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Report of Meeting

The Twenty-fourth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities Hajdúszoboszló (Hungary), February 6 – 9, 2025

The Twenty-fourth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities was held in Hotel Aurum, Hajdúszoboszló, Hungary, from February 6 to February 9, 2025. It was organized by the Department of Analysis of the Institute of Mathematics of the University of Debrecen.

13 participants came from the University of Debrecen (Hungary), 5 from the University of Silesia in Katowice (Poland), 2 from the University of Miskolc (Hungary) and 1 from the University of Rzeszów (Poland),

Professor Zsolt Páles opened the Seminar and welcomed the participants to Hajdúszoboszló.

The scientific talks presented at the Seminar focused on the following topics: functional equations in a single variable and in several variables; alternative equations; functional equations on different algebraic structures; equations and inequalities for integrals, applications in risk management; functional equations and inequalities related to various types of means; generalized monotonicity and convexity; characterizations of differential operators; new methods in fractal approximation and linear programming and their computer implementation; summability in Walsh–Fourier analysis. Interesting discussions were generated by the talks.

The program included two Problems and Remarks sessions and a festive dinner.

The closing address was given by Professor Maciej Sablik. His invitation to the Twenty-fifth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities in 2026 in Poland was gratefully accepted.

Summaries of the talks in alphabetical order of the authors follow in Section 1, problems and remarks in chronological order in Section 2, and the list of participants in the final section.

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1. Abstracts of talks

Mihály Bessenyei: *Approximation of fractals via Kantorovich iteration* (Joint work with Bálint Szabó)

The Kantorovich process provides an alternative method to approximate fractals even in those cases when Hutchinson's approach does not work. In the talk, we present examples for such situations, the implementation of the approximations, and some facts about computational complexity.

Zoltán Boros: *Non-zero additive solutions for some alternative equations* (Joint work with Rayene Menzer)

Z. Kominek, L. Reich and J. Schwaiger [1] proved, for various particular choices of $D \subseteq \mathbb{R}^2$, that any additive function $f\colon \mathbb{R} \to \mathbb{R}$ fulfilling

$$f(x)f(y) = 0 \quad \text{for every} \quad (x, y) \in D \tag{1}$$

has to be equal to zero identically. For instance, their investigations covered the cases when D is the unit circle as well as when D is a one parameter family generated by a pair of ordinary polynomials. 20 years later P. Kutas [2] established the existence of a not identically zero additive function $f\colon \mathbb{R} \to \mathbb{R}$ fulfilling (1) when

$$D = \{\, (x, y) \in \mathbb{R}^2 \mid xy = 1 \,\}.$$

In our presentation, we discuss a couple of generalizations of this example. Some of these generalizations are mentioned by P. Kutas in his cited paper as direct applications of his arguments. Another generalization is suggested in the talk as a result of a slightly modified argument.

References

[1] Z. Kominek, L. Reich, and J. Schwaiger, *On additive functions fulfilling some additional condition*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **207** (1998), 35–42.

[2] P. Kutas, *Algebraic conditions for additive functions over the reals and over finite fields*, Aequationes Math. **92** (2018), no. 3, 563–575.

György Gát: *Problems and recent results in the theory of summability of Walsh-Fourier series*

Let $\hat{f}(k) := \int_0^1 f(x)\omega_k(x)dx$ be the kth Walsh-Fourier coefficient of the integrable function f. Let $T := (t_{i,j})_{i,j=0}^{\infty}$ be a doubly infinite triangular matrix of nonnegative numbers, where $\sum_{k=0}^{n-1} t_{k,n-1} = 1$ for every natural number n. Define the nth matrix transform mean determined by the matrix T of the Fourier series of f as

$$\sigma_n^T(f) := \sum_{k=0}^{n-1} t_{k,n-1} S_k(f).$$

We mention some well-known summation methods of this type:
– The Cesàro (or (C,α)) summation,
– The Riesz summation,
– The Weierstrass summation.

A much-studied question is whether various summation methods can be used to reconstruct an integrable function. And, if so, in what sense? In this presentation we mention some of the problems and recent achievements [1, 2] in this area. In addition to the Walsh system, we also discuss the trigonometric case.

References

[1] I. Blahota and G. Gát, *Approximation by subsequences of matrix transform means of some two-dimensional rectangle Walsh-Fourier series*, J. Fourier Anal. Appl. **30** (2024), no. 5, Paper No. 51, 35 pp.

[2] G. Gát, *Almost everywhere divergence of Cesàro means of subsequences of partial sums of trigonometric Fourier series*, Math. Ann. **389** (2024), no. 4, 4199–4231.

Attila Gilányi: *Bernstein–Doetsch theorem for (M,N)-convex functions* (Joint work with Zsolt Páles)

One of the fundamental results of the classical theory of convex functions is the Bernstein–Doetsch theorem ([1]) which states that if a real-valued Jensen-convex function defined on an open interval is locally bounded above at a point in its domain, then it is continuous. In this talk, we present a generalization of this theorem for (M,N)-convex functions, calling a function $f\colon I\to J$ (M,N)-convex (cf., e.g., [3]) if it satisfies the inequality $f(M(x,y))\leq N(f(x),f(y))$ for all $x,y\in I$, where I and J are open intervals, M and N are suitable means on I and J, respectively. Our statement contains Tomasz Zgraja's result on (M,M)-convex functions (cf. [4]) as a special case. It also generalizes the main theorem presented in [2].

References

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[2] A. Gilányi and Z. Páles, *Bernstein–Doetsch and Sierpiński theorems for (M,N)-convex functions*, Talk, 11^{th} International Conference on Functional Equations and Inequalities, Stefan Banach International Mathematical Center, Będlewo, Poland, September 17–23, 2006.

[3] C. Niculescu and L.-E. Persson, *Convex Functions and Their Applications*, CMS Books in Mathematics, Springer, New York, 2006.

[4] T. Zgraja, *Continuity of functions which are convex with respect to means*, Publ. Math. Debrecen **63** (2003), no. 3, 401–411.

Richárd Grünwald: *Comparison of Gini means with fixed number of variables* (Joint work with Zsolt Páles)

Let us recall the definition of the *n-variable Gini mean corresponding to* *the pair parameters* $(p,q)\in\mathbb{R}^2$:

$$G_{p,q}^{[n]}(x_1,\dots,x_n) := \begin{cases} \left(\dfrac{x_1^p+\dots+x_n^p}{x_1^q+\dots+x_n^q}\right)^{\frac{1}{p-q}} & \text{if } p\neq q,\\ \exp\left(\dfrac{x_1^p\ln(x_1)+\dots+x_n^p\ln(x_n)}{x_1^p+\dots+x_n^p}\right) & \text{if } p=q, \end{cases} \qquad (x_1,\dots,x_n\in\mathbb{R}_+).$$

Let us consider the global comparison problem of Gini means with fixed number of variables on a subinterval I of $\mathbb{R}_+$, i.e., the following inequality

$$G_{r,s}^{[n]}(x_1,\dots,x_n) \le G_{p,q}^{[n]}(x_1,\dots,x_n), \tag{1}$$

where $n\in\mathbb{N}, n\geq 2$ is fixed, $(p,q),(r,s)\in\mathbb{R}^2$ and $x_1,\dots,x_n\in I$.

Given a nonempty subinterval I of $\mathbb{R}_+$ and $n\in\mathbb{N}$, we introduce the sets

$$\Gamma_n(I) := \{((r,s),(p,q))\in\mathbb{R}^2\times\mathbb{R}^2 \mid (1) \text{ holds for all } x_1,\dots,x_n\in I\},$$
$$\Gamma_\infty(I) := \bigcap_{n=1}^{\infty}\Gamma_n(I).$$

In the talk, apart from a limiting case, we characterize the elements of $\Gamma_n(I)$ via a conditional minimum problem. Then we formulate a conjecture for the aforementioned limiting case.

References

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Eszter Gselmann: *A characterization of second-order differential operators* (Joint work with Włodzimierz Fechner)

An easy computation shows that if $f,g,h\in\mathcal{C}^2(\Omega)$, then we have

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(f\cdot g\cdot h) - f\frac{\mathrm{d}^2}{\mathrm{d}x^2}(g\cdot h) - g\frac{\mathrm{d}^2}{\mathrm{d}x^2}(f\cdot h) - h\frac{\mathrm{d}^2}{\mathrm{d}x^2}(f\cdot g) + f\cdot g\frac{\mathrm{d}^2}{\mathrm{d}x^2}h + f\cdot h\frac{\mathrm{d}^2}{\mathrm{d}x^2}g + g\cdot h\frac{\mathrm{d}^2}{\mathrm{d}x^2}f = 0. \tag{1}$$

As a generalization of the Leibniz rule, usually the identity

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(f\cdot g)=\frac{\mathrm{d}^2}{\mathrm{d}x^2}f\cdot g+2\frac{\mathrm{d}}{\mathrm{d}x}f\cdot\frac{\mathrm{d}}{\mathrm{d}x}g+f\cdot\frac{\mathrm{d}^2}{\mathrm{d}x^2}g \qquad \big(f,g\in\mathcal{C}^2(\Omega)\big)$$

is considered. In some sense, the disadvantage of the latter identity, compared to identity (1), is that it includes not only the second-order, but also the first-order differential operator.

Let k be a fixed nonnegative integer and $\Omega\subset\mathbb{R}$ be a nonempty and open set. In this talk, we study operators $D\colon \mathcal{C}^k(\Omega)\to\mathcal{C}(\Omega)$ that fulfil

$$\text{(2)}\quad D(f\cdot g\cdot h)-fD(g\cdot h)-gD(f\cdot h)-hD(f\cdot g)\\ +f\cdot gD(h)+f\cdot hD(g)+g\cdot hD(f)=0$$

for all $f,g,h\in\mathcal{C}^k(\Omega)$. We emphasize that unless stated otherwise, the operator D is *not assumed* to be linear. As we can observe, (2) turns out to be suitable for characterizing second-order linear differential operators in function spaces.

Mehak Iqbal: *A functional equation for monomial functions* (Joint work with Eszter Gselmann)

Let $\mathbb{K}$ be a field of characteristic zero, $\mathbb{F}\subset\mathbb{K}$ be a subfield, n be a positive integer and κ be in $\mathbb{K}$. Let further $f\colon\mathbb{F}\to\mathbb{K}$ be a generalized monomial of degree n. In this talk, we focus on the functional equation

$$f(x^2)=\kappa\cdot x^n f(x) \qquad (x\in\mathbb{F})$$

for the unknown generalized monomial f. The presented results extend the classical cases $n=1,2$, that is, the cases of additive and quadratic functions, respectively.

Tibor Kiss: *On continuously differentiable solutions of a functional equation* (Joint work with Péter Tóth)

The general solution of the composite functional equation

$$F\Big(\frac{x+y}{2}\Big)+f_1(x)+f_2(y)=G\big(g_1(x)+g_2(y)\big), \qquad x,y\in I,$$

was known if the functions involved are continuously differentiable and the derivatives of g_1 and g_2 are nowhere zero. In the talk we are going to demonstrate how to eliminate this latter condition and solve the equation in question under continuous differentiability assumption.

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[2] T. Kiss, *Regular solutions of a functional equation derived from the invariance problem of Matkowski means*, Aequationes Math. **96** (2022), no. 5, 1089–1124.

[3] T. Kiss, *A Pexider equation containing the aritmetic mean*, Aequationes Math. **98** (2024), no. 2, 579–589.
[4] P. Tóth, *Measurable solutions of an alternative functional equation*, submitted in 2024.

Judit Makó: *Approximate Jensen convexity*

In the talk, some new results concerning approximate Jensen convexity are presented. Let X be a normed space, $D \subset X$ be a convex set, and $c > 0$. A function $f\colon D \to \mathbb{R}$ is (c,α)-Jensen convex, if for all $x, y \in D$

$$f\Big(\frac{x+y}{2}\Big) \leq cf(x) + cf(y) + \alpha(\|x-y\|).$$

We investigate the (c,α)-Jensen convex functions in different concepts and answer questions: How can we strengthen this inequality? Is the above inequality superstable? How from a Hermite–Hadamard type inequality can we get a (c,α)-Jensen convex function? We also obtain Bernstein–Doetsch type results.

Magdalena Mamcarz: *Inequalities for generalized convex functions with the use of the Hesselager theorem*

We observe that the theorem of Hesselager [2] may be used to obtain new proofs of Hermite-Hadamard inequalities for functions that are convex with respect to a Chebyshev system. Such inequalities were proved by Bessenyei and Páles [1] for integrals with density function. In our approach, the first function from the system involved must be constant but the main theorem is valid for the integral with respect to any measure. Additionally, we present examples of inequalities that follow from Hesselager's theorem but are not of the Hermite–Hadamard type.

References

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Rayene Menzer: *An alternative equation for polynomial functions on hyperbolas* (Joint work with Zoltán Boros)

In our presentation, we establish the following result:

Theorem. *Let m denote a non-zero real number. Suppose that $f, g\colon \mathbb{R} \to \mathbb{R}$ are generalized polynomials and $f(x)g(y) = 0$ for all solutions of the equation*

$$x^2 - my^2 = 1. \tag{1}$$

Then f or g is identically equal to zero.

This research is motivated by analogous investigations in [4] for one additive function f fulfilling $f(x)f(y)=0$ under such constraints, by our previous results in [2] and [3], when we considered one generalized monomial f or a generalized polynomial of degree two, for some rational number m, as well as by similar investigations by Z. Boros, W. Fechner in [1] for one real generalized polynomial in the particular case $m=-1$ (when our algebraic constraint describes the unit circle).

References

[1] Z. Boros and W. Fechner, *An alternative equation for polynomial functions*, Aequationes Math. **89** (2015), no 1, 17–22.
[2] Z. Boros and R. Menzer, *An alternative equation for generalized monomials*, Aequationes Math. **97** (2023), no. 1, 113–120.
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Gergő Nagy: *On the σ-balancing property of generalized quasi-arithmetic means of n variables* (Joint work with Tibor Kiss)

We say that a mean $M\colon I^2\to\mathbb{R}$ is balanced if

$$M(M(x,M(x,y)),M(M(x,y),y))=M(x,y)$$

holds for all $x,y\in I$. In the 1930s, Aumann proved that the balancing property characterizes quasi-arithmetic means among analytic or Cauchy means. In the talk, we are going to extend the above concept to the $n>2$ variable case, and investigate it in the class of generalized quasi-arithmetic means. Our related result is analogous to Aumann's statement.

References

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[2] T. Kiss, *On the balancing property of Matkowski means*, Aequationes Math. **95** (2021), no. 1, 75–89.
[3] T. Kiss and G. Nagy, *On the σ-balancing property of multivariate generalized quasi-arithmetic means*, Math. Inequal. Appl. **27** (2024), no. 4, 1009–1019.

Andrzej Olbryś: *On some version of separation theorem*

Let X be an abelian group, let $f,\ g\colon X\to\mathbb{R}$, $g\le f$ and $\omega\colon X\times X\to\mathbb{R}$ be given functions satisfying condition:

$$a_f(x,y)\le\omega(x,y)\le a_g(x,y),\quad x,y\in X,$$

where $a_k\colon X\times X\to\mathbb{R}$ stands for a Cauchy's difference of the function $k\colon X\to\mathbb{R}$, i.e.

$$a_k(x,y) := k(x+y) - k(x) - k(y).$$

In our talk, we discuss sufficient conditions that guarantee the existence of a function $h\colon X\to\mathbb{R}$ such that $g(x)\le h(x)\le f(x)$, $x\in X$, and

$$\omega(x,y) = a_h(x,y), \quad x,y\in X.$$

References

[1] Z. Gajda and Z. Kominek, *On separations theorems for subadditive and superadditive functionals*, Studia Math. **100** (1991), no. 1, 25–38.

[2] P. Kranz, *Additive functionals on abelian semigroup*, Comment. Math. Prace Mat. **16** (1972), 239–246.

Zsolt Páles: *On the equality of generalized Bajraktarević means under first-order differentiability assumptions* (Joint work with Amr Zakaria)

In the talk, we consider the equality problem of generalized Bajraktarević means, i.e., we are going to solve the functional equation

$$(*)\quad f^{-1}\left(\frac{p_1(x_1)f(x_1)+\cdots+p_n(x_n)f(x_n)}{p_1(x_1)+\cdots+p_n(x_n)}\right) = g^{-1}\left(\frac{q_1(x_1)g(x_1)+\cdots+q_n(x_n)g(x_n)}{q_1(x_1)+\cdots+q_n(x_n)}\right),$$

which holds for all $x=(x_1,\dots,x_n)\in I^n$, where $n\ge 2$, I is a nonempty open real interval, the unknown functions $f,g\colon I\to\mathbb{R}$ are strictly monotone, and the vector-valued weight functions $p=(p_1,\dots,p_n)\colon I\to\mathbb{R}_+^n$ and $q=(q_1,\dots,q_n)\colon I\to\mathbb{R}_+^n$ are also unknown. This equality problem in the symmetric two-variable case (i.e., when $n=2$ and $p_1=p_2$, $q_1=q_2$) was solved under sixth-order regularity assumptions by Losonczi [2]. In [3] the same conclusion was established assuming only first-order differentiability. In [1] the nonsymmetric case was considered. Assuming third-order differentiability of f, g and the first-order differentiability of at least three of the functions $p_1,\dots,p_n$, it was proved that (*) holds if and only if there exist four constants $a,b,c,d\in\mathbb{R}$ with $ad\neq bc$ such that

$$cf+d>0,\quad g=\frac{af+b}{cf+d},\quad \text{and}\quad q_\ell=(cf+d)p_\ell \quad (\ell\in\{1,\dots,n\}).$$

The goal is to verify the same conclusion under first-order differentiability conditions.

References

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Patryk Rela: *The Orlicz risk measure under uncertainty*

Let $(\Omega, \mathcal{F})$ is a measurable space and $\mu\colon \mathcal{F} \to [0,1]$ is a capacity, that is a monotone set function, satisfying $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$. The risk is represented by the $\mathcal{F}$-measurable function $X\colon \Omega \to [0,\infty)$ such that $\mu(\{X > t\}) = 0$ for some $t \in \mathbb{R}$. The Orlicz risk measure under uncertainty for the risk X is defined as the solution $\pi_{(\mu,\alpha,\Phi)}[X,x]$ of the equation

$$E_\mu\left[\Phi\left(\frac{(X-x)_+}{\pi_{(\mu,\alpha,\Phi)}[X,x]-x}\right)\right] = 1-\alpha, \tag{1}$$

where $\alpha \in [0,1)$ is a given parameter, $x \in \mathbb{R}$ is fixed real number and $\Phi\colon [0,\infty) \to [0,\infty)$ is a normalized Young function, that is a strictly increasing, convex function $\Phi\colon [0,\infty) \to [0,\infty)$ satisfying $\Phi(0) = 0$, $\Phi(1) = 1$ and $\lim_{z\to\infty} \Phi(z) = \infty$. E_μ is the Choquet integral with respect to the capacity μ, defined by

$$E_\mu[X] := \int_0^\infty \mu\left(\{X > k\}\right) dk.$$

The aim of this talk is to prove the existence and uniquenes of the Orlicz risk measure defined by (1) and to characterize its several important properties.

References

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Maciej Sablik: *Portfolio selection based on a fuzzy measure* (Joint work with Timothy Nadhomi and Chisom P. Okeke)

Effective portfolio risk management is a critical component for both investors and fund managers to achieve their financial objectives while navigating asset selection. This paper explores the application of non-additive fuzzy measures as an alternative to traditional risk metrics like standard deviation.

We briefly discuss the limitations of Modern Portfolio Theory (MPT) and its reliance on normal distribution assumptions. To address these limitations, we introduce non-additive fuzzy measures, which do not assume specific probability distributions. This approach accommodates imprecision and uncertainty in financial markets, providing a more comprehensive understanding of portfolio risk. By considering diversification and asset characteristic dependencies, non-additive fuzzy measures offer a promising avenue for more accurate risk analysis and informed investment decisions.

László Székelyhidi: *On the powers of maximal ideals in Fourier algebras*

The Fourier algebra of a locally compact abelian group plays a basic role in abstract harmonic analysis. In particular, the maximal ideals of this algebra are the corner stones of spectral analysis and synthesis. In this work we present some useful descriptions of the powers of maximal ideals in the Fourier algebra of $\mathbb{R}^n$. Our descriptions are related to the Weierstrass Preparation Theorem. One possible form of this theorem can be formulated in the following way:

Let n be a positive integer and m a nonnegative integer. Let z_0 be in $\mathbb{C}$, further let $p_0 = (p_{01}, p_{02}, \dots, p_{0n})$ be in $\mathbb{C}^n$. Assume that f is a complex valued analytic function in a neighborhood of (z_0, p_0), and z_0 is an m-multiple root of the equation $f(z, p_0) = 0$, that is,

$$\partial_1^k f(z_0, p_0) = 0, \quad k = 0, 1, \dots, m-1, \quad \partial_1^m f(z_0, p_0) \neq 0.$$

Then there exists a neighborhood of (z_0, p_0) in which f can be expressed in the form

$$f(z,p) = [(z-z_0)^m + a_{m-1}(p)(z-z_0)^{m-1} + a_{m-2}(p)(z-z_0)^{m-2} + \cdots + a_0(p)]b(z,p),$$

where $a_i(p_0) = 0$, the functions a_i, b $(i = 0, 1, \dots, m-1)$ are analytic and uniquely determined by f, further $b(z_0, p_0) \neq 0$.

Tomasz Szostok: *Inequalities involving the Choquet integral* (Joint work with Jacek Chudziak)

For an increasing function f, we write the Choquet integral

$$(C)\int_a^b f(x)d\mu(x) = \int_0^\infty \mu(\{s \in [a,b] : f(s) \geq x\})dx$$

(μ is here a capacity) in the form of a Stieltjes integral. Then, using this representation, we study some basic inequalities (like Hermite–Hadamard, Jensen or Ostrowski).

Norbert Tóth: *Testing non-solvability of linear programs* (Joint work with Mihály Bessenyei)

It may occur that only the solvability issue of a linear program counts in a practical application (and the solution itself, if it exists, does not). Motivated by this phenomena, we suggest an implementable method for testing non-solvability of linear programs.

Péter Tóth: *Generalized inverses of strictly monotone transformations of the plane*

Our talk concerns a problem proposed by Z. Páles during the *60th International Symposium on Functional Equations* [2]. Let $n \in \mathbb{N}$ and $K \subseteq \mathbb{R}^n$ be a nonempty, convex set. Suppose that the mapping $f\colon K \to \mathbb{R}^n$ is strictly monotone, i.e., it fulfills the inequality

$$\langle\, f(x) - f(y)\,,\, x - y\,\rangle > 0$$

for all $x, y \in K$ such that $x \neq y$, where $\langle\cdot\,,\cdot\rangle$ denotes the standard inner product on the Euclidean space $\mathbb{R}^n$. Observe that for $n = 1$ this is indeed equivalent to the strict monotonicity of f on an interval.

The question is: does f have a generalized left inverse function defined on $\operatorname{conv}(f(K))$ which is monotone? More precisely, does there exist a function $f^{(-1)}\colon \operatorname{conv}(f(K)) \to K$ such that $f^{(-1)}(f(x)) = x$ for all $x \in K$, moreover

$$\langle\, f^{(-1)}(u) - f^{(-1)}(v)\,,\, u - v\,\rangle \geq 0$$

for all $u, v \in \operatorname{conv}(f(K))$?

It is known (see [1]) that for $n = 1$ such a function $f^{(-1)}$ exists and it is unique. In our talk we consider the case $n = 2$. We are going to demonstrate that this generalized inverse does not exist for an arbitrary convex domain $K \subseteq \mathbb{R}^2$. However, we show that if K is closed, convex then $f^{(-1)}$ exists and it is uniquely determined. Our proof is based on a classical convex geometric theorem of Helly.

References

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2. Problems and remarks

Remark. The remark is connected with the inequalities of the Hermite–Hadamard type satisfied by higher-order convex functions.

Let $f\colon I \to \mathbb{R}$ be defined on an open interval $\emptyset \neq I \subseteq \mathbb{R}$. Observe that if we expand the differences of order $n+1$ for an n-convex function f and make an appropriate rearrangement then, we obtain for $x \in I, h > 0$ such that $x + h \in I$

$$\Delta_h f(x) \geq 0,$$
$$f(x+h) - f(x) \geq 0,$$

taking here $y = x + h$ we get

$$f(x) \leq f(y),$$

(with $x < y$) for 0-convex (i.e. non-decreasing) functions.

Similarly,

$$\Delta_h^2 f(x) \geq 0,$$
$$f(x+2h) - 2f(x+h) + f(x) \geq 0,$$
$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2},$$

for 1-convex (i.e. convex) functions and

$$\Delta_h^3 f(x) \geq 0,$$
$$f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x) \geq 0,$$
$$\frac{1}{4}f(x) + \frac{3}{4}f\left(\frac{x+2y}{3}\right) \leq \frac{3}{4}f\left(\frac{2x+y}{3}\right) + \frac{1}{4}f(y),$$

(where $x < y$) for 2-convex functions. As it is well known, the expressions occurring in the above inequalities may be separated by the integral mean of f, i.e., the inequalities:

$$f(x) \leq \frac{1}{y-x}\int_x^y f(t)\,dt \leq f(y),$$
$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(t)\,dt \leq \frac{f(x)+f(y)}{2},$$
$$\frac{1}{4}f(x) + \frac{3}{4}f\left(\frac{x+2y}{3}\right) \leq \frac{1}{y-x}\int_x^y f(t)\,dt \leq \frac{3}{4}f\left(\frac{2x+y}{3}\right) + \frac{1}{4}f(y)$$

are fulfilled for 0-convex, 1-convex and 2-convex functions, respectively.

However, if we carry out the same process for 3-convex functions and get

$$\Delta_h^4 f(x) \geq 0,$$
$$f(x+4h) - 4f(x+3h) + 6f(x+2h) - 4f(x+h) + f(x) \geq 0,$$
$$\frac{1}{2} f\left(\frac{3x+y}{4}\right) + \frac{1}{2} f\left(\frac{x+3y}{4}\right) \leq \frac{1}{8} f(x) + \frac{3}{4} f\left(\frac{x+y}{2}\right) + \frac{1}{8} f(y),$$

then it is not true that for all 3−convex functions f, the integral mean

$$\frac{1}{y-x} \int_x^y f(t)\,dt$$

is between the two values from the above inequality.

Thus, the open problem is to find appropriate integral expressions $K_f(x,y)$ such that

$$\frac{1}{2} f\left(\frac{3x+y}{4}\right) + \frac{1}{2} f\left(\frac{x+3y}{4}\right) \leq K_f(x,y) \leq \frac{1}{8} f(x) + \frac{3}{4} f\left(\frac{x+y}{2}\right) + \frac{1}{8} f(y)$$

would be fulfilled, for all $x, y \in I$, $x < y$, for any 3-convex function $f\colon I \to \mathbb{R}$. The analogous problem could be investigated for arbitrary n-convex functions ($n \geq 3$) as well.

Tomasz Szostok

Problem. Let $(S, \cdot)$ be a semigroup and let $(G, +)$ be an abelian group. Find a general solution $\omega\colon S \times S \longrightarrow G$ of the functional equation

$$\omega(x^2, y^2) - 2\omega(x,y) = \omega(xy, xy) - \omega(x,x) - \omega(y,y) \quad \forall\, x, y \in S.$$

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