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ON (k_1A_1, k_2A_2, k_3A_3)-EDGE COLOURINGS IN GRAPHS AND GENERALIZED JACOBSTHAL NUMBERS

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Abstract. In this paper we introduce a new kind of generalized Jacobsthal numbers in a distance sense. We give the identities and matrix representations for them and their connections with the Fibonacci and the Pell numbers. We also describe the interpretations of these numbers in terms of some kind of (k_1A_1, k_2A_2, k_3A_3)-edge colouring and quasi colouring.

1. Introduction and preliminary results

We use the standard definitions and notations of the graph theory, see [4] and [6]. Let us recall definitions of several very important sequences in the numbers theory. The Fibonacci sequence (F_n) is defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. The Pell sequence (P_n) is also defined by recurrence relation $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with the initial conditions $P_0 = 0$, $P_1 = 1$. The Jacobsthal sequence (J_n) is defined recursively as follows $J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ with the initial conditions $J_0 = 0$, $J_1 = 1$. The first few Fibonacci, Pell and Jacobsthal numbers are $0, 1, 1, 2, 3, 5, 8, \ldots,$ $0, 1, 2, 5, 12, 29, 70, \ldots$ and $0, 1, 1, 3, 5, 11, 21, \ldots,$ respectively.

In recent years many interesting generalizations of above mentioned classical sequences appeared in the mathematical literature. For example generalizations of the Pell numbers were introduced and studied in [8]–[10], and in [13], [15]–[18], [20]. Moreover in [16] generalized Jacobsthal numbers were considered.

In this paper we study a new kind of generalization of Jacobsthal numbers in the distance sense, i.e., generalization by the k-th order linear recurrence relation. This concept of generalization of Jacobsthal numbers is inspired by

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results given in [1]–[3], [5], [15], [17] and [19] where generalizations of Fibonacci numbers, Lucas numbers and Pell numbers were introduced.

For fixed integers $k \geq 1$ and $n \geq 0$ by $J^{(i)}(k,n)$ we denote the $(2,k)$-*distance Jacobsthal numbers of the i-th kind,* $i = 1,2$, i.e., numbers defined as follows

$$J^{(i)}(k,n) = J^{(i)}(k,n-k) + 2J^{(i)}(k,n-2) \quad \text{for} \quad n \geq k+1 \tag{1}$$

with initial conditions $J^{(i)}(k,0) = 0$, $J^{(i)}(k,1) = 1$ and for $2 \leq n \leq k$:

$$J^{(1)}(k,n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2^{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \end{cases} \qquad J^{(2)}(k,n) = 2^{\lfloor \frac{n-1}{2} \rfloor}.$$

In the following tables, we present several initial terms of the $(2,k)$-distance Jacobsthal sequences $J^{(i)}(k,n)$, $i = 1,2$, for special values of k and n.

Table 1. The $(2,k)$-distance Jacobsthal numbers of the first kind $J^{(1)}(k,n)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$J^{(1)}(1,n)$	0	1	1	3	5	11	21	43	85	171	341	683	1365	2731	5461	10923
$J^{(1)}(2,n)$	0	1	0	3	0	9	0	27	0	81	0	243	0	729	0	2187
$J^{(1)}(3,n)$	0	1	0	2	1	4	4	9	12	22	33	56	88	145	232	378
$J^{(1)}(4,n)$	0	1	0	2	0	5	0	12	0	29	0	70	0	169	0	408
$J^{(1)}(5,n)$	0	1	0	2	0	4	1	8	4	16	12	33	32	70	80	152
$J^{(1)}(6,n)$	0	1	0	2	0	4	0	9	0	20	0	44	0	97	0	214
$J^{(1)}(7,n)$	0	1	0	2	0	4	0	8	1	16	4	32	12	64	32	129

Table 2. The $(2,k)$-distance Jacobsthal numbers of the second kind $J^{(2)}(k,n)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$J^{(2)}(1,n)$	0	1	1	3	5	11	21	43	85	171	341	683	1365	2731	5461	10923
$J^{(2)}(2,n)$	0	1	1	3	3	9	9	27	27	81	81	243	243	729	729	2187
$J^{(2)}(3,n)$	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$J^{(2)}(4,n)$	0	1	1	2	2	5	5	12	12	29	29	70	70	169	169	408
$J^{(2)}(5,n)$	0	1	1	2	2	4	5	9	12	20	28	45	65	102	150	232
$J^{(2)}(6,n)$	0	1	1	2	2	4	4	9	9	20	20	44	44	97	97	214
$J^{(2)}(7,n)$	0	1	1	2	2	4	4	8	9	17	20	36	44	76	96	161

From the definition of the $(2,k)$-distance Jacobsthal numbers it follows immediately that $J^{(1)}(1,n) = J^{(2)}(1,n) = J_n$. The numbers $J^{(i)}(2,n)$, $i = 1,2$, are the elements of geometrical sequence, namely $J^{(1)}(2,n) = 3^{\frac{n-1}{2}}$ for odd n and $J^{(2)}(2,n) = 3^{\lfloor \frac{n-1}{2} \rfloor}$. Moreover if k is even and n is odd then $J^{(2)}(k,n) = J^{(2)}(k,n+1)$.

For $k = 3$ we have the following relationships between $J^{(1)}(k,n)$, $J^{(2)}(k,n)$ and the Fibonacci numbers F_n while for $k = 4$ we have their connections with the Pell numbers P_n.

THEOREM 1. *Let n be an integer. Then*

(i) $J^{(1)}(3,n) = F_{n-1} - (-1)^n$ *for* $n \geq 1$,

(ii) $J^{(2)}(3,n) = F_n$ *for* $n \geq 0$,

(iii) $J^{(1)}(4,2n-1) = P_n$ *for* $n \geq 1$,

(iv) $J^{(2)}(4,2n-1) = J^{(2)}(4,2n) = P_n$ *for* $n \geq 1$.

PROOF. In the proof of formula (i) we use the induction on n. For $n = 1$ or $n = 2$ the result follows immediately by the definitions of $J^{(1)}(k,n)$ and the Fibonacci numbers F_n.

Assume now that formula (i) is true for $t = 1, 2, \ldots, n$ with arbitrary $n \geq 2$. We will prove that $J^{(1)}(3,n+1) = F_n - (-1)^{n+1}$. By the induction hypothesis and the definitions of $J^{(1)}(k,n)$ and F_n, we have

$$\begin{aligned} J^{(1)}(3,n+1) &= J^{(1)}(3,n-2) + 2J^{(1)}(3,n-1) \\ &= F_{n-3} - (-1)^{n-2} + 2(F_{n-2} - (-1)^{n-1}) \\ &= 2F_{n-2} + F_{n-3} + (-1)^n = F_{n-2} + F_{n-1} + (-1^n) \\ &= F_n - (-1)^{n+1}, \end{aligned}$$

which ends the proof of (i). To prove (ii)–(iv) we can use the same method. □

The following theorem shows the relation between numbers $J^{(i)}(k,n)$ for $i = 1, 2$.

THEOREM 2. *Let $k \geq 2$, $n \geq 1$ be integers. Then*

(i) $J^{(2)}(k,n) = J^{(1)}(k,n) + J^{(1)}(k,n-1)$,

(ii) $J^{(1)}(k,n) = \sum_{i=0}^{n} (-1)^i J^{(2)}(k,n-i)$.

PROOF. In the proof of formula (i) we use the induction on n. Let $n = 1, 2, \ldots, k-1$. For $k = 2$ the result is obvious. If $k \geq 3$ then we have (i) from initial conditions for $J^{(1)}(k,n)$ and $J^{(2)}(k,n)$. Assume now that $n \geq k-1$ and that the equality (i) holds for all integers $k \leq t \leq n$. We shall prove that it is true for the integer $t = n+1$. Using (1) and the induction hypothesis, we obtain that

$$\begin{aligned} J^{(2)}(k,n+1) &= J^{(2)}(k,n+1-k) + 2J^{(2)}(k,n-1) \\ &= J^{(1)}(k,n+1-k) + J^{(1)}(k,n-k) + 2\left(J^{(1)}(k,n-1) + J^{(1)}(k,n-2)\right) \\ &= J^{(1)}(k,n+1-k) + 2J^{(1)}(k,n-1) + J^{(1)}(k,n-k) + 2J^{(1)}(k,n-2) \\ &= J^{(1)}(k,n+1) + J^{(1)}(k,n), \end{aligned}$$

which ends the proof of (i). The equality (ii) can be proved in the same manner. □

In this paper we give the graph interpretations of considered two types of the $(2,k)$-distance Jacobsthal numbers with respect to a special edge colouring of a graph. Next, using these interpretations, we obtain several identities and direct formulas for them. We also give matrix representations for $J^{(i)}(k,n)$, $i = 1,2$.

2. Graph interpretations

Let us consider an edge coloured graph G where the set of colours is $\{A_1, A_2, A_3\}$. For $\alpha \in \{A_1, A_2, A_3\}$ a subgraph of G will be said *α-monochromatic* if all its edges are coloured by colour α. We will write $\alpha(xy)$ if the edge xy of the graph has the colour α.

In [17], Trojnar-Spelina and Włoch defined so called (k_1A_1, k_2A_2, k_3A_3)-*edge colouring by monochromatic paths* in a graph G where $k_i \geq 1$, $i = 1,2,3$, are integers, as a generalization of the edge-colouring introduced and studied by Piejko and Włoch in [15]. Let us recall that the (k_1A_1, k_2A_2, k_3A_3)-edge colouring by monochromatic paths in a graph G is defined in such a way that every maximal, with respect to the set inclusion, A_i-monochromatic subgraph of G, can be partitioned into edge-disjoint paths of the length k_i, $i = 1,2,3$ (see [17]). Many interesting properties of (k_1A_1, k_2A_2, k_3A_3)-edge colouring by monochromatic paths were given in [17]. For example, it was proved that the number of $(kA_1, kA_2, 2A_3)$-edge colourings by monochromatic paths is closely related to $(2,k)$-distance Pell numbers of the i-th kind, $i = 1,2$, defined recursively as follows

$$P^{(i)}(k,n) = 2P^{(i)}(k,n-k) + P^{(i)}(k,n-2) \quad \text{for} \quad n \geq k$$

with initial conditions $P^{(i)}(k,0) = 0$, $P^{(i)}(k,1) = 1$ and for $2 \leq n \leq k-1$:

$$P^{(1)}(k,n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \qquad \text{and} \quad P^{(2)}(k,n) = 1.$$

In this section, we give graph interpretations for the $(2,k)$-distance Jacobsthal numbers $J^{(i)}(k,n)$ in terms of (k_1A_1, k_2A_2, k_3A_3)-edge colouring by monochromatic paths with $k_1 = k$, $k \geq 1$ and $k_2 = k_3 = 2$. We will use standard notation $\mathcal{P}_n$ for a path with n vertices.

Theorem 3. *Let $k \geq 1$, $n \geq 2$ be integers. The number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$ is equal to $J^{(1)}(k,n)$.*

Proof. Let $k \geq 1$, $n \geq 2$ be integers and let $V(\mathcal{P}_n) = \{x_1, x_2, \ldots, x_n\}$ with the numbering of vertices in the natural fashion. Then $E(\mathcal{P}_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$. The number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$ will be denoted by $\sigma(k,n)$. One can easy verify that $\sigma(k,n) = J^{(1)}(k,n)$ for $n = 2, \ldots, k+2$.

Assume now that $n \geq k+3$. Denote by $\sigma_{A_1}(k,n)$, $\sigma_{A_2}(k,n)$, $\sigma_{A_3}(k,n)$, the number of $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$ in which the colour of the last edge is determined respectively as follows: $A_1(x_{n-1}x_n)$, $A_2(x_{n-1}x_n)$, $A_3(x_{n-1}x_n)$. It is clear that $\sigma_{A_1}(k,n)$ is equal to the number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_{n-k}$ and $\sigma_{A_2}(k,n)$, $\sigma_{A_3}(k,n)$ are equal to the number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_{n-2}$. In other words $\sigma_{A_1}(k,n) = \sigma(k,n-k)$, $\sigma_{A_2}(k,n) = \sigma(k,n-2)$ and $\sigma_{A_3}(k,n) = \sigma(k,n-2)$. From the above we have

$$\sigma(k,n) = \sigma_{A_1}(k,n) + \sigma_{A_2}(k,n) + \sigma_{A_3}(k,n) = \sigma(k,n-k) + 2\sigma(k,n-2).$$

Taking into account the initial conditions we obtain that $\sigma(k,n) = J^{(1)}(k,n)$ for all $n \geq 2$. The proof of Theorem 3 is completed. □

Hence we obtain the following graph interpretation of Jacobsthal numbers.

Corollary 1. *Let $n \geq 2$ be an integer. The number of all $(A_1, 2A_2, 2A_3)$-edge colourings of the graph $\mathcal{P}_n$ is equal to J_n.*

Before we give the graph interpretation for the $(2,k)$-distance Jacobsthal numbers of the second kind $J^{(2)}(k,n)$ we introduce a *quasi* (k_1A_1, k_2A_2, k_3A_3)-*edge colouring by monochromatic paths* in a graph $\mathcal{P}_n$. Let $E(\mathcal{P}_n) = \{e_1, e_2, \ldots, e_{n-1}\}$ be numbered in the natural fashion. We say that the graph $\mathcal{P}_n$ is a quasi (k_1A_1, k_2A_2, k_3A_3)-edge coloured by monochromatic paths if the last r edges of $\mathcal{P}_n$ are uncoloured where $0 \leq r \leq t-1$, $t = \min\{k_1, k_2, k_3\}$ and the subpath $\mathcal{P}^* \subset \mathcal{P}_n$ induced by $\{e_1, e_2, \ldots, e_{n-r-1}\}$ is (k_1A_1, k_2A_2, k_3A_3)-edge coloured by monochromatic paths.

From above immediately follows that (k_1A_1, k_2A_2, k_3A_3)-edge coloured $\mathcal{P}_n$ is also quasi (k_1A_1, k_2A_2, k_3A_3)-edge coloured.

Using the same manner as that in the proof of Theorem 3 we can prove that the number of all quasi $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$ closely corresponds to the $(2,k)$-distance Jacobsthal numbers of the second kind $J^{(2)}(k,n)$. Namely we have

Theorem 4. *Let $k \geq 1$, $n \geq 2$ be integers. The number of all quasi $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$ is equal to $J^{(2)}(k,n)$.*

These graph interpretations will be used to derive the direct formulas for the $(2,k)$-distance Jacobsthal numbers $J^{(i)}(k,n)$, $i = 1, 2$.

Before that, we need to deduce certain preliminary results. Let $k \geq 1$, $n \geq 2$ and $0 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$ be integers. Let $p_k(n,t)$ denote the number of $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$ such that there are exactly t monochromatic paths of the length 2 coloured by A_2 or A_3. This means that in such edge colouring $2t_1$ edges have colour A_2, $2t_2$ edges have colour A_3 where $t_1 + t_2 = t$ and other $n - 1 - 2t$ edges have colour A_1.

THEOREM 5. *Let $n \geq 2$ and $0 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$ be integers. Then*

$$p_1(n,t) = \binom{n-1-t}{t} 2^t.$$

PROOF. Let us consider the graph $\mathcal{P}_n$ with the $(A_1, 2A_2, 2A_3)$-edge colouring by monochromatic paths in which there are exactly t monochromatic paths of the length 2 coloured by the colour A_2 or A_3. This edge colouring is related to a certain partition of the set $E(\mathcal{P}_n)$ into t edge-disjoint A_i monochromatic paths of the length 2 for $i = 2, 3$ and $n - 1 - 2t$ monochromatic paths of the length 1 for $i = 1$. We denote a number of all such partitions by $\delta(n,t)$. It is easy to observe that $\delta(n,t)$ equals to the number of all permutations with repetitions of $n - 1 - t$ elements that are equal to 1 or 2 where the number 1 is repeated $n - 1 - t$ times and the number 2 is repeated t times. Therefore, $\delta(n,t) = P_{n-1-t}^{t,n-1-2t}$, where the symbol $P_n^{n_1,n_2}$ denotes the number of all permutations with repetitions of the length n of two elements, where first of these elements is repeated n_1 times, the second is repeated n_2 times and $n_1 + n_2 = n$. Consequently we have

$$\delta(n,t) = \frac{(n-1-t)!}{t!(n-1-2t)!} = \binom{n-1-t}{t}.$$

Every of this partitions generates 2^t $(A_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths. Therefore, the desired result follows. □

THEOREM 6. *Let $k \geq 1$, $n \geq 2$ and $0 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$ be integers and let $\frac{n-1-2t}{k} \in N \cup \{0\}$. Then for $s = 0, 1, \ldots, n-1$ we have*

$$p_k(n,t) = p_{k-s}\Big(n - \frac{s(n-1-2t)}{k}, t\Big).$$

PROOF. Consider the $(kA_1, 2A_2, 2A_3)$-edge colouring by monochromatic paths of the graph $\mathcal{P}_n$ in which there are exactly t monochromatic paths of the length 2 coloured by the colour A_2 or A_3. Consequently, there are exactly $\frac{n-1-2t}{k}$ monochromatic paths of the length k coloured by A_1 in this edge colouring. If every of these paths is shorted to the A_1-monochromatic paths of the length $k - s$, then we obtain a $((k-s)A_1, 2A_2, 2A_3)$-edge colouring by monochromatic paths of the graph $\mathcal{P}_{n-s\frac{n-1-2t}{k}}$, in which there are $\frac{n-1-2t}{k}$ monochromatic paths of the length $k - s$ coloured by A_1 and the number of monochromatic paths of the length 2 coloured by A_i, $i = 2, 3$, remains the same as in the starting edge colouring. Therefore the proof is completed. $\square$

Putting $s = k-1$ in Theorem 6 we have $p_k(n,t) = p_1\big(n - \frac{(k-1)(n-1-2t)}{k}, t\big)$ and so using Theorem 5 we obtain the following

Corollary 2. *Let $k \geq 1$, $n \geq 2$ and $0 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$ be integers and let $\frac{n-1-2t}{k} \in N \cup \{0\}$. Then*

$$p_k(n,t) = \binom{\frac{1}{k}\left[n-1+t(k-2)\right]}{t} 2^t.$$

Let us consider again such $(kA_1, 2A_2, 2A_3)$-edge colouring by monochromatic paths of the graph $\mathcal{P}_n$, that there are exactly t monochromatic paths of the length 2 coloured by A_2 or A_3, i.e., in such edge colouring $2t_1$ edges have colour A_2, $2t_2$ edges have colour A_3 where $t_1 + t_2 = t$ and other $n-1-2t$ edges have colour A_1. We can observe that such $(kA_1, 2A_2, 2A_3)$-edge colouring by monochromatic paths of the graph $\mathcal{P}_n$ exists if and only if a number $\frac{n-1-2t}{k}$ is nonnegative integer. For given $k \geq 1$ and $n \geq 2$ we introduce the following notation:

$$I_k^n = \Big\{ t \in N \cup \{0\} : \frac{n-1-2t}{k} \in N \cup \{0\} \Big\}.$$

Note that for example: $I_{10}^3 = \{0, 3\}$, $I_{14}^4 = \emptyset$ and $I_1^n = \{0, 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ for all positive integers n. Moreover, if $I_k^n \neq \emptyset$ then $p_k(n,t) \neq 0$ for all $t \in I_k^n$ and otherwise $p_k(n,t) = 0$ for all $t \geq 0$.

Now we are able to present the following direct formula for the numbers $J^{(1)}(k,n)$.

Theorem 7. *Let $k \geq 1$, $n \geq 1$ and $0 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$ be integers. Then*

$$J^{(1)}(k,n) = \sum_{t \in I_k^n} \binom{\frac{1}{k}\left[n-1+t(k-2)\right]}{t} 2^t \quad \textit{if} \quad I_k^n \neq \emptyset$$

and

$$J^{(1)}(k,n) = 0 \quad \textit{if} \quad I_k^n = \emptyset.$$

Proof. Using Theorem 3 we have that $J^{(1)}(k,n)$ is equal to the number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$. Therefore for given $k \geq 1$, $n \geq 1$ it can be expressed as follows

$$J^{(1)}(k,n) = \sum_{t \in I_k^n} p_k(n,t).$$

Let us recall that if $I_k^n = \emptyset$ then $p_k(n,t) = 0$ for all $t \geq 0$ and therefore in this case the result follows. Otherwise, we have $p_k(n,t) \neq 0$ for all $t \in I_k^n$, so an application of Corollary 2 enables us to deduce the desired equality. □

Putting $k = 1$ in Theorem 7, we obtain the following well-known direct formula for Jacobsthal numbers.

COROLLARY 3. *Let $n \geq 1$ and $0 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$ be integers. Then*

$$J_n = \sum_{t=1}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-t}{t} 2^t.$$

To derive the direct formula for $J^{(2)}(k,n)$ we need new notations. Let $k \geq 1,\ n \geq 2$ and $0 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$ be integers. By $q_k(n,t)$ we denote the number of quasi $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$ such that exactly t monochromatic paths are coloured by A_2 or A_3. Let us note that in this edge colouring the number of A_1-monochromatic paths is equal to $\lfloor \frac{n-1-2t}{k} \rfloor$ and consequently $q_1(n,t) = p_1(n,t)$ so by Theorem 5 we have

$$q_1(n,t) = \binom{n-1-t}{t} 2^t. \tag{2}$$

THEOREM 8. *Let $k \geq 1$, $n \geq 2$ and $0 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$ be integers and $s \in N$. Then*

$$q_k(n,t) = q_s\Big(1 + 2t + \Big\lfloor \frac{n-1-2t}{k} \Big\rfloor s, t\Big).$$

PROOF. Let us consider the quasi $(kA_1, 2A_2, 2A_3)$-edge colouring by monochromatic paths of the graph $\mathcal{P}_n$ in which exactly t monochromatic paths of the length 2 have colour A_2 or A_3. Therefore there are $\left\lfloor \frac{n-1-2t}{k} \right\rfloor$ A_1-monochromatic paths of the length k in this edge-colouring. If, for a given positive integer n, every of these paths is replaced by A_1-monochromatic paths of the length s, then we obtain a quasi $(sA_1, 2A_2, 2A_3)$-edge colouring by monochromatic paths of the graph $\mathcal{P}_{\lfloor \frac{n-1-2t}{k} \rfloor s}$ in which the number of monochromatic paths of the length 2 coloured by A_i, $i = 2, 3$ remains the same as in the starting edge colouring. The proof is completed. □

Putting $s = 1$ in Theorem 8 and using the relation (2) we obtain the following result

COROLLARY 4. *Let $k \geq 1$, $n \geq 2$ and $0 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$ be integers. Then*

$$q_k(n,t) = \binom{t + \lfloor \frac{n-1-2t}{k} \rfloor}{t} 2^t.$$

Let us note that quasi $(kA_1, 2A_2, 2A_3)$-edge colouring by monochromatic paths of the graph $\mathcal{P}_n$, such that exactly t monochromatic paths are coloured by A_2 or A_3, exists if and only if one of the numbers $\frac{n-1-2t}{k}$ or $\frac{n-2-2t}{k}$ is nonnegative integer. In order to deduce a direct formula for the $(2,k)$-distance Jacobsthal numbers of the second kind we introduce the following new notation:

$$T_k^n = \Big\{t \in N \cup \{0\} : \frac{n-1-2t}{k} \in N \cup \{0\} \text{ or } \frac{n-2-2t}{k} \in N \cup \{0\}\Big\}.$$

Observe that for example $T_7^{20} = \{2, 6, 9\}$ and $T_1^n = T_2^n = \{0, 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ for all positive integers n. Moreover, $T_k^n \neq \emptyset$ for all positive integers k and $n \geq 2$.

Now we can state the analogue of Theorem 7 for the numbers $J^{(2)}(k,n)$.

THEOREM 9. *Let $k \geq 1$, $n \geq 2$ and $0 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$ be integers. Then*

$$J^{(2)}(k,n) = \sum_{t \in T_k^n} \binom{t + \lfloor \frac{n-1-2t}{k} \rfloor}{t} 2^t.$$

PROOF. As an immediate consequence of the graph interpretation of the numbers $J^{(2)}(k,n)$, given in Theorem 3, we obtain the following equality for $k \geq 1$ and $n \geq 2$:

$$J^{(2)}(k,n) = \sum_{t \in T_k^n} q_k(k,n).$$

Corollary 4 makes it obvious that the desired formula holds. □

3. Identities

In this part we give the identities for the $(2,k)$-distance Jacobsthal numbers. For special values of parameters we can reduce them to the well-known identities for the classical Jacobsthal numbers.

THEOREM 10. *Let $k \geq 1$, $m \geq 1$, $n \geq 0$ be integers and let $n + k - 2 \geq 0$. Then for $i = 1, 2$ we have*

(i) $J^{(i)}(k,n) + 2\sum_{i=1}^{m} J^{(i)}(k, n-2+ik) = J^{(i)}(k, n+mk)$,

(ii) $2^m J^{(i)}(k,n) + \sum_{i=1}^{m} 2^{m-i} J^{(i)}(k, n-k+2i) = J^{(i)}(k, n+2m)$.

PROOF. In the proof of the part (i) we use the induction on m. If $m = 1$ then the equation is obvious. Assume that formula in (i) is true for an arbitrary $m \geq 1$. We will prove that

$$J^{(i)}(k,n) + 2\sum_{i=1}^{m+1} J^{(i)}(k, n-2+ik) = J^{(i)}(k, n+(m+1)k).$$

By the induction hypothesis and the definition of $J^{(i)}(k,n)$, we have

$$\begin{aligned}
&J^{(i)}(k,n) + 2\sum_{i=1}^{m+1} J^{(i)}(k, n-2+ik)\\
&\quad = J^{(i)}(k,n) + 2\sum_{i=1}^{m} J^{(i)}(k, n-2+ik) + 2J^{(i)}(k, n-2+(m+1)k)\\
&\quad = J^{(i)}(k, n+mk) + 2J^{(i)}(k, n-2+(m+1)k) = J^{(i)}(k, n+(m+1)k),
\end{aligned}$$

which ends the proof of (i). The identity (ii) can be proved by the same method. □

Putting $n = 1$ in identity (i) of Theorem 10 we obtain

COROLLARY 5. *Let k, n be positive integers and let $i = 1, 2$. Then*

$$1 + 2\sum_{i=1}^{m} J^{(i)}(k, ik-1) = J^{(i)}(k, mk+1).$$

For $k = 1$ we obtain the well-known identity for the classical Jacobsthal numbers:

$$1 + 2\sum_{i=1}^{m} J_{i-1} = J_{m+1}.$$

Putting $k = 1$ and $n = 0$ or $n = 1$ respectively in Theorem 10, we obtain the well-known identities for the classical Jacobsthal numbers

$$\sum_{i=1}^{m} 2^{m-i} J_{2i-1} = J_{2m}, \quad 2^m + \sum_{i=1}^{m} 2^{m-i} J_{2i} = J_{2m+1}.$$

We can use the $(kA_1, 2A_2, 2A_3)$-edge colouring of the path $\mathcal{P}_n$ to obtain the following identity for the $(2,k)$-distance Jacobsthal numbers of the first kind.

THEOREM 11. *Let $k \geq 1$, $m \geq k$ and $n \geq k$ be integers. Then*

$$\begin{aligned}
J^{(1)}(k, m+n) &= 2J^{(1)}(k,m)J^{(1)}(k,n-1) + 2J^{(1)}(k,m-1)J^{(1)}(k,n)\\
&\quad + \sum_{i=1}^{k} J^{(1)}(k, m+1-i)J^{(1)}(k, n-k+i).
\end{aligned}$$

Proof. Consider the $(kA_1, 2A_2, 2A_3)$-edge colouring by monochromatic paths of the graph $\mathcal{P}_{m+n}$. Let

$$V(\mathcal{P}_{m+n}) = \{x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_{m+n}\}$$

be the set of vertices of this graph with the numbering in the natural fashion. Then

$$E(\mathcal{P}_{m+n}) = \{x_1x_2, x_2x_3, \dots, x_mx_{m+1}, \dots, x_{m+n-1}x_{m+n}\}.$$

By $\rho(k, m+n)$ we denote the number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_{m+n}$. Let $\rho_{A_1}(k, m+n)$, $\rho_{A_2}(k, m+n)$ and $\rho_{A_3}(k, m+n)$ denote the number of $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_{m+n}$, with $A_1(x_mx_{m+1})$, $A_2(x_mx_{m+1})$ and $A_3(x_mx_{m+1})$, respectively. In the case $A_1(x_mx_{m+1})$ we have k possibilities of the position in the graph $\mathcal{P}_{m+n}$ of the A_1-monochromatic path that includes an edge x_mx_{m+1}. It may be each of the following paths

$$x_{m+1-i}x_{m+2-i}\dots x_{m+k+1-i}, \quad i = 0, 1, \dots k.$$

For a fixed $i = 0, 1, \dots k$, the number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_{m+n}$ equals to the product of the number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_{m+1-i}$ and the number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_{n-k+i}$. Therefore $\rho_{A_1}(k, m+n)$ is equal to the sum of this products. In both cases $A_2(x_mx_{m+1})$ or $A_3(x_mx_{m+1})$ there are two possibilities of the position in the graph $\mathcal{P}_{m+n}$ of the A_i-monochromatic path, $i = 1, 2$, that includes an edge x_mx_{m+1}. It can be any of the paths: $x_{m-1}x_mx_{m+1}$ or $x_mx_{m+1}x_{m+2}$. Consequently $\rho_{A_2}(k, m+n)$ and $\rho_{A_3}(k, m+n)$ both are equal to the number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_m$ multiplied by the number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_{n-1}$ plus the number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_{m-1}$ multiplied by the number of all $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of the graph $\mathcal{P}_n$.

Thus from Theorem 3 we have

$$\begin{aligned}\rho_{A_2}(k, m+n) &= \rho_{A_3}(k, m+n)\\ &= J^{(1)}(k, m)J^{(1)}(k, n-1) + J^{(1)}(k, m-1)J^{(1)}(k, n)\end{aligned}$$

and

$$\rho_{A_1}(k, m+n) = \sum_{i=1}^{k} J^{(1)}(k, m+1-i)J^{(1)}(k, n-k+i).$$

Since $\rho(k,m+n)=J^{(1)}(k,m+n)$ and

$$\rho(k,m+n)=\rho_{A_1}(k,m+n)+\rho_{A_2}(k,m+n)+\rho_{A_3}(k,m+n),$$

then the assertion follows. □

Putting $k=1$ in Theorem 11 we obtain the well-known formula for the classical Jacobsthal numbers $J_{m+n}=2J_mJ_{n-1}+2J_{m-1}J_n+J_mJ_n=J_mJ_{n+1}+2J_{m-1}J_n$.

The following identity for the $(2,k)$-distance Jacobsthal numbers of the second kind can be deduced by Theorem 2 and Theorem 11.

Corollary 6. *Let $k\geq 1$, $m\geq k$ and $n\geq k$ be integers. Then*

$$\begin{aligned}J^{(2)}(k,m+n)&=2\Big(\sum_{j=0}^{m-1}(-1)^jJ^{(2)}(k,m-1-j)\Big)J^{(2)}(k,n)\\&\quad+2\Big(\sum_{j=0}^{m}(-1)^jJ^{(2)}(k,m-j)\Big)J^{(2)}(k,n-1)\\&\quad+\sum_{i=1}^{k}\Big(\sum_{j=0}^{m+1-i}(-1)^jJ^{(2)}(k,m+1-i-j)\Big)J^{(2)}(k,n-k+i).\end{aligned}$$

4. Matrix representations

Matrix representations of sequences give the possibility of deducing some properties of the terms of these sequences. For matrix generators of the Fibonacci numbers and the like, see [7] and [11].

In this section, we give matrix representations for the $(2,k)$-distance Jacobsthal numbers. Using these representations we obtain among other things Cassini-like formulas and some interesting identities for these numbers.

At the beginning, basing on the method used in [3], we introduce the matrix generator M_k for the numbers $J^{(i)}(k,n)$, $i=1,2$ where $k\geq 2$. Let us recall the recurrence relation for these numbers:

$$J^{(i)}(k,n)=J^{(i)}(k,n-k)+2J^{(i)}(k,n-2)\quad\text{for}\quad n\geq k+1.$$

Let for a positive integer $k\geq 2$, M_k be the matrix of the form $[m_{lj}]_{k\times k}$ where for a fixed $1\leq j\leq k$, a number m_{1j} is the coefficient of $J^{(i)}(k,n-j)$ in the above recurrence formula. Moreover, for $2\leq s\leq k$ we have

$$m_{sj}=\begin{cases}1 & \text{if } j=l-1,\\ 0 & \text{otherwise.}\end{cases}$$

According to this definition we obtain the following matrices for $k = 2, 3, 4, \dots$

$$M_2 = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \dots,$$

and in general

$$M_k = \begin{bmatrix} 0 & 2 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Now, for a fixed integer $k \geq 2$, we introduce the matrix $A_k^{(i)}$ of initial conditions. It is the following matrix of order k:

$$A_k^{(i)} = \begin{bmatrix} J^{(i)}(k, 2k-2) & J^{(i)}(k, 2k-3) & \dots & J^{(i)}(k, k) & J^{(i)}(k, k-1) \\ J^{(i)}(k, 2k-3) & J^{(i)}(k, 2k-4) & \dots & J^{(i)}(k, k-1) & J^{(i)}(k, k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ J^{(i)}(k, k) & J^{(i)}(k, k-1) & \dots & J^{(i)}(k, 2) & J^{(i)}(k, 1) \\ J^{(i)}(k, k-1) & J^{(i)}(k, k-2) & \dots & J^{(i)}(k, 1) & J^{(i)}(k, 0) \end{bmatrix},$$

where $i = 1, 2$, $k \geq 2$.

Using the same method as in [3], we obtain the following result.

THEOREM 12. *Let $k \geq 2$, $n \geq 1$ be integers. Then for $i = 1, 2$ we have*

$$(M_k)^n \cdot A_k^{(i)}$$
$$= \begin{bmatrix} J^{(i)}(k, n+2k-2) & J^{(i)}(k, n+2k-3) & \dots & J^{(i)}(k, n+k) & J^{(i)}(k, n+k-1) \\ J^{(i)}(k, n+2k-3) & J^{(i)}(k, n+2k-4) & \dots & J^{(i)}(k, n+k-1) & J^{(i)}(k, n+k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ J^{(i)}(k, n+k) & J^{(i)}(k, n+k-1) & \dots & J^{(i)}(k, n+2) & J^{(i)}(k, n+1) \\ J^{(i)}(k, n+k-1) & J^{(i)}(k, n+k-2) & \dots & J^{(i)}(k, n+1) & J^{(i)}(k, n) \end{bmatrix}.$$

Two next theorems will be helpful in formulating Cassini-like formulas for the $(2, k)$-distance Jacobsthal numbers $J^{(i)}(k, n)$.

THEOREM 13. *For all integers $k \geq 2$ the following equality holds*

$$\det M_k = \begin{cases} -3 & \text{for } k = 2, \\ (-1)^{k+1} & \text{for } k \geq 3. \end{cases}$$

Proof. It is easy to see that $\det M_2 = \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} = -3$. For $k \geq 3$ we calculate the determinant $|M_k|$ using the Laplace expansion by the last column of the matrix M_k. This expansion gives

$$\det M_k = \begin{vmatrix} 0 & 2 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{vmatrix} = (-1)^{k+1} \cdot \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & 0 \dots & 1 \end{vmatrix} = (-1)^{k+1}.$$

Thus the theorem is proved. □

Theorem 14. *Let $k \geq 2$ be an integer. Then*

$$\det A_k^{(1)} = (-1)^{\frac{k(k-1)}{2}} \tag{3}$$

and

$$\det A_k^{(2)} = \begin{cases} -1 & \text{for } k = 2, \\ 0 & \text{for odd } k, \\ 2(-1)^{\frac{1}{2}k} & \text{for even } k > 2. \end{cases} \tag{4}$$

Proof. Let $k \geq 2$ be an integer. In the proof of equality (3) the auxiliary sequence $J_B^{(1)}(k,n)$ will be very helpful. We define it as follows

$$J_B^{(1)}(k,0) = J_B^{(1)}(k,1) = \dots = J_B^{(1)}(k,k-2) = 0, \quad J_B^{(1)}(k,k-1) = 1$$

and

$$J_B^{(1)}(k,n) = J_B^{(1)}(k,n-2) + 2J_B^{(1)}(k,n-k) \quad \text{for} \quad n \geq k. \tag{5}$$

Now we define a matrix $B_k^{(1)}$ of order k, $k \geq 2$, whose elements are the terms of the sequence $J_B^{(1)}(k,n)$:

$$B_k^{(1)} = \begin{bmatrix} J_B^{(1)}(k,2k-2) & J_B^{(1)}(k,2k-3) & \dots & J_B^{(1)}(k,k) & J_B^{(1)}(k,k-1) \\ J_B^{(1)}(k,2k-3) & J_B^{(1)}(k,2k-4) & \dots & J_B^{(1)}(k,k-1) & J_B^{(1)}(k,k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ J_B^{(1)}(k,k) & J_B^{(1)}(k,k-1) & \dots & J_B^{(1)}(k,2) & J_B^{(1)}(k,1) \\ J_B^{(1)}(k,k-1) & J_B^{(1)}(k,k-2) & \dots & J_B^{(1)}(k,1) & J_B^{(1)}(k,0) \end{bmatrix}.$$

From the definition of the sequence $J_B^{(1)}(k,n)$ it follows that

$$B_k^{(1)} = \begin{bmatrix} J_B^{(1)}(k,2k-2) & J_B^{(1)}(k,2k-3) & \dots & J_B^{(1)}(k,k) & 1 \\ J_B^{(1)}(k,2k-3) & J_B^{(1)}(k,2k-4) & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ J_B^{(1)}(k,k) & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Using $k-1$ times the Laplace expansion by the last column we can calculate the determinant of the matrix $B_k^{(1)}$ as follows

$$\begin{aligned} \det B_k^{(1)} &= (-1)^{k+1} \cdot (-1)^k \cdot \dots \cdot (-1)^3 \\ &= (-1)^{k-1} \cdot (-1)^{k-2} \cdot \dots \cdot (-1) \\ &= (-1)^{1+2+\dots+(k-1)} = (-1)^{\frac{k(k-1)}{2}}. \end{aligned}$$

Using the recurrence formula (5) and definitions of matrices $A_k^{(1)}$ and $B_k^{(1)}$ one can prove that for $k \geq 2$ the equality $A_k^{(1)} = B_k^{(1)} \left(M_k^T\right)^{k-2}$ holds where M_k^T denotes the transpose of the matrix M_k. Therefore by properties of determinants we obtain

$$\det A_k^{(1)} = \det B_k^{(1)} \cdot \left(\det \left(M_k^T\right)\right)^{k-2}.$$

Consequently, for $k = 2$ we have $\det A_k^{(1)} = -1$ and for $k \geq 2$ by Theorem 13 we get

$$\det A_k^{(1)} = (-1)^{\frac{k(k-1)}{2}} \cdot (-1)^{(k+1)(k-2)}.$$

Note that the expression $(k+1)(k-2)$ is even for all integers $k \geq 3$, hence

$$\det A_k^{(1)} = (-1)^{\frac{k(k-1)}{2}}$$

which completes the proof of (3). For the proof of the equalities (4) we define a new sequence $J_B^{(2)}(k,n)$:

$$J_B^{(2)}(k,0) = J_B^{(2)}(k,k-1) = 1,$$

$$J_B^{(2)}(k,1) = J_B^{(2)}(k,2) = \dots = J_B^{(2)}(k,k-2) = 0,$$

(6) $$J_B^{(2)}(k,n) = J_B^{(2)}(k,n-k) + 2J_B^{(2)}(k,n-2) \quad \text{for} \quad k > 2,\ n \geq k,$$

and an auxiliary matrix $B_k^{(2)}$ of order k:

$$B_k^{(2)} = \begin{bmatrix} J_B^{(2)}(k,2k-2) & J_B^{(2)}(k,2k-3) & \dots & J_B^{(2)}(k,k) & J_B^{(2)}(k,k-1) \\ J_B^{(2)}(k,2k-3) & J_B^{(2)}(k,2k-4) & \dots & J_B^{(2)}(k,k-1) & J_B^{(2)}(k,k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ J_B^{(2)}(k,k) & J_B^{(2)}(k,k-1) & \dots & J_B^{(2)}(k,2) & J_B^{(2)}(k,1) \\ J_B^{(2)}(k,k-1) & J_B^{(2)}(k,k-2) & \dots & J_B^{(2)}(k,1) & J_B^{(2)}(k,0) \end{bmatrix}$$

$$= \begin{bmatrix} J_B^{(2)}(k,2k-2) & J_B^{(2)}(k,2k-3) & \dots & J_B^{(2)}(k,k) & 1 \\ J_B^{(2)}(k,2k-3) & J_B^{(2)}(k,2k-4) & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ J_B^{(2)}(k,k) & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

By definitions of matrices $A_k^{(2)}$ and $B_k^{(2)}$ and by the recurrence formula (6) we can deduce the following relationship between $A_k^{(2)}$ and $B_k^{(2)}$

$$A_k^{(2)} = B_k^{(2)} \left(M_k^T\right)^{k-2}. \tag{7}$$

Using basic properties of determinants one can prove that

$$\det B_k^{(2)} = \begin{cases} 0 & \text{for odd } k, \\ 2(-1)^{\frac{1}{2}k} & \text{for even } k. \end{cases}$$

From this and from the formula (7) it follows immediately that $\det A_k^{(2)} = 0$ for odd k. For even $k > 2$ by applying Theorem 13 we get

$$\det A_k^{(2)} = 2(-1)^{\frac{1}{2}k}(-1)^{(k+1)(k-2)} = 2(-1)^{\frac{1}{2}k}.$$

Moreover we can see that for $k = 2$ we have $\det A_k^{(2)} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = -1$, thus the proof is completed. □

As a consequence of Theorem 13 and Theorem 14 we obtain Cassini-like formulas for the $(2,k)$-distance Jacobsthal numbers.

Corollary 7. *Let $k \geq 2$, $n \geq 2$ be integers and let $i = 1, 2$. Then*

(i) $\det\left[(M_k)^n \cdot A_k^{(1)}\right] = \begin{cases} (-1)^{n+1}3^n & \textit{for } k = 2, \\ (-1)^{n(k+1)+\frac{k(k-1)}{2}} & \textit{for } k \geq 3, \end{cases}$

(ii) $\det\left[(M_k)^n \cdot A_k^{(2)}\right] = \begin{cases} -(-3)^n & \textit{for } k = 2, \\ 0 & \textit{for odd } k, \\ 2(-1)^{\frac{3}{2}k+1} & \textit{for even } k > 2. \end{cases}$

THEOREM 15. *Let $k \geq 3$ and $n \geq 2k-4$ be integers. Then $(M_k)^n$ is of the form*

$$\begin{bmatrix} J^{(1)}(k,n+1) & J^{(1)}(k,n+2) & J^{(1)}(k,n-k+3) \dots & J^{(1)}(k,n) \\ J^{(1)}(k,n) & J^{(1)}(k,n+1) & J^{(1)}(k,n-k+2) \dots & J^{(i)}(k,n-1) \\ \vdots & \vdots & \vdots & \vdots \\ J^{(1)}(k,n-k+3) & J^{(1)}(k,n-k+4) & J^{(1)}(k,n-2k+5) \dots & J^{(i)}(k,n-k+2) \\ J^{(1)}(k,n-k+2) & J^{(1)}(k,n-k+3) & J^{(1)}(k,n-2k+4) \dots & J^{(i)}(k,n-k+1) \end{bmatrix}.$$

PROOF. (By induction on n.) Let $k \geq 3$ be a fixed integer. For $n = 2k-4$ we can check the equation by inspection. Assume that the equation is true for all integers $2k-3, 2k-2, \dots, n$. To show that it is true also for $n+1$ it is enough to use the induction hypothesis, definition of $J^{(1)}(k,n)$ and the equation $(M_k)^{n+1} = (M_k)^n M_k$. □ □

By Theorem 13 and Theorem 15 we obtain new Cassini-like formulas for the $(2,k)$-distance Jacobsthal numbers of the first kind $J^{(1)}(k,n)$.

COROLLARY 8. *For all positive integers k, n, we have*

$$\det(M_k)^n = \begin{cases} (-3)^n & \textit{if } k = 2 \textit{ and } n \geq 1, \\ (-1)^{n(k+1)} & \textit{if } k \geq 3 \textit{ and } n \geq 2k-4. \end{cases}$$

Note that from Corollary 8 it follows that for all integers $n \geq 2k-4$ the determinant of the matrix $(M_k)^n$ can be expressed as follows

$$\det(M_k)^n = \begin{cases} (-1)^n & \text{if } k \text{ is even}, \\ 1 & \text{if } k \text{ is odd}. \end{cases} \tag{8}$$

For example putting $k = 3$ in (8) we obtain the following identity for the numbers $J^{(1)}(k,n)$.

COROLLARY 9. *For every integer $n \geq 2$ we have*

$$\begin{aligned} &\left(J^{(1)}(3,n+1)\right)^2 J^{(1)}(3,n-2) + \left(J^{(1)}(3,n)\right)^3 \\ &+ \left(J^{(1)}(3,n-1)\right)^2 J^{(1)}(3,n+2) - 2J^{(1)}(3,n-1)J^{(1)}(3,n)J^{(1)}(3,n+1) \\ &\qquad - J^{(1)}(3,n-2)J^{(1)}(3,n)J^{(1)}(3,n+2) = 1. \end{aligned}$$

5. Concluding remarks

The interpretation of the $(2,k)$-distance Jacobsthal numbers with respect to the number of $(kA_1, 2A_2, 2A_3)$-edge colourings by monochromatic paths of some graphs gives the motivation for studying different kinds of $(a_1A_1, a_2A_2,$

a_3A_3)-edge colourings of special graphs. For an arbitrary positive integer k some interesting results connected with the number of (A_1, A_2, kA_3)-edge colourings, $((k-1)A_1, (k-1)A_2, kA_3)$-edge colourings and $(kA_1, kA_2, 2A_3)$-edge colourings of some trees are recently obtained in [12]–[14] and [17].

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