xi = \lambda \circ \chi_R(C)$. Since $\tilde{\iota}$ is compatible with the equivalence of cuts, the embedding ι is well-defined and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}(F) & \stackrel{\lambda\circ\chi_F}{\longrightarrow} & M(F(x)) \\ \tilde{\iota}\Big\uparrow & & \iota\Big\uparrow \\ \mathcal{C}(R) & \stackrel{\lambda\circ\chi_R}{\longrightarrow} & M(R(x)) \end{array}$$

Theorem 31 [5, Theorem 5.1] *Take an extension $F|R$ of real closed fields. If vR is convex in vF, then the embedding ι as defined above does not depend on the particular choice of $\tilde{\iota}$ and is continuous and compatible with restriction. Conversely, if $\iota : M(R(x)) \to M(F(x))$ is continuous and compatible with restriction, then it induces an embedding $\tilde{\iota} : \mathcal{C}(R) \to \mathcal{C}(F)$ continuous with respect to the full topology and compatible with restriction, such that the above diagram commutes, and vR is convex in vF.*

Now assume that F is not real closed but admits an $\mathbb{R}$-place ξ such that vR is convex in $v_\xi F$. Choose a real closure R' of F with respect to some ordering P of F compatible with v_ξ. We have a continuous restriction

$$M(R'(x)) \stackrel{\mathrm{res}_{F(x)}}{\longrightarrow} M(F(x)) \stackrel{\mathrm{res}_{R(x)}}{\longrightarrow} M(R(x)),$$

which allows us to define an embedding $\iota \colon M(R(x)) \to M(F(x))$ by setting $\iota := \mathrm{res}_{F(x)} \circ \iota'$, where $\iota' : M(R(x)) \to M(R'(y))$ is an embedding as in Theorem 31. Note that ι is continuous, injective and compatible with restriction.

As the real closure R' can be taken with respect to any ordering on F which is compatible with v_ξ, we may lose the uniqueness of ι. However, we are able to prove the following partial uniqueness result.

Theorem 32 [5, Theorem 5.2] *Take two orderings P_1 and P_2 of F which induce the same $\mathbb{R}$-place, R_1' and R_2' the respective real closures of F, and $\iota_i' : M(R(x)) \to M(R_i'(x))$, $i = 1, 2$, the continuous embeddings compatible with restriction. Consider the following diagram:*

$$\begin{array}{ccccc} & & M(R_1'(x)) & & \\ & \stackrel{\iota_1'}{\nearrow} & & \stackrel{\mathrm{res}_1}{\searrow} & \\ M(R(x)) & \longleftarrow & \mathrm{res} & \longleftarrow & M(F(x)) \\ & \stackrel{\iota_2'}{\searrow} & & \stackrel{\mathrm{res}_2}{\nearrow} & \\ & & M(R_2'(x)) & & \end{array}$$

Then $\mathrm{res}_1 \circ \iota_1' \;=\; \mathrm{res}_2 \circ \iota_2'$.

If R is an archimedean real closed field, then $vR = \{0\}$ is always a convex subgroup of vF, so an embedding $\iota : M(R(x)) \hookrightarrow M(F(x))$ always exists. Moreover, it can be written more explicitly. Choose any $\mathbb{R}$-place ξ of F. Let $\overline{F} \subset \mathbb{R}$ be the residue field of the valuation corresponding to ξ. We can see $\overline{F}$ as a field extension of R. Let v_x be the *Gauss valuation* on $F(x)$, which is the unique extension of ξ that is trivial on $R(x)$. Let ξ_x be the corresponding place. The residue field of ξ_x is $\overline{F}(x)$. Since R is archimedean, every $\zeta \in M(R(x))$ is trivial on R. Therefore ζ is a place associated with an f-adic valuation where f is an irreducible polynomial in $R[x]$ or $f = 1/x$. Since R is real closed and $\overline{F}$ is formally real, such a polynomial f remains irreducible over $\overline{F}$ and thus, f (or $1/x$, respectively) determines a unique extension $\zeta_{\overline{F}}$ of ζ to $\overline{F}(x)$ that is trivial on $\overline{F}$. We define $\iota'(\zeta) := \zeta_{\overline{F}}$.

Lemma 33 [5, Lemma 6.1] *The map $\iota' : M(R(x)) \to M(\overline{F}(x))$ is a continuous embedding compatible with restriction. If $\overline{F}$ is real closed, then it is a homeomorphism.*

Theorem 34 [5, Theorem 6.2] *The map $\iota : M(R(x)) \to M(F(x))$ defined by $\iota(\zeta) := \zeta_{\overline{F}} \circ \xi_x$ is a continuous embedding.*

This theorem together with Theorem 32 gives:

Theorem 35 [5, Theorem 6.3] *The map $\iota : M(R(x)) \to M(R(x, y))$ is the unique continuous embedding compatible with restriction and such that all places in the image of ι have the same restriction to $R(y)$.*

The restriction map induces a map $M(\mathbb{R}(x, y)) \to M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$. Consider the product topology on $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$. If there was a continuous embedding of $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$ in $M(\mathbb{R}(x, y))$, then Theorem 11 would let us obtain the realizability of the torus.

We consider the more general case of n variables and the restriction map

$$\mathrm{res} : M(\mathbb{R}(x_1, \ldots, x_n)) \ni \xi \;\mapsto\; (\xi|_{\mathbb{R}(x_1)}, \ldots, \xi|_{\mathbb{R}(x_n)}) \in \prod_{i=1}^{n} M(\mathbb{R}(x_i)) \,.$$

In **[5, Lemma 7.1]** we prove that res is surjective. However, for $(\xi_1, \ldots, \xi_n) \in \prod_{i=1}^n M(\mathbb{R}(x_i))$ there are many possible choices for $\xi \in M(\mathbb{R}(x_1, \ldots, x_n))$ with image $(\xi_1, \ldots, \xi_n)$. The surjectivity of res shows that there exists an embedding

$$\iota : \prod_{i=1}^{n} M(\mathbb{R}(x_i)) \;\hookrightarrow\; M(\mathbb{R}(x_1, \ldots, x_n)) \,.$$

Such an embedding is called *compatible* if $\mathrm{res} \circ \iota$ is the identity map. Using the Tarski Transfer Principle, we prove:

Theorem 36 [5, Theorem 7.3] *The image of every compatible embedding ι lies dense in $M(\mathbb{R}(x_1, \ldots, x_n))$. But for $n > 1$, every nonempty basic open subset of $M(\mathbb{R}(x_1, \ldots, x_n))$ contains infinitely many places that are not in the image of ι.*

As a consequence of this theorem, we obtain:

Corollary 37 [5, Corollary 7.4] *A compatible embedding ι of $\prod_{i=1}^n M(\mathbb{R}(x_i))$ in $M(\mathbb{R}(x_1, \ldots, x_n))$ cannot be continuous with respect to the product topology on $\prod_{i=1}^n M(\mathbb{R}(x_i))$.*

Theorem 38 [5, Theorem 7.6] *For every compatible embedding ι, the topology induced on the product $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$ is finer than the product topology.*

The results above can be extended to a more general case. Assume that the fields F_1 and F_2 are function fields of transcendence degree ≥ 1 over $\mathbb{R}$, embedded in some extension E of $\mathbb{R}$ in such a way that they are linearly disjoint over $\mathbb{R}$. Denote by F the field compositum of F_1 and F_2 in E. As before, we consider the corresponding restriction map

$$\text{res}:\ M(F) \ni \xi \ \mapsto\ (\xi|_{F_1}, \xi|_{F_2}) \in M(F_1) \times M(F_2),$$

and we show that res is surjective but not injective. The surjectivity shows that there is an embedding

$$\iota:\ M(F_1) \times M(F_2) \ \longrightarrow\ M(F). \tag{2}$$

As before, ι is called *compatible* if res $\circ\, \iota$ is the identity.

Theorem 39 [5, Theorem 8.2] *If $F_1|\mathbb{R}$ and $F_2|\mathbb{R}$ are function fields of transcendence degree ≥ 1, then the image of every compatible embedding ι as in (2) lies dense in $M(F)$. But every nonempty basic open subset of $M(F)$ contains infinitely many places that are not in the image of ι.*

The theorem above shows that a compatible embedding cannot be continuous with respect to the product topology on $M(F_1) \times M(F_2)$. In the proof we use the Tarski Transfer Principle and a lemma obtained from [KP, p. 190].

In the last section of **[5]** we show how to use the previous constructions to embed $M(K)$ in $M(L)$, for an arbitrary formally real field K and a suitable transcendental extension L of K.

Theorem 40 [5, Theorem 9.1] *Assume that L admits a K-rational place ξ. Then $\iota:\ M(K) \ni \zeta \ \mapsto\ \zeta \circ \xi \in M(L)$ is a continuous embedding compatible with restriction.*

Note that in the proof of Proposition 5 we construct fields L of arbitrary transcendence degree over K which allow a unique K-rational place ξ.

Corollary 41 [5, Corollary 9.2] *Take a collection x_i, $i \in I$, of elements algebraically independent over K. Then there are at least $|K|^{|I|}$ many distinct continuous embeddings of $M(K)$ in $M(K(x_i\colon i \in I))$, all of them compatible with restriction and having pairwise disjoint images.*

This follows from the fact that for every choice of elements $a_i \in K$ there is a K-rational place ξ of $M(K(x_i\colon i \in I))$ such that $\xi(x_i) = a_i$.

Corollary 42 [5, Corollary 9.3] *There are at least $2^{\aleph_0}$ many continuous embeddings of $M(\mathbb{R}(x))$ in $M(\mathbb{R}(x,y))$, all of them compatible with restriction and having pairwise disjoint images.*

Paper [6]

The results obtained in papers **[3]** and **[5]** allow us to see more clearly the structure of the space of $\mathbb{R}$-places of the rational function field $R(x)$ over a nonarchimedean real closed field R. This structure is described in **[6]**.

First we show that a subbasis for the topology of the space $M(R(x))$ can be given by a rather small collection of sets, the cardinality of which depends on the cardinality of a chosen dense subfield of R.

Let F be a fixed dense subfield of R. Consider the following family of functions:

$$\mathcal{F} = \{a + bx, \frac{x-a}{x-b}\colon a, b \in F\}. \tag{3}$$

Considering the properties of ball and non-ball cuts in R and their relations with $\mathbb{R}$-places exhibited in **[3]**, we obtain:

Theorem 43 [6, Theorem 2.5] *The family $\{U(f)\colon f \in \mathcal{F}\}$ forms a subbasis for the Harrison topology on $M(R(x))$.*

The family $\mathcal{F}$ weakly separates points in $M(R(x))$, i.e., if $\xi, \eta \in M(R(x))$ with $\xi \neq \eta$, then there is $f \in \mathcal{F}$ such that $\xi(f) \neq \eta(f)$.

Assume that $M(R(x))$ is metrizable, which is equivalent to the existence of a countable real closed field F that lies dense in R. By countability of F, $M(F(x))$ is metrizable. On the other hand, one can see $M(F(x))$ as a subset of the space $\overline{\mathbb{R}}^{F(x)}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. It was shown in [13] that $M(F(x))$ is closed in $\overline{\mathbb{R}}^{F(x)}$. Therefore, the topology on $M(F(x))$ is induced by the restriction of the (up to equivalence) canonical metric of the product $\overline{\mathbb{R}}^{F(x)}$. This restriction can be defined in the following way. First choose any bijection $\sigma : F(x) \to \mathbb{N}$. Then a metric $\rho : M(F(x)) \times M(F(x)) \to [0, \infty)$ is given by

$$\rho(\xi, \eta) = \sup_{f \in F(x)} \{2^{-\sigma(f)} d_0(\xi(f), \eta(f))\},$$

where d_0 is any fixed metric on the circle $\overline{\mathbb{R}}$.

We show that in the above definition of the metric, one can restrict the functions f to the family $\mathcal{F}$ defined in (3). The map d thus obtained is a metric on $M(F(x))$.

Proposition 44 [6, Proposition 3.2] *The Harrison topology of the space $M(F(x))$ is equal to the topology induced by the metric d defined above.*

Using the homeomorphism from $M(F(x))$ to $M(R(x))$ (see Theorem 16), we obtain:

Theorem 45 [6, Theorem 3.3] *Let R be a real closed field and F a countable, real closed, dense subfield of R. Let $\mathcal{F} \subset F(x)$ be the family defined in (3). Take any bijection $\sigma : \mathcal{F} \to \mathbb{N}$. Then the map $d : M(R(x)) \times M(R(x)) \to [0, \infty)$ given by*

$$d(\xi, \eta) = \sup_{f \in \mathcal{F}} \{2^{-\sigma(f)} d_0(\xi(f), \eta(f))\}$$

is a metric on $M(R(x))$.

From now on we do not assume $M(R(x))$ to be metrizable. We want to determine the topological dimension of the space $M(R(x))$. We consider the covering dimension (dim), the small inductive dimension (ind), and the strong inductive dimension (Ind). These three cardinals are not always equal, especially for spaces which are not metrizable. But we have:

Theorem 46 [6, Theorem 1.2] *If R is any real closed field, then the (small or large) inductive dimension as well as the covering dimension of $M(R(x))$ is 1.*

The proof is based on a few important theorems from dimension theory and on [NTT, Theorem 5].

We take a closer look at the structure of the space $M(R(x))$. We observe that it carries a lot of self-similarities, so the structure is very rich. Every automorphism σ of a formally real field K induces the homeomorphism of $M(K)$ onto itself by the composition $\xi \mapsto \xi \circ \sigma$. Every R-automorphism σ of $R(x)$ is given by

$$x \mapsto \frac{ax + b}{cx + d} \quad \text{with } ad - bc \neq 0.$$

Such an automorphism can be achieved by a composition of the following operations: $x \mapsto x + c$ for $c \in R$, $x \mapsto cx$ for $c \in \dot{R}$ and $x \mapsto x^{-1}$. Every such operation determines a continuous and bijective action on $R \cup \{\infty\}$. This gives us a corresponding continuous action on the set of cuts of R; it is described in Section 5 of [6]. We observe that

1) The automorphism induced by $x \mapsto x + c$ for $c \in R$ maps an ultrametric ball $B_S(a)$ to the ball $B_S(a+c)$.

2) The automorphism induced by $x \mapsto cx$ for $c \in R \setminus \{0\}$ maps an ultrametric ball $B_S(a)$ to the ball $B_{S+v(c)}(ca)$.

3) The automorphism induced by $x \mapsto x^{-1}$ maps an ultrametric ball $B_S(a)$ to the ball $B_{S-2va}(\frac{1}{a})$ if $0 \notin B_S(a)$ and to the complement of the ball $B_{-(vR\setminus S)}(0)$ if $0 \in B_S(a)$.

We observe that all three operations send equivalent cuts to equivalent cuts; therefore we have:

Proposition 47 [6, Proposition 5.1] *All three operations induce homeomorphisms on $\mathcal{C}(R)$ that are compatible with equivalence.*

For any subset $T \subseteq R$, we define $\hat{T}$ to be the closure of the set $\{a^-, a^+ \colon a \in T\}$ in $\mathcal{C}(R)$ (where a^+, a^- are the principal cuts in a). If $T = B$ is a ball in R, then $\hat{T}$ is the interval $[B^-, B^+]$. If T is a ball complement $B^c := R \setminus B$ for some ball B, then $\hat{T} = [R^-, B^-] \cup [B^+, R^+]$. Let $\overline{T}$ be the set of $\mathbb{R}$-places determined by the cuts in $\hat{T}$. We observe that for any $r \in R$, the set $\{\overline{B_{s^-}(r)} \colon s \in vR\}$ forms a cofinal and coinitial chain of subspaces of $M(R(x))$ which are all homeomorphic. The order type of this chain is equal to that of vR. The same is true for the chains $\{\overline{B_{s^+}(r)} \colon s \in vR\}$ and $\{\overline{B_{S+s}(r)} \colon s \in vR\}$ where S is any upper cut set in vR and $S + s = \{s' + s \colon s' \in S\}$.

A topological space M is called *self-homeomorphic* if every open subset contains a homeomorphic copy of M. In particular cases the space $M(R(x))$ can be self-homeomorphic. Consider the power series field $R = \mathbb{R}((t^{\mathbb{Q}}))$. This is a real closed field. Since any two countable dense linear orderings without endpoints are order isomorphic, for every nonempty upper cut set S of $\mathbb{Q}$ that does not have a smallest element there exists an order isomorphism φ_S from $\mathbb{Q}$ onto S. Any such isomorphism induces an isomorphism

$$\psi_S : \sum_{q\in\mathbb{Q}} c_q t^q \mapsto \sum_{q\in\mathbb{Q}} c_q t^{\varphi_S(q)}$$

from the ordered additive group of R onto its convex subgroup $B_S(0)$. This isomorphism induces a homeomorphism $\widehat{\psi_S} : \mathcal{C}(R) \to \widehat{B_S(0)}$ which is compatible with equivalence. If r is any element in R, then we can compose the homeomorphism $\widehat{\psi_S}$ with the homeomorphism that sends $\widehat{B_S(0)}$ to $\widehat{B_S(r)}$, in order to obtain a homeomorphism $\overline{\psi_{S,r}} : M(R(x)) \to \overline{B_S(r)}$. As the nonempty upper cut sets S of $\mathbb{Q}$ without smallest element form a dense linear ordering under inclusion and correspond bijectively to the real numbers, and since their intersection is empty, we obtain:

Theorem 48 [6, Theorem 5.2] *Take the field $R = \mathbb{R}((t^{\mathbb{Q}}))$ and $r \in R$. Then there exists a set of subspaces of $M(R(x))$, all homeomorphic to $M(R(x))$, on which inclusion induces the dense linear ordering of $\mathbb{R}$, and such that the place ξ_r is the only $\mathbb{R}$-place of $R(x)$ contained in all of them.*

We show in [6, Lemma 5.3] that for an arbitrary real closed field R, every nonempty open subset of $M(R(x))$ contains $\overline{B_{s^+}(r)}$ for some $s \in vR$ and $r \in R$. Applying this result to the field $R = \mathbb{R}((t^{\mathbb{Q}}))$, we obtain:

Corollary 49 [6, Corollary 5.4] *The space $M(\mathbb{R}((t^{\mathbb{Q}}))(x))$ is self-homeomorphic.*

In the last section of **[6]** we describe the "fractal" structure of the space $M(R(x))$. On the set of cuts $\mathcal{C}(R)$ we first identify equivalent principal cuts and the two improper cuts. In that way we obtain an embedding of the circular order $R \cup \{\infty\}$ in $M(R(x))$. Then we add all images of the non-ball cuts, on which λ is injective. If R is archimedean, then there are no more $\mathbb{R}$-places to be added and we are done, having obtained the usual circle. For a non-archimedean R we still have to identify equivalent ball cuts. We observe that for each $s \in vR$ and $a \in R$,

$$B_{s^-}(a) \;=\; \bigcup_{b \in B_{s^-}(a)} B_{s^+}(b)\;.$$

By the properties of ultrametric balls, this union is disjoint. Then $\overline{B_{s^-}(a)}$ (which we call a *subnecklace* of $M(R(x))$) is the disjoint union of homeomorphic balls $\overline{B_{s^+}(b)}$ (which we call *pearls*) and the single place induced by the ball cuts of the ball $B_{s^-}(a)$ which can be seen as the connection of the subnecklace with $M(R(x)) \setminus \overline{B_{s^-}(a)}$. The latter set is homeomorphic to a pearl again. Moreover, every pearl contains again the subnecklace $\overline{B_{t^-}(a)}$, for every $t > s$, which is homeomorphic to $\overline{B_{s^-}(a)}$. Note that the chain of subnecklaces $\overline{B_{t^-}(a)}$, $t \in vK$ is densely ordered because vK is divisible. This fact distinguishes $M(R(x))$ from usual fractals. We call it the *densely fractal pearl necklace*.

Theoretically it would be possible to determine the Hausdorff dimension of $M(R(x))$ in the metrizable case. But results of [HR] show that this dimension strongly depends on the choice of the metric, and as we have seen in Theorem 45 we have several equivalent metrics on $M(R(x))$, which depend on the choice of the bijection $\sigma : \mathcal{F} \to \mathbb{N}$.

Paper [7]

In this paper we generalize a result of R. Gilmer [G] and some of the results of [**3**] to the case of an algebraic function field F of transcendence degree 1 over an arbitrary real closed field R. Our goal is to determine the structure of the space of orderings of F and to find which orderings induce the same $\mathbb{R}$-place of F.

Consider the set of all proper valuation rings of F containing R. The maximal ideals of these valuation rings may be regarded as closed points of the scheme associated with F. The set of all real points (i.e., places of F with residue field R) is a complete smooth real algebraic curve $\mathfrak{c}$. The elements of F can be seen as functions on $\mathfrak{c}$. Every embedding of $\mathfrak{c}$ in the projective space $\mathbb{P}^n R$ induces a *euclidean topology* (or *strong topology*) on $\mathfrak{c}$, i.e., the coarsest topology with respect to which all functions in F are continuous.

In the papers [Kn1] and [Kn2], M. Knebusch described the structure of $\mathfrak{c}$. It is a disjoint union of finitely many semi-algebraically connected components $\mathfrak{c}_1, \ldots, \mathfrak{c}_N$ which can be separated by *component separating functions* $\eta_i \in F$ as follows:

$$\operatorname{sgn} \eta_i(\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathfrak{p} \in \mathfrak{c} \setminus \mathfrak{c}_i, \\ -1 & \text{if } \mathfrak{p} \in \mathfrak{c}_i. \end{cases}$$

The η_i are determined uniquely up to multiplication by nonzero sums of squares. Each component is homeomorphic to the projective line $\mathbb{P}^1 R$, hence it admits two orientations. Consequently, $\mathfrak{c}$ admits a total of 2^N possible orientations. Assume that the orientation on $\mathfrak{c}$ is fixed. One can equip $\mathfrak{c}_i$ with the interval topology, with respect to the cyclic order given by the orientation. For every interval $(\mathfrak{p}, \mathfrak{q})$ there is an *interval function* $\chi_{(\mathfrak{p},\mathfrak{q})} \in F$ which satisfies:

$$\operatorname{sgn} \chi_{(\mathfrak{p},\mathfrak{q})}(\mathfrak{r}) = \begin{cases} 1 & \text{if } \mathfrak{r} \notin [\mathfrak{p}, \mathfrak{q}], \\ 0 & \text{if } \mathfrak{r} \in \{\mathfrak{p}, \mathfrak{q}\}, \\ -1 & \text{if } \mathfrak{r} \in (\mathfrak{p}, \mathfrak{q}). \end{cases}$$

This function is unique up to multiplication by nonzero sums of squares.

In each component $\mathfrak{c}_i$ of $\mathfrak{c}$ we fix one point and denote it by ∞_i. The set $\mathfrak{c}_i \setminus \{\infty_i\}$ is linearly ordered by the fixed orientation of the curve, so we may talk about cuts on $\mathfrak{c}_i$. A *cut in a component* $\mathfrak{c}_i$ is a pair $(\mathfrak{L}, \mathfrak{U})$ of subsets $\mathfrak{L}, \mathfrak{U} \subset \mathfrak{c}_i$ such that

- $\mathfrak{c}_i$ is the disjoint union $\mathfrak{L} \,\dot\cup\, \mathfrak{U} \,\dot\cup\, \{\infty_i\}$, and
- for every $\mathfrak{l} \in \mathfrak{L}$ and every $\mathfrak{u} \in \mathfrak{U}$, the point ∞_i lies in the interval $(\mathfrak{u}, \mathfrak{l})$.

Proposition 50 [7, Proposition 3.4] *Every cut $(\mathfrak{L}, \mathfrak{U})$ of $\mathfrak{c}_i$ defines an ordering P of the field F by the formula:*

$$P = \Psi\big((\mathfrak{L}, \mathfrak{U})\big) = \Big\{ f \in F \colon \exists_{\mathfrak{l} \in \mathfrak{L} \cup \{\infty_i\}} \exists_{\mathfrak{u} \in \mathfrak{U} \cup \{\infty_i\}} \forall_{\mathfrak{p} \in (\mathfrak{l}, \mathfrak{u})} \ f(\mathfrak{p}) > 0 \Big\}.$$

Every point $\mathfrak{p} \in \mathfrak{c}_i$ determines two principal cuts on $\mathfrak{c}_i$. On the other hand $\mathfrak{p}$ corresponds to some R-rational place $F \to R \cup \{\infty\}$. Composition of this place with the unique $\mathbb{R}$-place of R gives an $\mathbb{R}$-place of F, with the two corresponding orderings determined by the two principal cuts in $\mathfrak{p}$. As shown in [P, Theorem 9.9], the set of orderings corresponding to R-rational places is dense in $X(F)$. We use this fact to prove:

Proposition 51 [7, Proposition 2.7] *For every ordering P of F there is exactly one component $\mathfrak{c}_i$ of $\mathfrak{c}$ such that $\eta_i \in -P$.*

We say that the component $\mathfrak{c}_i$ of the above proposition is *associated with the ordering P*.

Proposition 52 [7, Proposition 3.2] *Every ordering P of F defines a cut on the associated component $\mathfrak{c}_i$ by the formula:*

$$\Phi(P) = (\mathfrak{L}, \mathfrak{U}) \ \text{with} \ \begin{cases} \mathfrak{U} = \big\{ \mathfrak{p} \in \mathfrak{c}_i \setminus \{\infty_i\} \colon \chi_{(\mathfrak{p}, \infty_i)} \in P \big\} \\ \mathfrak{L} = \big\{ \mathfrak{p} \in \mathfrak{c}_i \setminus \{\infty_i\} \colon \chi_{(\infty_i, \mathfrak{p})} \in P \big\}. \end{cases}$$

In this way we obtain two functions $\Phi : X(F) \to \mathcal{C}(\mathfrak{c})$ and $\Psi : \mathcal{C}(\mathfrak{c}) \to X(F)$, where $\mathcal{C}(\mathfrak{c})$ is the set of cuts of $\mathfrak{c}$. In **[7, Lemma 3.6]** and **[7, Proposition 3.8]** we prove that they are bijections inverting each other. Both maps are continuous, and we obtain the first main result of **[7]**.

Theorem 53 [7, Theorem 3.10] *The space $\mathcal{C}(\mathfrak{c})$ of cuts on $\mathfrak{c}$ is homeomorphic to the space $X(F)$ of orderings of F.*

Fix an element $x \in F \setminus R$. It is transcendental over R and $R(x) \subseteq F$. We have the following proposition:

Proposition 54 [7, Proposition 2.3] *For every nonconstant function $x \in F$ and every component $\mathfrak{c}_i \subseteq \mathfrak{c}$ there are finitely many points $\mathfrak{p}_0, \ldots, \mathfrak{p}_m \in \mathfrak{c}_i$ such that on every interval between two consecutive points, x is monotonic and has no poles.*

We use this proposition to define projections from the set of cuts of $\mathfrak{c}$ to the set of cuts of R. For the interval $I = (a, b)$ in an ordered set X, we denote by $\mathcal{C}^*(I)$ the set of all cuts of I, i.e., the interval $[a^+, b^-]$. If $I = \{c \mid c \geq a\}$ or $I = \{c \mid c \leq b\}$, then we take $\mathcal{C}^*(I)$ to be the interval $[a^+, X^+]$ or $[X^-, b^-]$, respectively. Take an interval $(\mathfrak{p}, \mathfrak{q}) \subset \mathfrak{c}_i$ on which $x \in F$ is monotonic and without poles. By [Kn2, Theorem 8.2], the projection $\mathfrak{r} \mapsto x(\mathfrak{r})$ is an order isomorphism of $(\mathfrak{p}, \mathfrak{q})$ onto the interval $I := \big(x(\mathfrak{p}), x(\mathfrak{q})\big)$ or $I := \big(x(\mathfrak{q}), x(\mathfrak{p})\big)$ in R. This order isomorphism induces an order isomorphism π_x from $\mathcal{C}^*\big((\mathfrak{p}, \mathfrak{q})\big)$ onto $\mathcal{C}^*(I)$. The decomposition

$$\mathfrak{c}_i = \{\infty_i\} \,\dot\cup\, (\infty_i, \mathfrak{p}_1) \,\dot\cup\, \{\mathfrak{p}_1\} \,\dot\cup\, \cdots \,\dot\cup\, (\mathfrak{p}_{m-1}, \mathfrak{p}_m) \,\dot\cup\, \{\mathfrak{p}_m\} \,\dot\cup\, (\mathfrak{p}_m, \infty_i)$$

gives us a decomposition

$$\mathcal{C}(\mathfrak{c}_i) = \mathcal{C}^*\big((\infty_i, \mathfrak{p}_1)\big) \,\dot\cup\, \cdots \,\dot\cup\, \mathcal{C}^*\big((\mathfrak{p}_{m-1}, \mathfrak{p}_m)\big) \,\dot\cup\, \mathcal{C}^*\big((\mathfrak{p}_m, \infty_i)\big).$$

We apply the order isomorphism π_x to every interval to obtain a map

$$\pi_x : \mathcal{C}(\mathfrak{c}) \to \mathcal{C}(R).$$

Proposition 55 [7, Proposition 3.12] *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C}(\mathfrak{c}) & \longrightarrow & X(F) \\ \Big\downarrow{\scriptstyle \pi_x} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\ \mathcal{C}(R) & \longrightarrow & X(R(x)) \end{array}$$

In this diagram all maps are continuous and the horizontal maps are homeomorphisms. We use it to give an alternative proof of the following proposition stated by C. Scheiderer in the appendix to [GBH]:

Proposition 56 [7, Proposition 3.13] *The space $X(F)$ is homeomorphic to $X(R(x))$.*

Now our goal is to determine which cuts (or more exactly, corresponding orderings) of $\mathfrak{c}$ determine the same $\mathbb{R}$-place. In the case of the rational function field $R(x)$ such cuts are determined by the ultrametric balls in R. The ball cuts in R can be characterized also in another way.

Proposition 57 [7, Proposition 5.1] *Take a cut C in R and the corresponding ordering P of $R(x)$. Let v_P be the natural valuation of the rational function field $R(x)$ associated with P. Then C is a ball cut if and only if $\big[v_P R(x) : 2v_P R(x)\big] = 2$.*

For any $x \in F \setminus R$, F is a finite extension of $R(x)$. From [Kn, §3] it follows that

$$[v_P F : 2v_P F] \;=\; [v_{\mathrm{res} P} R(x) : 2v_{\mathrm{res} P} R(x)]$$

for every ordering P of F, independently of the choice of x. This allows us to introduce the definition of a ball cut on the curve. A cut C of the curve $\mathfrak{c}$ is called a *ball cut* if for one (or equivalently, every) $x \in F$ transcendental over R, the projection $\pi_x(C)$ is a ball cut in R.

Theorem 58 [7, Theorem 5.3] *Let C_1 and C_2 be two ball cuts on $\mathfrak{c}$. The corresponding orderings determine the same $\mathbb{R}$-place of F if and only if for every $x \in F \setminus R$ the cuts $\pi_x(C_1)$ and $\pi_x(C_2)$ are induced by the same ultrametric ball.*

Once we embed our curve in an affine space we obtain a clearer picture, which will justify our notion of ball cuts on $\mathfrak{c}$.

The ultrametric determined by the natural valuation v of the non-archimedean real closed field R allows us to define an ultrametric on the finitely dimensional affine space $\mathbb{A}^n R$ over R in the following way:

$$d_p\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) := v\Big(\sum_{i \le n} |x_i - y_i|^p\Big)^{\frac{1}{p}}, \qquad \text{for } p = 1, 2, \ldots$$

$$d_\infty\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) := \min_{i \le n}\big\{v(x_i - y_i)\big\}.$$

In **[7, Proposition 4.2]** we prove that all of these ultrametrics are not only equivalent, but actually equal. Having an ultrametric on $\mathbb{A}^n R$ we can define balls in the natural way.

Proposition 59 [7, Proposition 6.1] *Fix a smooth and complete real affine curve $\mathfrak{c} \subset \mathbb{A}^n R$. Assume that there is a component $\mathfrak{c}_i$ of the curve which has nonempty intersections with both an ultrametric ball B in $\mathbb{A}^n R$ and its complement $\mathbb{A}^n R \setminus B$. Then B induces a ball cut on $\mathfrak{c}_i$ (possibly more than one).*

Now we can formulate another important result of [**7**].

Theorem 60 [7, Theorem 6.2] *Let $\mathfrak{c}$ be a smooth and complete real affine curve. Then every ball cut on $\mathfrak{c}$ is induced by some ultrametric ball in $\mathbb{A}^n R$.*

The idea of the proof is as follows. First we observe that the assertion is true for the principal cuts, which are induced by singleton balls. Then we consider a non-principal ball cut C on a component $\mathfrak{c}_k \subseteq \mathfrak{c}$. For every coordinate x_i, $i \in \{1, \dots, n\}$, the projection $\pi_i(C) = \pi_{x_i}(C)$ is a ball cut of R. We construct an interval $[\mathfrak{p}, \mathfrak{q}]$ on $\mathfrak{c}_k$ such that C is a proper cut of $[\mathfrak{p}, \mathfrak{q}]$ and for every $i \in \{1, \dots, n\}$,

- x_i is monotonic on $(\mathfrak{p}, \mathfrak{q})$,
- the ultrametric ball in R determining $\pi_i(C)$ is centered in either $x_i(\mathfrak{p})$ or $x_i(\mathfrak{q})$.

In this way we obtain a finite set $\mathcal{B}$ of ultrametric balls in R centered in $x_i(\mathfrak{p})$ or $x_i(\mathfrak{q})$ for some i. We compare the radii of these balls and choose the maximal one (w.r.t. inclusion), say S. Next we show that there is exactly one ball B_0 in $\mathcal{B}$ with radius S. Without loss of generality we may assume that B_0 is centered in $x_i(\mathfrak{p})$. We choose a point $\mathfrak{p}_0 > \mathfrak{p}$ in the lower cut set of C such that $v(x_i(\mathfrak{p}_0) - x_i(\mathfrak{q})) \in S$ for all i such that $x_i(\mathfrak{q})$ is the center of the ball which determines $\pi_i(C)$. Then we show that C is induced by $B_S(\mathfrak{p}_0)$. As a consequence we obtain:

Theorem 61 [7, Theorem 6.3] *Assume that $\mathfrak{c} \subset \mathbb{A}^n R$ is a smooth and complete real affine curve. Let C_1 and C_2 be two ball cuts on $\mathfrak{c}$. If the corresponding orderings determine the same $\mathbb{R}$-place of F, then there is an ultrametric ball $B \subset \mathbb{A}^n R$ inducing C_1 and C_2 on $\mathfrak{c}$.*

At the end of paper [7] we give an example which shows that the converse of the above theorem is not true. In general, to distinguish cuts which induce the same $\mathbb{R}$-place, we have to choose a particular embedding of $\mathfrak{c}$ in $\mathbb{A}^n R$.

C) Description of the most important results not included in the habilitation thesis

Apart from real algebra I also worked in the area of pure valuation theory and on generalizations of fixed point and coincidence point theorems. Here is a list of papers not included in the habilitation thesis:

[8] K. Osiak, *A Cantor cube as a space of higher level orderings*, Tatra Mt. Math. Publ. **32** (2005), 71–84

[9] K. Osiak, A. Sładek, *A note on number of orderings of higher level*, Arch. Math. (Basel) **86** (2006), no. 2, 101–110

[10] K. Osiak, *The Boolean space of higher level orderings*, Fund. Math. **196** (2007), no. 2, 101–117

[11] M. Machura, K. Osiak, *The extensions of $\mathbb{R}$-places and application*, Quadratic forms—algebra, arithmetic, and geometry, 289–297, Contemp. Math. **493**, Amer. Math. Soc., Providence, RI, 2009

[12] S. Kuhlmann, M. Marshall, K. Osiak, *Cyclic 2-structures and spaces of orderings of power series fields in two variables*, J. Algebra **335** (2011), 36–48

[13] T. Banakh, Y. Kholyavka, O. Potyatynyk, M. Machura, K. Kuhlmann, *On the dimension of the space of $\mathbb{R}$-places of certain rational function fields*, Cent. Eur. J. Math. **12** (2014), no. 8, 1239–1248

[14] F.-V. Kuhlmann, K. Kuhlmann, *A common generalization of metric, ultrametric and topological fixed point theorems.* Forum Math. **27** (2015), no. 1, 303–327

[15] F.-V. Kuhlmann, K. Kuhlmann, *Correction to A common generalization of metric, ultrametric and topological fixed point theorems*, Forum Math. **27** (2015), no. 1, 329–330

[16] F.-V. Kuhlmann, K. Kuhlmann, S. Shelah, *Symmetrically complete ordered sets, abelian groups and fields*, Israel J. Math. **208** (2015), no. 1, 261–290

[17] F.-V. Kuhlmann, K. Kuhlmann, C. Vişan, *Valuations on rational function fields that are invariant under permutation of the variables*, J. Algebra **464** (2016), 279–296

[18] F.-V. Kuhlmann, K. Kuhlmann, *Fixed point theorems for spaces with a transitive relation*, accepted for publication in Fixed Point Theory

[19] F.-V. Kuhlmann, K. Kuhlmann, F. Sonaallah, *Coincidence point theorems for ball spaces and their applications*, submitted

The first three papers present the results of my Ph.D. thesis. They contain the generalization of Craven's result for the realizability of boolean spaces as spaces of orderings to the higher level orderings introduced by E. Becker in [Be0]. To this end we show in **[8]** that every Cantor cube can be realized as a space of orderings of level n, and in **[9]** we show how to realize finite spaces. Paper **[10]** contains the final result.

Paper **[11]** contains some new results in the extension theory of $\mathbb{R}$-places. We describe the possible numbers of extensions of a given $\mathbb{R}$-place of K to finite Galois extensions of K. Then we use these results to prove that an arc can be realized as a space of $\mathbb{R}$-places. This result was later generalized in **[2]**.

Paper **[12]** contains the results of a research project I was working on during my visit as a postdoctoral fellow at the University of Saskatchewan in Canada. The goal of this project was to describe spaces of orderings and $\mathbb{R}$-places of a formal power series field $R((x,y))$ in two variables over any real closed field R. The space of orderings of the rational function field $R((x))(y)$ is the disjoint union of the spaces of orderings of the fields $R_1(y)$ and $R_2(y)$, where R_1, R_2 are the real closures of $R((x))$ with respect to its unique two orderings. Therefore, we can see $X(R((x))(y))$ as a disjoint union $\mathcal{C}(R_1)\,\dot\cup\,\mathcal{C}(R_2)$ of spaces of cuts in R_1 and R_2. The restriction map res : $X(R((x,y))) \to X(R((x))(y))$ is injective (**[12, Lemma 4.3]**), and since y is infinitesimal with respect to each ordering of $R((x))$, we obtain that the restriction of any ordering in $X(R((x,y)))$ to the rational function field defines a cut in the ideal I_1 or I_2 of infinitesimals of R_1 or R_2, respectively. Therefore, $X(R((x,y)))$ can be identified with $\mathcal{C}(I_1)\,\dot\cup\,\mathcal{C}(I_2)$ (**[12, Lemma 4.5]**). Consider the set $S = R_1 \,\dot\cup\, R_2 \,\dot\cup\, \{+\infty, -\infty\}$. It admits a cyclic ordering, derived from the orderings of R_1 and R_2 by joining them at newly introduced points $\pm\infty$. The cuts in this cyclically ordered set S are defined as before. The orderings of $R((x))(y)$ correspond bijectively to the cuts in S. We show that every irreducible polynomial in $R((x))[y]$ defining a formally real extension of $R((x))$ has exactly two roots in $R_1 \,\dot\cup\, R_2$. This gives rise to a well-defined map $r \mapsto r'$ from $R_1 \,\dot\cup\, R_2$ onto itself, which we call the *conjugation map*. We extend it by sending $+\infty$ to $-\infty$ and vice versa, and prove that the so-obtained conjugation map is continuous with respect to the cyclic order topology (interval topology) on S (**[12, Theorem 3.2]**). A *cyclic 2-structure* is defined as a pair (S, Φ) where S is a cyclically ordered set and Φ is an equivalence relation on S such that each equivalence class has exactly two elements. Then the set $S = R_1 \,\dot\cup\, R_2 \,\dot\cup\, \{+\infty, -\infty\}$ with the equivalence relation determined by the conjugacy classes is a cyclic 2-structure. Each equivalence class $\{r, r'\}$ determines two arcs $(r, r') = \{s \in S : r < s < r'\}$ and $(r', r) = \{s \in S : r' < s < r\}$ and functions $f_1, f_2 : \mathcal{C}(S) \to \{-1, 1\}$ (called the *atoms* associated to the equivalence class $\{r, r'\}$) defined by

$$f_1(x) := \begin{cases} 1 \text{ if } x \text{ is a cut of } (r, r'), \\ -1 \text{ if } x \text{ is a cut of } (r', r) \end{cases}$$

and $f_2 := -f_1$. Denote by $G_{(S,\Phi)}$ the group of functions $f : \mathcal{C}(S) \to \{-1, 1\}$ generated by the constant functions $1, -1$ and the atoms associated to the various equivalence classes of S. A pair (X, G), where X is a set and G is a group of functions from X to $\{-1, 1\}$, is said to be *described by the cyclic 2-structure* (S, Φ) if there exists a bijection $p : X \to \mathcal{C}(S)$ such that $G = \{f \circ p : f \in G_{(S,\Phi)}\}$. The first main result of **[12]** is:

Theorem 62 **[12, Theorem 5.1]** *For any real closed field R, the spaces of orderings of the fields $R((x))(y)$ and $R((x, y))$ are described by cyclic 2-structures in a natural way.*

The second main result of **[12]** shows that if the real closed field R is archimedean, then also the spaces of $\mathbb{R}$-places of $R((x))(y)$ and $R((x, y))$ can be described in terms of cyclic 2-structures.

Theorem 63 **[12, Theorem 6.5]** *Let P and Q be two distinct orderings of $R((x))(y)$ or of $R((x, y))$.*

(1) A sufficient condition for P and Q to have the same associated $\mathbb{R}$-place is that for each pair of intervals (r_1, s_1) and (r_2, s_2) of the cyclically ordered set S with $P \in \mathcal{C}((r_1, s_1))$ and $Q \in \mathcal{C}((r_2, s_2))$, there exist conjugate elements $r, r' \in S$ such that $r_1 < r < s_1$ and $r_2 < r' < s_2$.

(2) If the real closed field R is archimedean, then the sufficient condition described in (1) is also necessary.

In paper **[13]** we investigate the dimension of the space of $\mathbb{R}$-places of rational function fields $K(x_1, \ldots, x_n)$ in several variables over a totally archimedean field K. We prove that for every $n \in \mathbb{N}$ the space $M(K(x_1, \ldots, x_n))$ has topological covering dimension $\dim M(K(x_1, \ldots, x_n)) \leq n$. The main result of the paper states:

Theorem 64 **[13, Theorem 2]** *For any totally archimedean field K the space $M(K(x, y))$ has integral cohomological dimension $\dim_{\mathbb{Z}} M(K(x, y)) = \dim M(K(x, y)) = 2$ and cohomological dimension $\dim_G M(K(x, y)) = 1$ for any nontrivial 2-divisible abelian group G.*

This result shows that the space $M(K(x, y))$ is a natural example of a compact space that is not dimensionally full-valued (which means that the cohomological dimensions of $M(K(x, y))$ for various coefficient groups G do not coincide). The proof is based on the notion and properties of *graphoids* introduced by T. Banakh and O. Potyatynyk in [BP]. This is a very strong result in the theory of $\mathbb{R}$-places, but is not included in the habilitation thesis because apart from bringing the topic to the attention of the authors, the habilitant's only input was the proof of Theorem 2.2 of **[13]** which identifies points of a proper graphoid with $\mathbb{R}$-places of $K(x, y)$.

The seemingly "fractal" structure of the space of $\mathbb{R}$-places of the rational function field over a non-archimedean real closed field R motivated us to look at possible generalizations of fixed point theorems for contractive or non-expanding maps known separately for (generalized) metric, ultrametric and topological spaces. In all cases a notion of *completeness* is needed. In paper **[14]** we introduced the notion of a *ball space* which is just a nonempty set X with a nonempty collection $\mathcal{B}$ of nonempty subsets, which we call *balls*. A ball space is *spherically complete* if every nonempty nest (a collection totally ordered by inclusion) of its balls has a nonempty intersection. The terminology is taken from ultrametric spaces, and the notion of ultrametric spherical completeness coincides with the spherical completeness of the ultrametric space as a ball space where the balls are taken to be the nonempty ultrametric balls. For topological spaces with the family of all nonempty closed subsets, spherical completeness is equivalent to compactness. Completeness of a metric space is equivalent to the spherical completeness of families of closed metric balls for which their radii form subsets of $\mathbb{R}^+$ with 0 as their unique accumulation point. For a function $f : X \to X$, a subset $B \subseteq X$ is called *f-contracting* if it is either a singleton containing a fixed point or satisfies $f(B) \subsetneq B$. The following are two main theorems of **[14]**.

Theorem 65 *Take a function f on a ball space $(X, \mathcal{B})$ which satisfies the following conditions:*

(C1) *there is at least one f-contracting ball,*
(C2) *for every f-contracting ball $B \in \mathcal{B}$, the image $f(B)$ contains an f-contracting ball,*
(C3) *the intersection of every nest of f-contracting balls contains an f-contracting ball.*

Then f admits a fixed point.

Theorem 66 *Take a function f on a ball space $(X, \mathcal{B})$ which satisfies the following conditions:*

(CU1) *X is an f-contracting ball,*
(CU2) *for every f-contracting ball $B \in \mathcal{B}$, the image $f(B)$ is again an f-contracting ball,*
(CU3) *the intersection of every nest of f-contracting balls is again an f-contracting ball.*

Then f has a unique fixed point.

In **[14]** we show how from the above theorems one obtains known fixed point theorems for (adequately defined) contracting maps: Banach's Fixed Point Theorem for metric spaces, its ultrametric version proved by S. Prieß-Crampe and P. Ribenboim in [PR], and a topological version for connected topological spaces proved in [SWJ].

The flexibility of the notion of "ball space" allows us to carry Banach's Fixed Point Theorem over to generalized metric spaces (i.e., metrics with values in the non-negative part of some not necessarily archimedean ordered group). A natural example of such a space is any non-archimedean ordered abelian group or field. Associated with them are two natural ball spaces:

- the order ball space, where the balls are closed bounded intervals, and
- the ultrametric ball space, where the balls are the ultrametric balls derived from the natural valuation.

We discuss these ball spaces and present corresponding fixed point theorems. Moreover, we consider hybrid ball spaces, in which we use order balls and ultrametric balls simultaneously. We use this concept to give a simple characterization of those ordered fields which are power series fields with residue field $\mathbb{R}$.

Ordered groups and fields which are spherically complete with respect to the order balls are the main topic of paper **[16]**. Let X be an ordered set. Take a nest $\mathcal{N} = ([a_i, b_i])_{i\in I}$ of closed bounded intervals in X and assume that the intersection of $\mathcal{N}$ is empty. Then there is a cut (D, E) in X such that the sequence $(a_i)_{i\in I}$ is cofinal in D and the sequence $(b_i)_{i\in I}$ is coinitial in E. This situation will not happen if for every cut C in X the cofinality of the lower cut set and the coinitiality of the upper cut set of C differ; such a cut is called *asymmetric*. Already Hausdorff ([Hd]) in 1907 proved that there are ordered sets in which every cut is asymmetric. Such sets are now called *symmetrically complete*. For ordered fields, this notion was introduced by S. Shelah in [S], where he proved that every ordered field can be extended to a symmetrically complete ordered field. In paper **[16]** we extend this result to ordered abelian groups. It turns out that for an ordered abelian group G to be symmetrically complete, the same must be true for the value set vG of its natural valuation v. In fact, it must have an even stronger property. A cut C of X is called *strongly asymmetric* if it is asymmetric and the cofinality of its lower cut set or the coinitiality of its upper cut set is uncountable. An ordered set X is called *strongly symmetrically complete* if every cut in X which is not a jump is strongly asymmetric, and we call X *extremely symmetrically complete* if in addition, the coinitiality and cofinality of X are both uncountable. The following two theorems give full characterizations of symmetrically complete ordered groups and fields.

Theorem 67 *A nontrivial densely ordered abelian group G is symmetrically complete if and only if it is spherically complete with respect to its natural valuation v, has a densely ordered strongly symmetrically complete value set vG, and all its archimedean components are isomorphic to $\mathbb{R}$. It is strongly symmetrically complete if and only if in addition, vG has uncountable cofinality, and it is extremely symmetrically complete if and only if in addition, vG is extremely symmetrically complete.*

Theorem 68 *An ordered field K is symmetrically complete if and only if it is spherically complete with respect to its natural valuation v, has residue field $\mathbb{R}$ and a dense strongly symmetrically complete value group vK. Further, the following are equivalent:*

a) K is strongly symmetrically complete,

b) K is extremely symmetrically complete,

c) K is spherically complete w.r.t. its natural valuation v, has residue field $\mathbb{R}$ and a dense extremely symmetrically complete value group vK.

The following corollary states important properties of symmetrically complete ordered groups and fields.

Corollary 69 *Every dense symmetrically complete ordered abelian group is divisible and isomorphic to a Hahn product. Every symmetrically complete ordered field is real closed and isomorphic to a power series field with residue field $\mathbb{R}$ and divisible value group.*

The research on ball spaces is continuing. In **[18]** we use ball spaces to prove fixed point theorems for spaces with a transitive relation (which can be used to encode path connection in an underlying graph). In **[19]** we prove coincidence point theorems for ball spaces which generalize our fixed point theorems, and we present several applications.

On the class of ball spaces we introduce a hierarchy depending on the level of their completeness. We consider intersections of nests and of directed and centered systems of balls. The criterion for the level is whether such intersections are nonempty, contain a ball, contain a largest ball, or are themselves balls. Examples of the strongest ball spaces, i.e., ball spaces for which the intersection of every centered system of balls is a ball (we call them S^*), are ultrametric spaces with the family of all nonempty ultrametric balls, compact topological spaces with the family of all nonempty closed subsets, and complete lattices with the family of all nonempty closed bounded intervals.

Our fixed point theorems for ball spaces can be used to prove the Bourbaki-Witt Fixed Point Theorem for increasing functions on pointed posets as well as the Knaster-Tarski Fixed Point Theorem for order preserving functions on complete lattices. The latter theorem does not only state the existence of fixed points, but also that the set of all fixed points is itself a complete lattice. We generalize this result to S^* ball spaces. This allows us to shift the Knaster-Tarski Theorem to other settings, like ultrametric spaces and topological spaces.

Similarly, it is possible to prove an equivalent of Tychonoff's theorem for ball spaces, stating that the product of spherically complete ball spaces, defined in a natural way, is itself spherically complete. Again, this enables us to shift the result to other settings, like ultrametric spaces.

All of these results were presented at the 29th Summer Conference on Topology and its Applications at the City University of New York in July 2014 and at the Summer School Around Valuation Theory at Sirince (Turkey) in May 2014. Our next goal is to write a book in the "Lecture Notes in Mathematics" series about ball spaces and their applications.

References

[AS] E. Artin, O. Schreier, *Algebraische Konstruktion reeller Körper*, Abh. Math. Sem. Univ. Hamburg **5** (1927), 85–99

[B1] R. Baer, *Über nicht-archimedisch geordnete Körper*, Sitz. Ber. der Heidelberger Akad. **8** Abh. (1927), 3–13

[B2] R. Baer, *Zur Topologie der Gruppen*, J. reine angew. Math. **160** (1929), 208–226

[BP] T. Banakh, O. Potyatynyk, *Dimension of graphoids of rational vector-functions*, Topology Appl. **160** (2013), 24–44.

[Be0] E. Becker, *Hereditarily Pythagorean Fields and Orderings of Higher Level*, Monograf. Mat. **29**, Ins. Mat. Pura Aplicada, Rio de Janero (1978)

[Be1] E. Becker, *Valuations and real places in the theory of formally real fields*, Real algebraic geometry and quadratic forms (Rennes, 1981), 1–40, Lecture Notes in Math. **959**, Springer, Berlin-New York, 1982

[Be2] E. Becker, *The real holomorphy ring and sums of* $2n$*-th powers*, Real algebraic geometry and quadratic forms (Rennes, 1981), 139–181, Lecture Notes in Math. **959**, Springer, Berlin-New York, 1982

[Be3] E. Becker, *Sums of powers in fields and orderings of higher level*, preprint

[BG] E. Becker, D. Gondard, *Notes on the space of real places of a formally real field*, Real Analytic and Algebraic Geometry (Trento, 1992), 21–46, de Gruyter, Berlin, 1995

[BCR] J. Bochnak, M. Coste, and M. Roy, *Real algebraic geometry*, volume 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, Springer-Verlag, Berlin, 1998.

[Br] R. Brown, *Real-valued places on the function field of an algebraic curve*, Houston J. Math. **6** (1980), 227–243

[Br1] R. Brown, *Real places and ordered fields.* Rocky Mount. J. Math. **1** (1971), 633–636

[BM] R. Brown and J. Merzel, *The space of real places on* $\mathbb{R}(x, y)$, preprint

[Cr] T. C. Craven, *The topological space of orderings of a rational function field*, Duke Math. J. **41**, (1974), 339–347

[C] T. C. Craven, *The Boolean space of orderings of a field*, Trans. Amer. Math. Soc. **209** (1975), 225–235

[D] D. W. Dubois, *Infinite primes and ordered fields*, Dissertationes Math. **69** (1970), 1–43

[E] Y. L. Ershov, *The number of linear orders on a field*, Mat. Zametki **6** (1969), 201–211; English transl. in Math. Notes **6** (1969)

[G] R. Gilmer, *Extension of an order to a simple transcendental extension*, Ordered fields and real algebraic geometry (San Francisco, Calif., 1981), 113–118, Contemp. Math. **8**, Amer. Math. Soc., Providence, R.I., 1982

[GM] D. Gondard, M. Marshall, *Towards an abstract description of the space of real places*, Contemp. Math. **253** (2000), 77–113

[GBH] N. Grenier-Boley, D. Hoffmann, *Isomorphism criteria for Witt rings of real fields*, Forum Math. **25** (2013), 1–18 (with an appendix by Claus Scheiderer)

[HR] K. E. Hare, J. Rönning, *Fractal dimensions of infinite product spaces*, Int. J. Pure Appl. Math. **14** (2004), no. 2, 139–169

[H] J. Harman, *Chains of higher level orderings*, Contemp. Math. **8** (1982), 141–174

[Hd] F. Hausdorff, *Grundzüge einer Theorie der geordneten Mengen*, Math. Ann. **65** (1908), 435–505

[K] W. Krull, *Allgemeine Bewertungstheorie*, J. reine angew. Math. **167** (1931), 160–196

[Kn] M. Knebusch, *On the extension of real places*, Comment. Math. Helv. **48** (1973), 354–369

[Kn1] M. Knebusch, *On algebraic curves over real closed fields. I*, Math. Zeit. **150** (1976), 49–70

[Kn2] M. Knebusch, *On algebraic curves over real closed fields. II*, Math. Zeit. **151** (1976), 189–205

[KP] F.-V. Kuhlmann and A. Prestel, *On places of algebraic function fields*, J. reine angew. Math. **353** (1984), 181–195

[L] T. Y. Lam, *Orderings, valuations and quadratic forms*, CBMS Regional Conf. Ser. Math. **52**. Published for the Conf. Board of the Math. Sciences, Washington (1983)

[LL] F. Lorenz, J. Leicht, *Die Primideale des Wittschen Ringes*, Invent. Math. **10** (1970), 82–88

[N] J. Nagata, *Modern dimension theory*, Bibliotheca Mathematica, Vol. VI. Interscience Publishers John Wiley & Sons, Inc., New York 1965

[NTT] J. Nikiel, H. M. Tuncali, E. D. Tymchatyn, *On the rim-structure of continuous images of ordered compacta*, Pacific J. Math. **149** (1991), 145–155

[P] A. Prestel, *Lectures on formally real fields*, Lecture Notes in Mathematics **1093**, Springer-Verlag, Berlin, 1984

[PR] S. Prieß-Crampe, P. Ribenboim, *Fixed Point and Attractor Theorems for Ultrametric Spaces*, Forum Math. **12** (2000), 53–64

[Sch] H.-W. Schülting, *Real points and real places*, Ordered fields and real algebraic geometry (San Francisco, Calif., 1981), 289–295, Contemp. Math. **8**, Amer. Math. Soc., Providence, R.I., 1982

[S] S. Shelah, *Quite Complete Real Closed Fields*, Israel J. Math. **142** (2004), 261–272

[SWJ] J. Steprāns, S. Watson, W. Just, *A topological Banach fixed point theorem for compact Hausdorff spaces*, Canad. Bull. Math. **37** (1994), 552–555

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