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A NOTE ON BIFURCATION OF EQUILIBRIUM FORMS OF A GAS BALLOON PARACHUTE

Anita Zgorzelska, Hanna Guze

Abstract. We will be concerned with deformations of a free elastic top rim of a parachute of a gas balloon. The top rim is connected with the circular deflation port of the balloon envelope by heavy duty flexible load tapes. The inside part of the balloon is filled with compressed gas. Equilibrium forms of the parachute may be found as solutions of a certain nonlinear functional-differential equation with two physical parameters: an elasticity coefficient of tapes and a physical parameter describing compressed gas. This equation possesses radially symmetric solutions corresponding to circular shapes of the top rim. Our goal is to study the existence of symmetry breaking bifurcation of the top rim of parachute.

1. Mathematical model

Bifurcation theory is a powerful tool in studying deformations of elastic beams, shells or plates. Lots of works have been devoted to the study of bifurcation in elasticity theory, see for instance [1], [4], [5], [11], [12], [14] and references therein.

Our study was motivated by gas balloons. Precisely, we are interested in the behaviour of the part of a balloon that is called an envelope. Following the description in [8], the fabric of the envelope is flexible (elastic). It is composed of large vertical sections called gores. Each gore is made up of the same number of horizontal sections called panels. The panels and gores are held together by stitching and by heavy duty flexible load tapes which help support the weight of the balloon and minimize a strain on the fabric. The top part of the envelope consisting of one panel of each gore is named a parachute (see Fig. 1). The standard parachute possesses a circular deflation port – a crown

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ring that is closed off by a circular panel which is held sealed during a flight by a flexible hook-and-loop closure. Moreover, we can treat a parachute in a balloon envelope as a two-dimensional object, because its height is much smaller than the length of a top rim - a horizontal tape between the parachute and the rest of envelope. An equilibrium form of the gas balloon parachute is described in polar coordinates by a 2π- periodic C^{m+2}-smooth positive function $r(\theta)$, $m \in \mathbb{N} \cup \{0\}$.

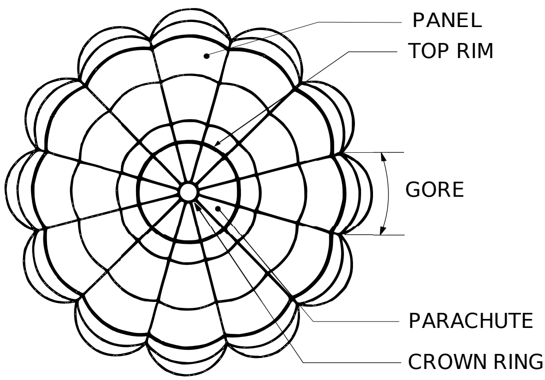

Figure 1. A balloon envelope: view from above

Let $C^m(2\pi)$, $m \in \mathbb{N} \cup \{0\}$, denote the Banach space of 2π-periodic C^m-smooth functions $r(\theta)$ with the standard norm

$$\|r\|_m = \sum_{k=0}^{m} \max_{\theta \in [0,2\pi]} |r^{(k)}(\theta)|,$$

where $r^{(k)}(\theta)$ denotes the k-th derivative of $r(\theta)$ and $r^{(0)}(\theta) = r(\theta)$. It is well known that $C^m(2\pi)$ is continuously embedded into the Hilbert space $L^2(2\pi)$ with the scalar product

$$\langle f, g \rangle = \int_0^{2\pi} f(\theta) g(\theta) d\theta.$$

The total energy of the parachute is given by:

$$E(r, \alpha, \beta) = \int_0^{2\pi} \left(\sqrt{r^2(\theta) + r'^2(\theta)} + \alpha r(\theta) \right) d\theta - \beta \ln S, \tag{1.1}$$

where $\alpha > 0$ is an elasticity coefficient of heavy duty load tapes, $\beta > 0$ is a physical parameter describing a compressed gas inside the gas balloon, and S denotes the area of parachute, i.e.

$$S = S(r) = \frac{1}{2} \int_0^{2\pi} r^2(\theta) d\theta.$$

Gas balloons are inflated with a gas of lower molecular weight than the ambient atmosphere. The most popular gas here is helium. Let us point out that a similar model was investigated by the second author and J. Janczewska in [7] and [8]. As a result of conversations with J. Janczewska, the formula for the energy functional has been improved (simplified). The component of the energy functional corresponding to the energy of compresses gas inside the gas balloon depends only on one parameter. However the main conclusions concerning deformations of the parachute are the same.

It can be easily calculated that the Fréchet derivative of the energy functional E with respect to the variable r is of the form:

$$\begin{aligned} E'_r(r,\alpha,\beta)h = &\int_0^{2\pi} \frac{r^3(\theta)+2r(\theta)r'^2(\theta)-r^2(\theta)r''(\theta)}{(r^2(\theta)+r'^2(\theta))^{3/2}} h(\theta)d\theta \\ &+ \int_0^{2\pi} \left(\alpha - \frac{\beta}{S} r(\theta)\right) h(\theta)d\theta, \end{aligned}$$

where $\alpha,\beta \in \mathbb{R}_+$, $r,h \in C^{m+2}(2\pi)$ and $r(\theta) > 0$ for $\theta \in [0,2\pi]$. Critical points of the energy functional $E(r,\alpha,\beta)$ are 2π-periodic C^{m+2}-smooth positive solutions of the equation

$$\frac{r^3(\theta)+2r(\theta)r'^2(\theta)-r^2(\theta)r''(\theta)}{(r^2(\theta)+r'^2(\theta))^{3/2}} + \alpha - \frac{\beta}{S} r(\theta) = 0. \tag{1.2}$$

We are interested in radially symmetric solutions of the equation (1.2). Substituting $r(\theta) \equiv r$ into (1.2), we get an algebraic equation

$$1 + \alpha - \frac{\beta}{r\pi} = 0$$

with a solution given by

$$R_\alpha = \frac{\beta}{\pi(1+\alpha)}, \tag{1.3}$$

which corresponds to a circular shape of the top rim of radius R_α. To sum up, for all $\beta \in \mathbb{R}_+$ there exists a family of radially symmetric solutions of the equation (1.2) given by

$$\Gamma^\beta = \{(R_\alpha,\alpha)\colon \alpha \in \mathbb{R}_+\},$$

where R_α is defined by (1.3).

2. Symmetry-breaking bifurcation problem

We now want to find all values of the parameter α for which the radially symmetric solution R_α loses its stability. For this purpose we will study bifurcation from the set of radial solutions with respect to α.

DEFINITION 2.1. $(R_\alpha, \alpha) \in \Gamma^\beta$ is called a *symmetry-breaking bifurcation point* of the equation (1.2) with respect to the set Γ^β if there exists a branch of non-radially symmetric solutions $(r(t), \alpha(t))$ of (1.2), depending on $|t| < \varepsilon$, with $r(0) = R_\alpha$ and $\alpha(0) = \alpha$.

Set

$$\alpha_k = k^2 - 1, \quad k \geq 1.$$

THEOREM 2.2. *For each $k \geq 2$, there exists a smooth family of non-radially symmetric solutions $(r(t), \alpha(t))$ of (1.2), defined for $|t| < \varepsilon$, satisfying*

$$r(t)(\theta) = R_{\alpha(t)} + t \cdot \frac{1}{\sqrt{\pi}} \cos(k\theta) + o(|t|).$$

In particular, at $t = 0$ we have $r(0) = R_{\alpha_k}$ and $\alpha(0) = \alpha_k$. This implies that $(R_{\alpha_k}, \alpha_k) \in \Gamma^\beta$ is a symmetry-breaking bifurcation point for the equation (1.2).

The proof of the above theorem relies on the Crandall-Rabinowitz theorem concerning simple bifurcation points (see [6]). We will apply the gradient (variational) version of this theorem, developed by A.Yu. Borisovich (see [2], [3]). To enhance clarity, let us state this theorem.

THEOREM 2.3. *Assume that H is a Hilbert space with a scalar product $(\cdot,\cdot)_H$. Let X and Y be Banach spaces continuously embedded in H. Suppose that a C^r-operator $F\colon X_\delta(0) \times \mathbb{R}_\delta(\alpha_0) \to Y$ and a C^{r+1}-functional $E\colon X_\delta(0) \times \mathbb{R}_\delta(\alpha_0) \to \mathbb{R}$, where $r \geq 2$, satisfy the following conditions:*
1. $F(0, \alpha) = 0$ *for* $\alpha \in \mathbb{R}_\delta(\alpha_0)$,
2. $\dim\ker F'_x(0, \alpha_0) = 1$, $F'_x(0, \alpha_0)e = 0$, $(e, e)_H = 1$,
3. $\operatorname{codim}\ \operatorname{im} F'_x(0, \alpha_0) = 1$,
4. $E'_x(x, \alpha)h = (F(x, \alpha), h)_H$ *for* $(x, \alpha) \in X_\delta(0) \times \mathbb{R}_\delta(\alpha_0)$ *and* $h \in X$,
5. $E'''_{xx\alpha}(0, \alpha_0)(e, e, 1) \neq 0$.

Then $(0, \alpha_0)$ is a bifurcation point of the equation

$$F(x, \alpha) = 0. \tag{2.1}$$

In fact, the solution set of this equation in a certain neighborhood of $(0, \alpha_0)$ consists of the curve $\Gamma_1 = \{(0, \alpha)\colon \alpha \in \mathbb{R}_\delta(\alpha_0)\}$ and a C^{r-2}-curve Γ_2, inter-

secting only at $(0,\alpha_0)$. *Moreover, if* $r \geq 3$, *the curve* Γ_2 *can be parametrized by a variable* t, $|t| \leq \varepsilon$, *as*

$$\Gamma_2 = \{(x(t),\alpha(t))\colon t \in \mathbb{R}_\varepsilon(0)\}, \quad \textit{where } x(0)=0,\ \alpha(0)=\alpha_0 \textit{ and } x'(0)=e.$$

Let us introduce the symbol $C_e^m(2\pi)$, $m \in \mathbb{N}\cup\{0\}$ be the subspace of $C^m(2\pi)$ of even functions. Set

$$X = C_e^m(2\pi) \quad \text{and} \quad Y = C^m(2\pi), \quad m \in \mathbb{N}\cup\{0\}.$$

Given any $\alpha_0 \in \mathbb{R}_+$ take $(R_{\alpha_0},\alpha_0) \in \Gamma^\beta$. Starting now, $X_\delta(0)$ and $(\mathbb{R}_+)_\delta(\alpha_0)$ denote balls of radius δ around 0 in X and $\alpha_0 \in \mathbb{R}_+$, correspondingly. For $\varrho \in X_\delta(0)$ and $\alpha \in (\mathbb{R}_+)_\delta(\alpha_0)$ define

$$r(\theta) = R_\alpha + \varrho(\theta). \tag{2.2}$$

Note that $r(\theta)$ represents a small perturbation in X from R_α given by (1.3). Substituting (2.2) in (1.1), we get the new energy functional $\hat{E}$ given by

$$\hat{E}(\varrho,\alpha,\beta) = \int_0^{2\pi} \Big(\sqrt{(R_\alpha+\varrho)^2+\varrho'^2} + \alpha(R_\alpha+\varrho)\Big)\, d\theta - \beta \ln \hat{S},$$

where $\varrho \in X_\delta(0)$, $\alpha \in (\mathbb{R}_+)_\delta(\alpha_0)$ and

$$\hat{S} = \hat{S}(\varrho,\alpha) = \frac{1}{2}\int_0^{2\pi} (R_\alpha+\varrho)^2 d\theta.$$

Furthermore, the Fréchet derivative of $\hat{E}$ with respect to ϱ is expressed by

$$\hat{E}'_\varrho(\varrho,\alpha,\beta)h = \int_0^{2\pi} \frac{(R_\alpha+\varrho)^3 + 2(R_\alpha+\varrho)\varrho'^2 - (R_\alpha+\varrho)^2\varrho''}{((R_\alpha+\varrho)^2+\varrho'^2)^{3/2}} h d\theta + \int_0^{2\pi} \Big(\alpha - \frac{\beta}{\hat{S}}(R_\alpha+\varrho)\Big) h d\theta.$$

Let us define the mapping $\hat{F} : X_\delta(0) \times (\mathbb{R}_+)_\delta(\alpha_0) \to Y$ by the formula

$$\hat{F}(\varrho,\alpha,\beta) = \frac{(R_\alpha+\varrho)^3 + 2(R_\alpha+\varrho)\varrho'^2 - (R_\alpha+\varrho)^2\varrho''}{((R_\alpha+\varrho)^2+\varrho'^2)^{3/2}} + \alpha - \frac{\beta}{\hat{S}}(R_\alpha+\varrho).$$

Of course $\hat{F}$ is smooth. It is easy to notice the following

LEMMA 2.4. *The mapping* $\hat{F}$ *is the variational gradient of* $\hat{E}$ *with respect to the inner product in* $L^2(2\pi)$, *i.e.,*

$$\hat{E}'_\varrho(\varrho,\alpha,\beta)h = \int_0^{2\pi} \hat{F}(\varrho,\alpha,\beta) h d\theta = \big\langle \hat{F}(\varrho,\alpha,\beta), h\big\rangle$$

fol all $\varrho \in X_\delta(0)$, $h \in X$ *and* $\alpha \in (\mathbb{R}_+)_\delta(\alpha_0)$.

Consider the equation of the form

$$\hat{F}(\varrho, \alpha, \beta) = 0. \tag{2.3}$$

The equation (2.3) has a trivial family of solutions

$$\hat{\Gamma}^{\beta} = \{(0, \alpha) \in X \times \mathbb{R}_+ : \alpha \in (\mathbb{R}_+)_\delta(\alpha_0)\}.$$

In order to prove the existence of a symmetry-breaking bifurcation branch of solutions of the equation (1.2) at (R_{α_k}, α_k), we will investigate bifurcation from the set of trivial solutions $\hat{\Gamma}^{\beta}$ of the equation (2.3).

LEMMA 2.5. *For each* $\alpha \in (\mathbb{R}_+)_\delta(\alpha_0)$, $\hat{F}'_\varrho(0, \alpha, \beta) \colon X \to Y$ *given by the formula*

$$\hat{F}'_\varrho(0, \alpha, \beta)h = -\frac{1}{R_\alpha}h'' - \beta\left(\frac{1}{\pi R_\alpha^2}h - \frac{1}{\pi^2 R_\alpha^2}\int_0^{2\pi} h\,d\theta\right) \tag{2.4}$$

is a Fredholm map of index 0.

The proof is similar to that in [8]. It is sufficient to show that $\hat{F}'_\varrho(0, \alpha, \beta)$ is the sum of a Fredholm map of index 0 and a completely continuous map. According to the implicit function theorem, a necessary condition for bifurcation from the trivial solutions of (2.3) at $(0, \alpha_0)$ is that $\dim\ker \hat{F}'_\varrho(0, \alpha_0, \beta) > 0$. To determine the critical values of the bifurcation parameter, we need to solve the equation

$$\hat{F}'_\varrho(0, \alpha, \beta)h = 0 \tag{2.5}$$

considering two additional constraints

$$\int_0^{2\pi} h(\theta)\cos\theta\, d\theta = 0 \tag{2.6}$$

and

$$\int_0^{2\pi} h(\theta)\, d\theta = 0. \tag{2.7}$$

The evenness of $h(\theta)$ and the condition (2.6) prevent any displacement of the mass center of the parachute. Additionally, conditions (2.6) and (2.7) rule out $h(\theta) = cos(\theta)$ and $h(\theta) = const \neq 0$. Consequently, assumption (2.7) leads to a loss of radial symmetry. Furthermore, assumption (2.7) reduces equation (2.5) to

$$-\frac{1}{R_\alpha}h'' - \beta\frac{1}{\pi R_\alpha^2}h = 0.$$

We choose the bifurcation mode $e_k(\theta) = \frac{1}{\sqrt{\pi}} \cos(k\theta)$ for $k \geq 2$. Using formula (1.3), we finally obtain

$$\alpha = \alpha_k = k^2 - 1, \quad k \geq 2.$$

Now, it is enough to show that

$$\hat{E}'''_{\varrho\varrho\alpha}(0, \alpha_k, \beta) e_k e_k \neq 0.$$

It follows from Lemma 2.4 that

$$\hat{E}''_{\varrho\varrho}(0, \alpha, \beta) hg = \left\langle \hat{F}'_{\varrho}(0, \alpha, \beta) h, g \right\rangle.$$

Applying (2.4) and substituting $h = g = e_k$, we obtain

$$\hat{E}''_{\varrho\varrho}(0, \alpha, \beta) e_k e_k = \frac{1}{R_\alpha} \left(k^2 - 1 - \alpha \right). \tag{2.8}$$

Finally, differentiating (2.8) with respect to α we have

$$\hat{E}'''_{\varrho\varrho\alpha}(0, \alpha, \beta) e_k e_k = \frac{\pi}{\beta} \left(k^2 - 2 - 2\alpha \right),$$

and thus

$$\hat{E}'''_{\varrho\varrho\alpha}(0, \alpha_k, \beta) e_k e_k = -\frac{\pi}{\beta} k^2 < 0, \quad k \geq 2.$$

It follows from Theorem 2.3 that $(0, \alpha_k)$ is a bifurcation point of (2.3) and the solution set of (2.3) in a neighbourhood of this point is the sum of $\hat{\Gamma}^\beta$ and a smooth curve $(\varrho(t), \alpha(t))$, $|t| < \varepsilon$, such that $\varrho(0) = 0$, $\alpha(0) = \alpha_k$ and

$$\varrho(\theta) = \frac{t}{\sqrt{\pi}} \cos(k\theta) + o(|t|),$$

which, together with (2.2), proves Theorem 2.2.

3. Subcritical behaviour of the parachute of gas balloon

In the previous section, using the Crandall–Rabinowitz theorem, we proved the existence of a family of non-radially symmetric solutions of the equation (1.2) at the point (R_{α_k}, α_k), parameterized by a real parameter $t \in (-\varepsilon, \varepsilon)$.

As the next step, our goal is to parameterize the non-radially symmetric branches of solutions of the equation (1.2) using the bifurcation parameter α. To achieve this, we will apply the Lyapunov–Schmidt finite-dimensional reduction and Sapronov's key function method. For the convenience of the reader, we state this theorem.

Assume that the assumptions of Theorem 2.3 are satisfied. Let us consider the equation

$$F(x,\alpha) + (\xi - (x,e)_H)e = 0, \tag{3.1}$$

where $x \in X_\delta(0)$, $\alpha \in \mathbb{R}_\delta(\alpha_0)$ and $\xi \in \mathbb{R}$. By the implicit function theorem there are open sets $S \subset \mathbb{R} \times \mathbb{R}_\delta(\alpha_0)$ and $U \subset X_\delta(0)$ and there exists a C^r-smooth function $\tilde{x} : S \to U$ such that the solution set of (3.1) in a neighbourhood of $(0,0,\alpha_0) \in X_\delta(0) \times \mathbb{R} \times \mathbb{R}_\delta(\alpha_0)$ is the graph of $\tilde{x}$ and $\tilde{x}(0,\alpha_0) = 0$.

Define $\Phi\colon S \to \mathbb{R}$ by

$$\Phi(\xi,\alpha) = -E(\tilde{x}(\xi,\alpha),\alpha) + \frac{1}{2}\left(\xi - (\tilde{x}(\xi,\alpha),e)_H\right)^2.$$

Φ is called the key function. It is known that $(0,\alpha_0) \in X \times \mathbb{R}$ is a bifurcation point of (2.1) if and only if $(0,\alpha_0) \in \mathbb{R} \times \mathbb{R}$ is a bifurcation point of the equation $\Phi_\xi(\xi,\alpha) = 0$ (see [9], Proposition 2.3).

Theorem 3.1 (The key function method, [13]). *Under the conditions stated above:*

(i) *If* $\Phi'''_{\xi\xi\alpha}(0,\alpha_0) \neq 0$ *then* $(0,\alpha_0) \in X \times \mathbb{R}$ *is a bifurcation point of the equation* (2.1) *and the solution set of* (2.1) *in a small neighbourhood of* $(0,\alpha_0)$ *is a union of two branches:* $\Gamma_1 = \{(0,\alpha)\colon \alpha \in \mathbb{R}_\delta(\alpha_0)\}$ *and a* C^{r-2}*-curve* Γ_2*, intersecting only at* $(0,\alpha_0)$.

(ii) *If* $\Phi'''_{\xi\xi\alpha}(0,\alpha_0) \neq 0$ *and* $\Phi'''_{\xi\xi\xi}(0,\alpha_0) \neq 0$ *then* $(0,\alpha_0) \in X \times \mathbb{R}$ *is said to be a transcritical bifurcation point of* (2.1)*, and the curve* Γ_2 *can be parametrized as follows:*

$$\Gamma_2\colon x(\alpha) = C(\alpha - \alpha_0)e + o(|\alpha - \alpha_0|),\ \alpha \in (\alpha_0 - \eta, \alpha_0 + \eta),$$

where

$$C = -2\frac{\Phi'''_{\xi\xi\alpha}(0,\alpha_0)}{\Phi'''_{\xi\xi\xi}(0,\alpha_0)}$$

and $0 < \eta \leq \delta$ *(see Fig.* 2(a)*).*

(iii) *Let*

$$\Phi'''_{\xi\xi\alpha}(0,\alpha_0) \neq 0,$$
$$\Phi'''_{\xi\xi\xi}(0,\alpha_0) = 0,$$
$$\Phi''''_{\xi\xi\xi\xi}(0,\alpha_0) \neq 0.$$

Set

$$D = -6\frac{\Phi'''_{\xi\xi\alpha}(0,\alpha_0)}{\Phi''''_{\xi\xi\xi\xi}(0,\alpha_0)}.$$

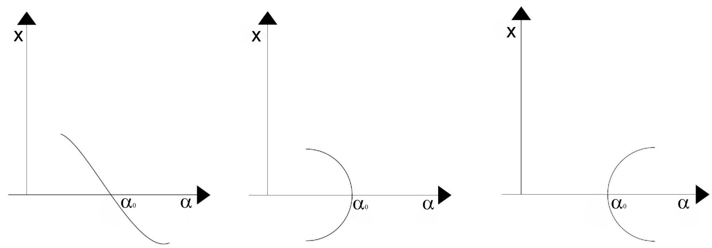

Figure 2. Transcritical, subcritical and postcritical bifurcation diagrams

If $D < 0$ *then* $(0, \alpha_0) \in X \times \mathbb{R}$ *is said to be a subcritical bifurcation point of* (2.1)*, and the curve* Γ_2 *can be parametrized as follows:*

$$\Gamma_2 : \; x^{\pm}(\alpha) = \pm\sqrt{|D|}(\alpha_0 - \alpha)^{\frac{1}{2}} e + o(|\alpha - \alpha_0|^{\frac{1}{2}}), \; \tau \in (\alpha_0 - \eta, \alpha_0],$$

where $0 < \eta \leq \delta$ *(see Fig.* 2(b)*).*

If $D > 0$ *then* $(0, \alpha_0) \in X \times \mathbb{R}$ *is said to be a postcritical bifurcation point of* (2.1)*, and the curve* Γ_2 *can be parametrized as follows:*

$$\Gamma_2 : \; x^{\pm}(\alpha) = \pm\sqrt{D}(\alpha - \alpha_0)^{\frac{1}{2}} e + o(|\alpha - \alpha_0|^{\frac{1}{2}}), \; \alpha \in [\alpha_0, \alpha_0 + \eta),$$

where $0 < \eta \leq \delta$ *(see Fig.* 2(c)*).*

Fact 3.2 (see [10]). *The first few terms of the Taylor series of the key function* $\Phi(\xi, \alpha)$ *at the point* $(0, \alpha_0)$ *are given by the following formulae:*

$$\begin{aligned}
\Phi(0, \alpha_0) &= -E(0, \alpha_0), \\
\Phi'_{\xi}(0, \alpha_0) &= 0, \\
\Phi''_{\xi\xi}(0, \alpha_0) &= 0, \\
\Phi^{(1+k)}_{\xi\alpha\ldots\alpha}(0, \alpha_0) &= 0 \quad \textit{for all } k = 1, 2, \ldots, \\
\Phi'''_{\xi\xi\alpha}(0, \alpha_0) &= -E'''_{xx\alpha}(0, \alpha_0)ee, \\
\Phi'''_{\xi\xi\xi}(0, \alpha_0) &= -E'''_{xxx}(0, \alpha_0)eee, \\
\Phi''''_{\xi\xi\xi\xi}(0, \alpha_0) &= -E''''_{xxxx}(0, \alpha_0)eeee - 3E'''_{xxx}(0, \alpha_0)eeh,
\end{aligned}$$

where h *is a unique solution of the equation*

$$F'_x(0, \alpha_0)h - (h, e)_H \, e = -F''_{xx}(0, \alpha_0)ee.$$

Theorem 3.3. *Let* $\alpha_k = k^2 - 1, \; k \geq 2$ *be a critical value of bifurcation parameter* $\alpha \in \mathbb{R}_+$*. Then* $(0, \alpha_k) \in X \times \mathbb{R}_+$ *is a subcritical bifurcation point of the equation* (2.3)*.*

PROOF. Fix $k \geq 2$. Consider the key function corresponding to the equation (2.3) in a neighbourhood of $(0, \alpha_k)$. According to Theorem 3.1 we have to show that $\Phi'''_{\xi\xi\xi}(0, \alpha_k) = 0$ and $\Phi'''_{\xi\xi\alpha}(0, \alpha_k) \cdot \Phi''''_{\xi\xi\xi\xi}(0, \alpha_k) > 0$. For this purpose we will use Fact 3.2. From the previous section, we have

$$\hat{E}'''_{\varrho\varrho\alpha}(0, \alpha_k, \beta)e_k e_k = -\frac{\pi}{\beta}k^2 < 0.$$

Using Fact 3.2 it follows that

$$\Phi'''_{\xi\xi\alpha}(0, \alpha_k) = \frac{\pi}{\beta}k^2 > 0.$$

A straightforward calculation gives

$$\hat{F}''_{\varrho\varrho}(0, \alpha_k, \beta)e_k e_k = \frac{k^2}{R^2_{\alpha_k}}\left(\frac{1}{\pi}\sin^2(k\theta) - \frac{2}{\pi}\cos^2(k\theta) + \frac{1}{\pi}\right).$$

According to Lemma 2.4, we obtain

$$\hat{E}'''_{\varrho\varrho\varrho}(0, \alpha_k, \beta)e_k e_k e_k = \left\langle \hat{F}''_{\varrho\varrho}(0, \alpha_k, \beta)e_k e_k, e_k\right\rangle = 0.$$

Combining this with Fact 3.2, we conclude that $\Phi'''_{\xi\xi\xi}(0, \alpha_k) = 0$. We check at once that

$$h_k(\theta) = \frac{1}{R_{\alpha_k}\pi}\cos^2(k\theta) - \frac{1}{R_{\alpha_k}\pi}$$

is a unique solution of the equation

$$\hat{F}'_{\varrho}(0, \alpha_k, \beta)h - \langle h, e_k\rangle e_k = -\hat{F}''_{\varrho\varrho}(0, \alpha_k, \beta)e_k e_k.$$

It follows that

$$\hat{E}'''_{\varrho\varrho\varrho}(0, \alpha_k, \beta)e_k e_k h_k = \left\langle \hat{F}''_{\varrho\varrho}(0, \alpha_k, \beta)e_k e_k, h_k\right\rangle = -\frac{7k^2}{4R^3_{\alpha_k}\pi}.$$

Finally, we obtain

$$\hat{E}''''_{\varrho\varrho\varrho\varrho}(0, \alpha_k, \beta)e_k e_k e_k e_k = -\frac{9k^4}{4R^3_{\alpha_k}\pi},$$

and as a result, using Fact 3.2, we get

$$\Phi''''_{\xi\xi\xi\xi}(0, \alpha_k) = \frac{k^2}{4R^3_{\alpha_k}\pi}(9k^2 + 21) > 0.$$

Hence

$$D = -6\frac{\Phi'''_{\xi\xi\alpha}(0, \alpha_k)}{\Phi''''_{\xi\xi\xi\xi}(0, \alpha_k)} = -\frac{8R^3_{\alpha_k}\pi^2}{\beta(3k^2 + 7)} < 0.$$

According to Theorem 3.1 we conclude that $(0, \alpha_k)$ is a subcritical bifurcation point of the equation (2.3). Furthermore, in a small neighbourhood of this point, the solution set consists of the trivial branch $\hat{\Gamma}^\beta$ and a C^∞-curve $\hat{\Gamma}_2^\beta$, which is parametrized as follows:

$$\hat{\Gamma}_2^\beta : \ \varrho^\pm(\alpha) = \pm\sqrt{|D|}(\alpha_k - \alpha)^{\frac{1}{2}} e_k + o(|\alpha - \alpha_k|^{\frac{1}{2}}) \quad \text{for } \alpha \in (\alpha_k - \eta, \alpha_k],$$

where $0 < \eta \leq \delta$. □

4. Graphical representation of the parachute of gas balloon in the neighbourhood of bifurcation points

We have shown that for the elasticity parameter values $\alpha_k = k^2 - 1$, $k \in \mathbb{N}$, $k \geq 2$, non-radially symmetric solutions $(r(t), \alpha(t))$ of our problem appear. Using the simplified formula

$$r(t)(\theta) \approx R_{\alpha_k} + t \cdot \frac{1}{\sqrt{\pi}} \cos(k\theta)$$

and the Mathematica software, we present below how the top rim of a parachute may behave for $k = 3, 4, 7, 8, 9, 12$.

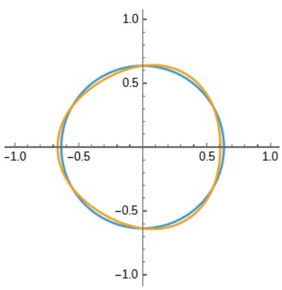

Figure 3. $k = 2$, $\beta = 18$, $t = -0.05$

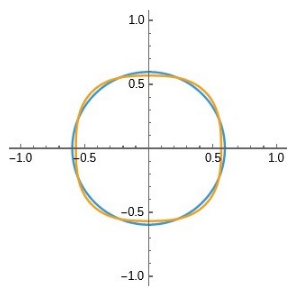

Figure 4. $k = 4$, $\beta = 30$, $t = -0.05$

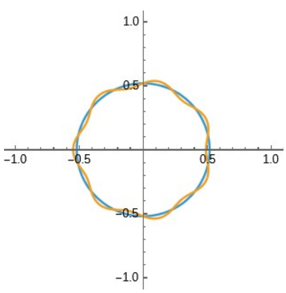

Figure 5. $k = 7$, $\beta = 80$, $t = -0.05$

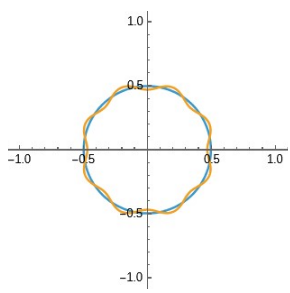

Figure 6. $k = 8$, $\beta = 100$, $t = -0.05$

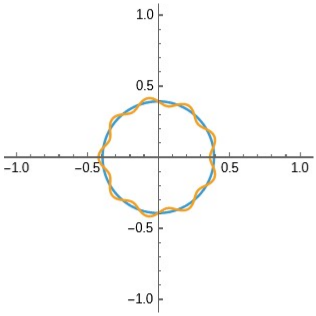

Figure 7. $k = 9$, $\beta = 100$, $t = -0.05$

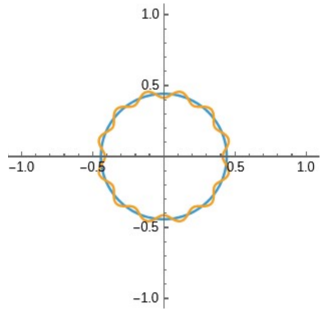

Figure 8. $k = 12$, $\beta = 200$, $t = -0.05$

References

[1] S.S. Antman, *Nonlinear Problems of Elasticity*, Appl. Math. Sci., 107, Springer-Verlag, New York, 1995.

[2] A.Yu. Borisovich, J. Dymkowska, and Cz. Szymczak, *Buckling and postcritical behaviour of the elastic infinite plate strip resting on linear elastic foundation*, J. Math. Anal. Appl. **307** (2005), no. 2, 480–495.

[3] A.Yu. Borisovich and J. Janczewska, *Stable and unstable bifurcation in the von Kármán problem for a circular plate*, Abstr. Appl. Anal. **2005**, no. 8, 889–899.

[4] I. Chueshow and I. Lasiecka, *Von Karman Evolution Equations. Well-posedness and Long-Time Dynamics*, Springer Monogr. Math., Springer, New York, 2010.

[5] P.G. Ciarlet, *Mathematical Elasticity. Volume III: Theory of Shells*, Stud. Math. Appl., 29, North-Holland Publishing Co., Amsterdam, 2000.

[6] M.G. Crandall and P.H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Functional Analysis **8** (1971), 321–340.

[7] H. Guze and J. Janczewska, *Symmetry-breaking bifurcation for free elastic shell of biological cluster, part 2*, Milan J. Math. **82** (2014), no. 2, 331–342.

[8] H. Guze and J. Janczewska, *Subcritical bifurcation of free elastic shell of biological cluster*, Nonlinear Anal. Real World Appl. **24** (2015), 61–72.

[9] J. Janczewska, *Local properties of the solution set of the operator equation in Banach spaces in a neighbourhood of a bifurcation point*, Cent. Eur. J. Math. **2** (2004), no. 4, 561–572.

[10] J. Janczewska, *Multiple bifurcation in the solution set of the von Kármán equations with S^1-symmetries*, Bull. Belg. Math. Soc. Simon Stevin **15** (2008), no. 1, 109–126.

[11] N.F. Morozov, *Selected Two-Dimensional Problems of Elasticity Theory* (in Russian), Univ. Press, Leningrad, 1978.

[12] J.N. Reddy, *Energy Principles and Variational Methods in Applied Mechanics*, John Wiley & Sons, Inc., Hoboken, New Jersey, 2002.

[13] Yu.I. Sapronov, *Branching of solutions of smooth Fredholm equations*, in: Yu.G. Borisovich (Ed.), *Equations on Manifolds* (in Russian), Novoe Global. Anal., Voronezh. Gos. Univ., Voronezh, 1982, pp. 60–82.

[14] I.I. Vorovich, *Mathematical Problems in the Nonlinear Theory of Shallow Shells* (in Russian), Nauka, Moscow, 1989.

Anita Zgorzelska
Institute of Applied Mathematics
Faculty of Applied Physics and Mathematics
Gdańsk University of Technology,
Narutowicza 11/12
80-233 Gdańsk
Poland
e-mail: anita.zgorzelska@pg.edu.pl

Hanna Guze
Mathematics Center
Gdańsk University of Technology
Narutowicza 11/12
80-233 Gdańsk
Poland
e-mail: hanna.guze@pg.edu.pl