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ALGEBRAIC PROPERTIES OF SEMI-DIRECT SUMS OF RINGS

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Abstract. Let R be an associative ring not necessarily with unity. We say that R is a semi-direct sum of rings S and I, if $R = S + I$, where S is a subring of a ring R, I is an ideal of R and $S \cap I = \{0\}$.

The aim of this paper is to investigate certain algebraic properties of semi-direct sums of associative rings with applications to amalgamated rings. We generalize several results from the literature to associative rings without unity. In particular we show that the class of semi-direct sums of rings is equal to the class of amalgamated rings, we provide a description of the Jacobson radical of semi-direct sums and we offer a characterization of semi-direct sums that are left Steinitz rings.

1. Introduction

A fundamental problem in Ring Theory is to determine or describe ideal extensions of a given ring A by another ring B. More precisely the problem can be formulated as follows: for given rings A, B describe all rings R such that $A \lhd R$ and $R/A \cong B$. An equally important problem concerns a description of all ideal extensions of a given ring. Many authors have considered this problem in various contexts e.g., C.J. Everett in [16] presented an axiomatic description of ideal extensions of a given ring. His research was continued by S. Mac Lane in [21] who used a homological treatment. Another approach to this problem was provided by M. Petrich in [27] and R.M. Raphael and W.D. Burgess in [6].

If $R = S + I$, where S is a subring of a ring R, I is an ideal of R and $S \cap I = \{0\}$, then a ring R is said to be a *semi-direct sum* of rings S and I and we write $R = S \oplus_{\triangleright} I$. A semi-direct sum is an example of an ideal

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extension. Moreover, it is a natural generalization of a direct sum of rings and corner rings (cf. [22]). Semi-direct sums of rings are also called general corner rings or general corner extensions (cf. [20]) and generalize the notion of Pierce decomposition. Furthermore, semi-direct sums are examples of Schreier extensions (cf. [28]).

In [11] M. D'Anna, C.A. Finocchiaro and M. Fontana introduced the notion of amalgamated rings in case of commutative unital rings (cf. [11]). Various classical constructions can be presented as particular cases of an amalgamation, for example amalgamated duplication of a ring along an ideal (cf. [10], [13]), Nagata's idealization also called the trivial ring extension (cf. [24]), $D + M$ constructions or CPI-extensions (cf. [11]). Furthermore, the amalgamation is related to a construction introduced by D.D. Anderson in [2] and motivated by a classical embedding of a ring without unity into a ring with unity (cf. [14]). Moreover, there are some applications of amalgamated algebras in algebraic geometry which follow from the fact that it is possible to iterate the amalgamation of algebras and the result is still an amalgamated algebra (cf. [11]). Additionally, an amalgamation can be realized as a pullback of mappings, and some pullbacks give rise to amalgamated rings (cf. [11]).

There are systematic studies concerning amalgamated rings, but most of them are restricted to commutative rings with unity. For more details, see for example [7], [8], [12], [15], [19], [23], [26].

We show that the class of semi-direct sums of rings and the class of amalgamated rings coincide. This is done in Section 3. This allows us to obtain characterizations of semi-direct sums which are closed under some algebraic constructions. Moreover, we improve some results from [26].

In Section 4, we provide a description of the Jacobson radical of a semi-direct sum of rings and amalgamated rings, applying [28, Theorem 3]. We show when the Jacobson radical of a semi-direct sum of rings A and J is a semi-direct sum of the Jacobson radicals of A and J.

The main theorem of Section 5 is a characterization of semi-direct sums of rings which are left Steinitz. This result extends [15, Theorem 2.1] to associative rings. We present a short and elementary proof of that fact using description of the Jacobson radical from Section 4. We also show when a semi-direct sum of rings is local and T-nilpotent.

2. Preliminaries

All rings in this paper are associative but not necceasarily with unity.

We write $I \lhd R$, if I is an ideal of a ring R. We say that an ideal I of R is *essential*, if $I \cap J \neq \{0\}$ for every $\{0\} \neq J \lhd R$.

If $R = S \oplus_{\triangleright} I$ is a direct-sum of S and I, then clearly $S \oplus_{\triangleright} I/I \cong S$. We will consider only the situation when $S \neq \{0\} \neq I$. Moreover, if $S \lhd R$, then $R = S \oplus I$, where $\oplus$ denotes a direct sum of rings S and I.

For a given ring R and a non-empty subset X of R we denote by $l_R(X) = \{r \in R \mid aX = \{0\}\}$ the *left annihilator* of X. If L is a left ideal of R, then clearly $l_R(L) \lhd R$.

A ring R is called *prime*, if for every $I, J \lhd R$, from the equality $IJ = \{0\}$ follows that either $I = \{0\}$ or $J = \{0\}$.

An ideal I of a ring R is called *semiprime*, if for every left ideal L of R, whenever $L^2 \subseteq I$, then $L \subseteq I$.

Let $\mathcal{R}$ be a class of rings. Then:

(a) $\mathcal{R}$ is *closed under extensions*, if the following implication holds:

$$I \lhd A,\ I \in \mathcal{R} \quad \text{and} \quad A/I \in \mathcal{R} \implies A \in \mathcal{R}.$$

(b) $\mathcal{R}$ is *homomorphically closed*, if every homomorphic image of a ring from $\mathcal{R}$ is in $\mathcal{R}$.

(c) $\mathcal{R}$ is *closed under subrings*, if every subring of a ring belonging to $\mathcal{R}$ is in $\mathcal{R}$.

(d) $\mathcal{R}$ is *hereditary*, if $I \lhd A \in \mathcal{R}$ implies that $I \in \mathcal{R}$.

A class of rings γ is called a *radical class* or shortly a *radical* (cf. [18]), if it is closed under extensions, homomorphically closed and $\gamma(A) := \sum\{I \lhd A \mid I \in \gamma\} \in \gamma$ for every ring A. If $A \in \gamma$, we say that A is γ*-radical*, i.e. $\gamma(A) = A$.

An element $r \in R$ is called *quasi-regular*, if there exists $s \in R$ such that $r + s - rs = 0$. An ideal I of a ring R is called *quasi-regular*, if its every element is quasi-regular.

By $\mathcal{J}(R)$ we denote the Jacobson radical of a ring R. It is well known that $\mathcal{J}(R)$ is a semiprime ideal and the largest quasi-regular ideal of R. Furthermore, the following equalities hold

$$\begin{aligned}\mathcal{J}(R) &= \{r \in R \mid Rr \text{ is quasi-regular}\} = \{r \in R \mid rR \text{ is quasi-regular}\} \\ &= \{r \in R \mid rR \subseteq \mathcal{J}(R)\} = \{r \in R \mid Rr \subseteq \mathcal{J}(R)\}.\end{aligned}$$

Clearly, if R is a $\mathcal{J}$-radical, then $\mathcal{J}(R) = R$.

Recall that R is a *local* ring, if the quotient ring $R/\mathcal{J}(R)$ is a division ring.

A ring R is left T*-nilpotent*, if for every sequence $(a_i)_{i \in \mathbb{N}}$ of elements of R there is an integer n such that $a_1 a_2 \cdot \ldots \cdot a_n = 0$. An ideal of a ring is called *left* T*-nilpotent*, if it is left T-nilpotent as a ring. [17, Theorem 1.1] shows that the class of T-nilpotent rings is closed under subrings, homomorphic images, extensions and direct sums.

A ring R is said to be a *left Steinitz* ring (cf. [9]), if every linearly independent subset of a finitely generated free left R-module F can be extended to a basis of F by adjoining elements of a given basis of F. Analogously we define a right Steinitz ring. If a ring is both left and right Steinitz, then we call a such ring a *Steinitz* ring. B.S. Chwe and J. Neggers in [9] proved that a ring is left Steinitz if and only if it is local and its Jacobson radical is left T-nilpotent.

Recall that a unital ring R is called (*uniquely*) *clean*, if every element of R can be written (uniquely) as the sum of a unit and an idempotent. The concept of clean rings was introduced by W.K. Nicholson in [24].

Let A, B be associative rings, J be an ideal of B and $f\colon A \longrightarrow B$ a ring homomorphism (we do not assume that f preserves identity even in case of unital rings). We consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A, j \in J\}$$

which is called the *amalgamation of A with B along J with respect to f*. We will say shortly that $A \bowtie^f J$ is the *amalgamated ring.*

Throughout the paper, following [26, Remark 2.2], we assume that

$$B = f(A) + J,\ A \neq \{0\} \quad \text{and} \quad J \neq \{0\}.$$

Moreover, by [26, Proposition 2.1] we know that $A \cong \{(a, f(a)) \mid a \in A\}$ is a subring of $A \bowtie^f J$, $J \cong \{0\} \times J$ is an ideal of $A \bowtie^f J$ and $(A \bowtie^f J)/J \cong A$.

3. Semi-direct sums are amalgamated rings

We start this section with a observation that follows directly from [26, Remark 2.2]. Namely from the forementioned remark and under the above notation we know that

$$A \oplus_{\triangleright} J \cong A \bowtie^f J. \tag{3.1}$$

Furthermore

$$A \oplus_{\triangleright} J \cong A \bowtie^{id} J, \tag{3.2}$$

since a mapping $\varphi\colon A \oplus_{\triangleright} J \longrightarrow A \bowtie^{id} J$ given by

$$\varphi((a + j)) = (a, a + j)$$

is a ring isomorphism. Therefore we immediately get the following corollary.

Corollary 3.1. *Let $\mathcal{A}$ and $\mathcal{S}$ denote the class of amalgamated rings and the class of rings which are semi-direct sums of rings. Then $\mathcal{A} = \mathcal{S}$.*

Theorems 3.2 and 3.3 and Corollary 3.4 present characterizations of semi-direct sums belonging to classes of rings closed under some constructions.

It is straightforward to see that [26, Theorem 2.4] and (3.1) imply the following result.

THEOREM 3.2. *Assume $\mathcal{R}$ is the class of rings closed under extensions and subrings. Under the above notation the following conditions are equivalent:*
(i) $A \bowtie^f J \in \mathcal{R}$,
(ii) $A \in \mathcal{R}$ *and* $J \in \mathcal{R}$,
(iii) $A \oplus_{\triangleright} J \in \mathcal{R}$.

Theorem 3.3 and Corollary 3.4 are new versions of [26, Theorem 2.5] and [26, Corollary 2.6], respectively.

THEOREM 3.3. *Assume $\mathcal{R}$ is the class of rings closed under extensions, homomorphically closed and hereditary. Under the above notation the following conditions are equivalent:*
(i) $A \bowtie^f J \in \mathcal{R}$,
(ii) $A \in \mathcal{R}$ *and* $J \in \mathcal{R}$,
(iii) $A \oplus_{\triangleright} J \in \mathcal{R}$.

PROOF. Since A is a homomorphic image of $A \bowtie^f J$ and $J \lhd A \bowtie^f J$, so the implication (i) $\Longrightarrow$ (ii) follows from [26, Lemma 2.3].

Assume (ii) holds. Then $(A \oplus_{\triangleright} J)/J \cong A \in \mathcal{R}$ and $J \in \mathcal{R}$. By the assumption that $\mathcal{R}$ is closed under extensions, we get $A \oplus_{\triangleright} J \in \mathcal{R}$, so we prove (iii).

The implication (iii) $\Longrightarrow$ (i) is clear by (3.1). □

Corollary 3.4 directly follows from Theorem 3.3.

COROLLARY 3.4. *Let γ be a hereditary radical class. Under the above notation the following conditions are equivalent:*
(i) $A \bowtie^f J \in \gamma$,
(ii) $A \in \gamma$ *and* $J \in \gamma$,
(iii) $A \oplus_{\triangleright} J \in \gamma$.

Next result gives necessary and sufficient conditions for a semi-direct sum to be a prime ring and a domain. It follows from well-known properies of prime rings, but we include the proof for completeness. We will use Andrunakievich's Lemma (see [4]) which says that, if $J \lhd K \lhd R$, then $\bar{J}^3 \subseteq J$, where $\bar{J}$ denotes an ideal of R generated by J.

COROLLARY 3.5. *Let $R = A \oplus_{\triangleright} J$ be a semi-direct sum of A and J.*
(i) *The following conditions are equivalent:*
  (1) *R is a prime ring,*
  (2) *J is a prime ring, $l_R(J) = \{0\}$,*
  (3) *J is a prime ring, J is an essential ideal of R.*
(ii) *The following conditions are equivalent:*
  (1) *R is a domain,*
  (2) *J is a domain, $l_R(J) = \{0\}$.*

Proof. (i) Let R be a prime ring. Obviously $l_R(J)J = \{0\}$ and $l_R(J) \triangleleft R$, so $l_R(J) = \{0\}$.

To show that J is a prime ring let $L, M \triangleleft J$ be such that $LM = \{0\}$. By Andrunakiewich's Lemma we have $\bar{L}^3 \subseteq L, \bar{M}^3 \subseteq M$. Hence $\bar{L}^3\bar{M}^3 \subseteq LM = \{0\}$. Since R is a prime ring, so either $\bar{L}^3 = \{0\}$ or $\bar{M}^3 = \{0\}$. However, again by the assumption we get that either $\bar{L} = \{0\}$ or $\bar{M} = \{0\}$, which finally means that either $L = \{0\}$ or $M = \{0\}$. So we have proved the implication (i1)$\Longrightarrow$(i2).

Assume (i2) is true and $\{0\} \neq K \triangleleft R$. Hence $\{0\} \neq KJ \subseteq K \cap J$, so J is an essential ideal of R and we get (i3).

To prove the implication (i3)$\Longrightarrow$(i1) let (i3) be true. Moreover, assume $L, M \triangleleft R$ and $LM = \{0\}$. Clearly $L \cap J \triangleleft J$ and $M \cap J \triangleleft J$. Note that $(L \cap J)(M \cap J) \subseteq LM = \{0\}$. By the assumption of J we have that either $L \cap J = \{0\}$ or $M \cap J = \{0\}$. Since J is an essential ideal of R, so either $L = \{0\}$ or $M = \{0\}$.

(ii) The implication (ii1)$\Longrightarrow$(ii2) is obvious. To prove the opposite implication assume that $ab = 0$ for some $a, b \in R$, J is a domain and $l_R(J) = \{0\}$. Then clearly J is a prime ring and $(bja)^2 = \{0\}$ for every $j \in J$. So $bja = 0$ for every $j \in J$ which implies $bJa = \{0\}$. This equality yields that $(JbJ)(JaJ) = \{0\}$. However J is a prime ring, hence either $JbJ = \{0\}$ or $JaJ = \{0\}$. Thus bJ and aJ are ideals of J such that $(bJ)(aJ) = \{0\}$. Now the assumption that J is a prime ring gives that either $aJ = \{0\}$ or $bJ = \{0\}$, but $l_R(J) = \{0\}$, so either $a = 0$ or $b = 0$. □

4. Jacobson radical

In this section we focus on the Jacobson radical of a semi-direct sum of rings.

Since semi-direct sums and amalgamated rings are Schreier extensions, so immediately from [28, Theorem 3] we get the following description of their Jacobson radical.

Proposition 4.1.

(i) *Let $A \oplus_{\triangleright} J$ be a semi-direct sum of rings A and J. Then*

$$\mathcal{J}(R) = \{a + j \mid a \in \mathcal{J}(A), (a + j)J \subseteq \mathcal{J}(J)\}.$$

(ii) *Let $A \bowtie^f J$ be an amalgamated ring. Then*

$$\mathcal{J}(A \bowtie^f J) = \{(a, f(a) + j) \mid a \in \mathcal{J}(A), (f(a) + j)J \subseteq \mathcal{J}(J)\}.$$

The following fact will play an important role in the rest of the paper.

Proposition 4.2. *Let $R = A \oplus_{\triangleright} J$ be a semi-direct sum of rings A and J. Then the following conditions are equivalent:*

(i) $\mathcal{J}(A)J \subseteq \mathcal{J}(J)$,
(ii) $\mathcal{J}(R) = \mathcal{J}(A) \oplus_{\triangleright} \mathcal{J}(J)$,
(iii) $J\mathcal{J}(A) \subseteq \mathcal{J}(J)$.

Proof. Assume (i) and take $a + j \in \mathcal{J}(R)$, $k \in J$. Then by Proposition 4.1(i) we have that $a \in \mathcal{J}(A)$ and $(a+j)k = ak + jk \in \mathcal{J}(J)$. The statement (i) yields that $ak \in \mathcal{J}(J)$, hence $jk \in \mathcal{J}(J)$ and therefore $jJ \subseteq \mathcal{J}(J)$. This means that $j \in \mathcal{J}(J)$ and $\mathcal{J}(R) \subseteq \mathcal{J}(A) \oplus_{\triangleright} \mathcal{J}(J)$.

To prove the opposite inclusion let $a \in \mathcal{J}(A)$, $j \in \mathcal{J}(J)$ and $k \in J$. Then $(a + j)k = ak + jk$. Since $ak \in \mathcal{J}(A)J \subseteq \mathcal{J}(J)$ and $jk \in \mathcal{J}(J)J \subseteq \mathcal{J}(J)$, so $(a + j)k \in \mathcal{J}(J)$. Consequently $a + j \in \mathcal{J}(R)$, by Proposition 4.1. In effect $\mathcal{J}(A) \oplus_{\triangleright} \mathcal{J}(J) \subseteq \mathcal{J}(R)$.

Assume that the condition (ii) holds and $a \in \mathcal{J}(A), k \in J$, $j \in \mathcal{J}(J)$. Then by (ii) we have $a+j \in \mathcal{J}(R)$. Moreover, Proposition 4.1(i) implies that $(a + j)k \in \mathcal{J}(J)$. Since $j \in \mathcal{J}(J)$, so $jk \in \mathcal{J}(J)$. Therefore $ak \in \mathcal{J}(J)$. This yields the inclusion $\mathcal{J}(A)J \subseteq \mathcal{J}(J)$.

Let the statement (i) be true. Then

$$(J\mathcal{J}(A))^2 = J\mathcal{J}(A)J\mathcal{J}(A) \subseteq J\mathcal{J}(A)J \subseteq J\mathcal{J}(J) \subseteq \mathcal{J}(J),$$

but $J\mathcal{J}(A)$ is a left ideal of J and $\mathcal{J}(J)$ is a semiprime ideal of J, so $J\mathcal{J}(A) \subseteq \mathcal{J}(J)$. Hence we get (iii). The implication (iii)$\Longrightarrow$(i) is proven similarly. □

Proposition 4.3. *Let $R = A \oplus_{\triangleright} J$ be a semi-direct sum of rings A and J and $\mathcal{J}(A)J \subseteq \mathcal{J}(J)$ (or equivalently $J\mathcal{J}(A) \subseteq \mathcal{J}(J)$). Then*

$$R/\mathcal{J}(R) \cong A/\mathcal{J}(A) \oplus_{\triangleright} J/\mathcal{J}(J).$$

Proof. Since $\mathcal{J}(J) \lhd J \lhd R$, so by [3, Theorem 1] we have that $\mathcal{J}(J) \lhd R$. Then $R/\mathcal{J}(J) \cong A \oplus_{\triangleright} (J/\mathcal{J}(J))$. Clearly $\mathcal{J}(J/\mathcal{J}(J)) = \{0\}$, thus we can replace J by $J/\mathcal{J}(J)$ and assume that $\mathcal{J}(J) = \{0\}$. In effect it is enough to prove that

$$R/\mathcal{J}(R) \cong A/\mathcal{J}(A) \oplus_{\triangleright} J.$$

The equality $\mathcal{J}(J) = \{0\}$ together with the inclusion $\mathcal{J}(A)J \subseteq \mathcal{J}(J)$ give $\mathcal{J}(A)J = \{0\}$. By Proposition 4.2 (iii) we obtain that the equality $J\mathcal{J}(A) = \{0\}$ is also true. Consequently

$$\mathcal{J}(A)R = \mathcal{J}(A)(A \oplus_{\triangleright} J) \subseteq \mathcal{J}(A)A + \mathcal{J}(A)J \subseteq \mathcal{J}(A)$$

and similarly $R\mathcal{J}(A) \subseteq \mathcal{J}(A)$, hence $\mathcal{J}(A) \lhd R$.

Proposition 4.2 implies that $\mathcal{J}(R) = \mathcal{J}(A) \oplus_{\triangleright} \mathcal{J}(J) = \mathcal{J}(A)$. Therefore

$$R/\mathcal{J}(R) = R/\mathcal{J}(A) = (A \oplus_{\triangleright} J)/\mathcal{J}(A) \cong (A/\mathcal{J}(A)) \oplus_{\triangleright} J. \qquad \square$$

The following result for amalgamated rings can be proven analogously as Proposition 4.2, Proposition 4.2(iii) and Proposition 4.3 for semi-direct sums applying Proposition 4.1(ii).

Corollary 4.4. *Let $A \bowtie^f J$ be an amalgamated ring.*

(i) *The following conditions are equivalent:*
 (1) $f(\mathcal{J}(A))J \subseteq \mathcal{J}(J)$,
 (2) $\mathcal{J}(A \bowtie^f J) = \mathcal{J}(A) \bowtie^f \mathcal{J}(J)$,
 (3) $Jf(\mathcal{J}(A)) \subseteq \mathcal{J}(J)$.

(ii) *If $f(\mathcal{J}(A))J \subseteq \mathcal{J}(J)$ (or equivalently $Jf(\mathcal{J}(A)) \subseteq \mathcal{J}(J)$), then*

$$(A \bowtie^f J)/\mathcal{J}(A \bowtie^f J) = A/\mathcal{J}(A) \bowtie^f J/\mathcal{J}(J).$$

Example 4.5. The equality of the statement (ii) of Proposition 4.2 holds for example for nil or nilpotent rings. However it does not hold for all rings R such that $R = A \oplus_{\triangleright} J$. To show that it is enough to consider a polynomial ring $R[x]$ in one variable x over a ring R which is a local domain with a unique maximal ideal M. Then clearly $\mathcal{J}(R) = M$ and $R[x] = R + I$, where I is an ideal of $R[x]$ generated by x. Moreover, $R \cap I = \{0\}$ and R is a subring of $R[x]$, so $R[x] = R \oplus_{\triangleright} I$.

Amitsur in [1] showed that $\mathcal{J}(R[x]) = N[x]$, where N is a nil ideal of R. However R is a domain, so R does not contain nonzero nil ideals. In effect $N = \{0\}$. Futhermore [18, Corollary 3.2.4] yields that $\mathcal{J}(I) = I \cap \mathcal{J}(R)$. Thus $\mathcal{J}(I) = \{0\}$. Consequently,

$$\{0\} = \mathcal{J}(R[x]) \neq \mathcal{J}(R) \oplus_{\triangleright} \mathcal{J}(I) = M.$$

From now on a ring $R = A \oplus_{\triangleright} J$ will be called a *semi-direct regular* ring, if it satisfies the equality:

$$\mathcal{J}(R) = \mathcal{J}(A) \oplus_{\triangleright} \mathcal{J}(J) \tag{4.1}$$

or equivalently, if any of these inclusions hold:

$$\mathcal{J}(A)J \subseteq \mathcal{J}(J), \quad J\mathcal{J}(A) \subseteq \mathcal{J}(J).$$

Analogously a ring $A \bowtie^f J$ will be called an *amalgamated regular* ring, if it satisfies the equality:

$$\mathcal{J}(A \bowtie^f J) = \mathcal{J}(A) \bowtie^f \mathcal{J}(J) \tag{4.2}$$

or equivalently, if any of these inclusions are true:

$$f(\mathcal{J}(A))J \subseteq \mathcal{J}(J), \quad Jf(\mathcal{J}(A)) \subseteq \mathcal{J}(J).$$

According to (3.2) we immediately obtain the following corollary.

Corollary 4.6. *A ring $A \oplus_{\triangleright} J$ is semi-direct regular if and only if $A \bowtie^{id} J$ is an amalgamated regular ring.*

5. Left Steinitz rings

In this section we extend some results from [8, 15] to associative rings without unity and give corresponding results for semi-direct regular rings. Recall that $B = f(A) + J$.

Below, we characterize local semi-direct regular rings.

PROPOSITION 5.1. *Let $R = A \oplus_{\triangleright} J$ be a semi-direct regular ring. Then the following conditions are equivalent:*

(i) *R is a local ring,*
(ii) *A is a local ring and J is $\mathcal{J}$-radical.*

PROOF. (i)$\Longrightarrow$(ii) Since R is a local ring, then so is A as a homomorphic image of R. Moreover, from Proposition 4.3 follows that

$$R/\mathcal{J}(R) \cong A/\mathcal{J}(A) \oplus_{\triangleright} J/\mathcal{J}(J).$$

This means that $A/\mathcal{J}(A) \oplus_{\triangleright} J/\mathcal{J}(J)$ is a division ring. However $J/\mathcal{J}(J) \lhd A/\mathcal{J}(A) \oplus_{\triangleright} J/\mathcal{J}(J)$ hence either $J/\mathcal{J}(J) = \{0\}$ or $J/\mathcal{J}(J) = A/\mathcal{J}(A) \oplus_{\triangleright} J/\mathcal{J}(J)$. The latter equality does not hold, since $A \cap J = \{0\}$. Thus $J = \mathcal{J}(J)$, so J is $\mathcal{J}$-radical.

Assume the statement (ii) holds. Then by Proposition 4.3 we have

$$R/\mathcal{J}(R) \cong A/\mathcal{J}(A) \oplus_{\triangleright} J/\mathcal{J}(J) = A/\mathcal{J}(A).$$

The assumption implies that $A/\mathcal{J}(A)$ is a division ring, so is $R/\mathcal{J}(R)$, hence R is a local ring. □

Next fact generalizes [8, Theorem 2.13] to associative rings satisfying (4.2). It can be proved analogously to Proposition 5.1 using Corollary 4.4.

COROLLARY 5.2. *Let $A \bowtie^f J$ be an amalgamated regular ring. Then the following conditions are equivalent:*

(i) *$A \bowtie^f J$ is a local ring,*
(ii) *A is a local ring and J is $\mathcal{J}$-radical.*

Note that the condition (ii) in Corollary 5.2 can be repalced by

$$A \text{ is a local ring and } J \subseteq \mathcal{J}(B).$$

Indeed, if J is $\mathcal{J}$-radical, then $\mathcal{J}(J) = J$. However $J \lhd B$, so [18, Corollary 3.2.4] gives that $\mathcal{J}(J) = J \cap \mathcal{J}(B)$. This means $J \subseteq \mathcal{J}(B)$.

Conversely, if $J \subseteq \mathcal{J}(B)$, then $\mathcal{J}(J) = J$ by the same corollary. Hence we have proved the following fact.

Corollary 5.3. *Let $A \bowtie^f J$ be an amalgamated regular ring. Then the following conditions are equivalent:*
(i) *$A \bowtie^f J$ is a local ring,*
(ii) *A is a local ring and $J \subseteq \mathcal{J}(B)$.*

[25, Theorem 15] together with Proposition 5.1 give a characterization of semi-direct regular rings which are local and uniquely clean.

Corollary 5.4. *Let $R = A \oplus_{\triangleright} J$ be a semi-direct regular ring with unity. Then the following conditions are equivalent:*
(i) *R is a local, uniquely clean ring,*
(ii) *A is a local, uniquely clean ring and J is $\mathcal{J}$-radical,*
(iii) *$R/\mathcal{J}(R) \cong \mathbb{Z}_2$ and J is $\mathcal{J}$-radical.*

Corollaries 5.2, 5.3 and 5.4 together with (3.1) yield the following conclusion which extends [8, Corollary 2.16] to non-commutative rings satisfying (4.2).

Corollary 5.5. *Let $A \bowtie^f J$ be an amalgamated regular ring with unity. Then the following conditions are equivalent:*
(i) *$A \oplus_{\triangleright} J$ is a local, uniquely clean ring,*
(ii) *A is a local, uniquely clean ring and J is $\mathcal{J}$-radical,*
(iii) *A is a local, uniquely clean ring and $J \subseteq \mathcal{J}(B)$,*
(iv) *$(A \oplus_{\triangleright} J)/\mathcal{J}(A \oplus_{\triangleright} J) \cong \mathbb{Z}_2$ and J is $\mathcal{J}$-radical,*
(v) *$(A \oplus_{\triangleright} J)/\mathcal{J}(A \oplus_{\triangleright} J) \cong \mathbb{Z}_2$ and $J \subseteq \mathcal{J}(B)$.*

Proposition 5.6 provides necessary and sufficient conditions for a ring that is a semi-direct sum of rings to be left T-nilpotent. In particular we obtain a characterization of left T-nilpotent amalgamated rings.

Proposition 5.6. *Under the previous notation the following conditions are equivalent:*
(i) *$A \oplus_{\triangleright} J$ is a left T-nilpotent ring,*
(ii) *A and J are left T-nilpotent,*
(iii) *$A \bowtie^f J$ is a left T-nilpotent ring.*

Proof. Let the statement (i) be true. Clearly A and J are subrings of $A \oplus_{\triangleright} J$, therefore [17, Theorem 1.1] yields that A, J are left T-nilpotent, since the class of left T-nilpotent rings is closed under subrings.

Assume that the statement (ii) holds. Obviously $(A \bowtie^f J)/J \cong A$, so by the assumptions we have that $(A \bowtie^f J)/J$ and J are left T-nilpotent. From [17, Theorem 1.1] follows that the class of left T-nilpotent rings is closed under extensions, so consequently $A \bowtie^f J$ is left T-nilpotent.

The implication (iii)$\Longrightarrow$(i) is clear by (3.1). □

Remark 5.7. It is worth to mention that the implication (ii)$\Longrightarrow$(i) in Proposition 5.6 can also be proved using [5, Lemma 2.4] which states that, if $R = A + B$, where A, B are left T-nilpotent subrings of R and A is a one-sided ideal of R, then R is a left T-nilpotent ring.

Now we are ready to present a characterization of semi-direct regular rings which are left Steinitz.

Theorem 5.8. *Let $R = A \oplus_{\triangleright} J$ be a semi-direct regular ring. Then the following conditions are equivalent:*

(i) *R is a left Steinitz ring,*
(ii) *A is a left Steinitz ring, J is $\mathcal{J}$-radical and left T-nilpotent.*

Proof. Assume R is a left Steinitz ring. By [9] this is equivalent to: R is a local ring and $\mathcal{J}(R)$ is a maximal left ideal which is left T-nilpotent. By the assumption that R is a semi-direct regular ring and by Proposition 5.1 this is equivalent to: A is a local ring,$J = \mathcal{J}(J)$ and $\mathcal{J}(R)$ is a left T-nilpotent maximal ideal. Since R is a semi-direct regular ring the equality $\mathcal{J}(R) = \mathcal{J}(A) \oplus \mathcal{J}(J)$ and Proposition 5.6 show that this is equivalent to: A is a local ring, $\mathcal{J}(A)$ is a left T-nilpotent maximal ideal of A and $J = \mathcal{J}(J)$ is left T-nilpotent. Finally this is equivalent to the statement (ii). □

Below, we provide a characterization of left Steinitz amalgamated regular rings. The proof of this result is similar to the proof of Theorem 5.8 using Corollaries 5.2, 5.3, and Proposition 5.6. This fact extends the main result in [15] (cf. Theorem 2.1 (2)) to associative rings without unity satisfying (4.2).

Corollary 5.9. *Let $A \bowtie^f J$ be an amalgamated regular ring. Then the following conditions are equivalent:*

(i) *$A \bowtie^f J$ is a left Steinitz ring.*
(ii) *A is a left Steinitz ring, J is $\mathcal{J}$-radical and left T-nilpotent.*
(iii) *A is a left Steinitz ring and J is left T-nilpotent ring such that $J \subseteq \mathcal{J}(B)$.*

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