Piotr Helbin, MSc
Institute of Mathematics
Faculty of Mathematics, Physics and Chemistry
University of Silesia in Katowice

Katowice, May 6th, 2017

Summary of PhD Thesis

Fuzzy implications generated from copulas

The aim of the following PhD thesis is to collect the information about fuzzy implications generated from two-valued copulas, or from more general functions (such as semicopulas). Copulas are very important functions in theory of probability. The importance of the copulas is clarifed be the Sklar theorem, which says that if H is a joint distribution of random varibles $(X_i)_{1\leq i\leq n}$ (for $n \geq 2$) with marginals $(F_i)_{1\leq i\leq n}$, then there exists a copula $C\colon [0,1]^n \to [0,1]$ such that

$$H(x_1, x_2, \ldots x_n) = C(F_1(x_1), F_2(x_2), \ldots F_n(x_n)), \qquad x_1, x_2, \ldots x_n \in \mathbb{R}.$$

If $(F_i)_{1\leq i\leq n}$ are continous functions, then C is unique. Conversely, if C is a copula and the function H is defined by the above formula, then H is a joint distribution function with margins $(X_i)_{1\leq i\leq n}$. In the articles [7], [8] Grzegorzewski introduced four new classes of fuzzy implications based on copulas using Sklar theorem. Namely, probabilistic implications, probabilistic s-implications, survival implications and survival s-implications. In the article [5] authors, in similar way, like Grzegorzewski introduced new family of fuzzy implications based on copulas, called conditional implications. It turns out that class of functions $J_{I,B}\colon [0,1]^2 \to [0,1]$ (presented for the first time at the conference IFSA-EUSFLAT 2015 [2]) the following form

$$J_{I,B}(x, y) = I(x, B(x, y)), \qquad x, y \in [0,1],$$

where I is a fuzzy implication, B is a semicopula, generalizes all four classes of implications generated from copulas introduced by Grzegorzewski.

Chapter I contains preliminary information on the basic logical connectives, copulas, quasicopulas and semicopulas with their the most important properties and a few useful information about continuous functions.

Chapter II is devoted to solutions of the Frank equation [6]. This proof is rarely presented in monographs, but Frank t-norms, which are the solutions of the Frank equation, are often quoted in many works. Moreover, many equations for resulting from appropriate properties of probabilistic s-implications can be solved using Frank t-norms. Therefore, we present complete proof of the solution of the Frank equation.

Chapter III is devoted to discuss two significant classes of fuzzy implications. The first one is the family of residual implications generated from semicopulas. Residual implications were known much earlier in the literature (see [4]). By the residual implication generated from t-norm T we understand the function $I_T\colon [0,1]^2 \to [0,1]$ defined in the following way

$$I_T(x, y) = \sup\{t \in [0,1] \mid T(x,t) \leq y\}, \qquad x, y \in [0,1].$$

In applications, the requirement of a t-norm for residual implication is quite “inconvenient” due to the associative of t-norms. Therefore, it is necessary to consider the residual implications obtained from semicopulas. Semicopulas do not require

condition of associativity. Moreover, t-norms, copulas and quasicopulas belong to the class of semicopulas. The second one is family of functions mentioned earlier, namely the functions of the the following form $I(x, B(x, y))$, where I is a fuzzy implications B is a semicopulas. Presented properties of this class of functions are based on the results of the article [3] obtained by the author in cooperation with M. Baczyński, R. Mesiar, P. Grzegorzewski and W. Niemyska.

In chapter IV we show how using the Sklar theorem we can obtain functions such as probabilistic implications, probabilistic s-implications, conditional implications, survival implications and survival s-implications. In addition, the basic properties of these classes of functions are presented.

In the last chapter, presented are new results from the work [1] obtained by the author in cooperation with M. Baczyński, P. Grzegorzewski, W. Niemyska and unpublished results obtained by the author. These results include the properites of implications from Chapter IV such as the law of contraposition, the law of importation, T-conditionality, and the intersections of classes of implications described in this dissertation with other known classes of fuzzy implications.

References

[1] M. Baczyński, P. Grzegorzewski, P. Helbin, and W. Niemyska. Properties of the Probabilistic Implications and S-Implications. *Information Sciences*, 331:2–14, 2016.

[2] M. Baczyński, P. Grzegorzewski, and R. Mesiar. Fuzzy implications based on semicopulas. In *IFSA-EUSFLAT 2015*, pages 792–798, Amsterdam, 2015.

[3] M. Baczyński, P. Grzegorzewski, R. Mesiar, P. Helbin, and W. Niemyska. Fuzzy implications based on semicopulas. *Fuzzy Sets and Systems*, 2016. accepted, doi: http://dx.doi.org/10.1016/j.fss.2016.09.009.

[4] G. Birkoff. *Lattice Theory.* American Mathematical Society, Providence, Rhode Island, 3rd edition, 1967.

[5] A. Dolati, J.F. Sánchez, and M. Úbeda Flores. A copula-based family of fuzzy implication operators. *Fuzzy Sets and Systems*, 211:51–61, 2013.

[6] M.J. Frank. On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$. *Aequationes Mathematicae*, 19:194–226, 1979.

[7] P. Grzegorzewski. Probabilistic implications. In *EUSFLAT-2011 and LFA-2011*, pages 254–258, Amsterdam, 2011.

[8] P. Grzegorzewski. Survival implications. In S. Greco, B. Bouchon-Meunier, G. Coletti, M. Fedrizzi, B. Matarazzo, and R.R. Yager, editors, *IPMU 2012, Part II*, volume 298 of *Communications in Computer and Information Science*, pages 335–344, 2012.