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RELATIVE UNIFORM CONVERGENCE OF QUANTUM DIFFERENCE SEQUENCE OF FUNCTIONS RELATED TO ℓ_p-SPACE DEFINED BY ORLICZ FUNCTION

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Abstract. The sequence spaces ${}_{ru}\ell_\infty(\mathcal{O},\nabla_q)$, ${}_{ru}\ell_p(\mathcal{O},\nabla_q)$, ${}_{ru}c(\mathcal{O},\nabla_q)$, ${}_{ru}c_0(\mathcal{O},\nabla_q)$, ${}_{ru}m_\phi(\mathcal{O},\nabla_q,p)$, ${}_{ru}n_\phi(\mathcal{O},\nabla_q,p)$, ${}_{ru}m_\phi(\mathcal{O},\nabla_q)$, ${}_{ru}n_\phi(\mathcal{O},\nabla_q)$ are defined by the Orlicz function in this article. We examine all of its characteristics, including symmetry, solidity, and completeness. A few geometric properties on convexity on the space ${}_{ru}m_\phi(\mathcal{O},\nabla_q,p)$ are also examined in this article.

1. Introduction

A sequence $(f_n(x))$ of functions, defined on a compact domain D, converges relatively uniformly to a limit function $f(x)$ if there exists a function $\mu(x)$, called a scale function, such that for every small positive number ε there is an integer n_ε such that for every $n \geq n_\varepsilon$ the inequality

$$|f(x) - f_n(x)| \leq \varepsilon|\mu(x)|,$$

holds uniformly in x on the interval D.

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The above definition of a relatively uniform convergence of sequence of functions was formulated by Chittenden [3] using the notion which was first used by Moore [13]. Many additional scholars, including Demirci et al. ([4], [5]), Demirci and Orhan [6], Sahin and Dirik [21], Devi and Tripathy ([7], [8], [9]), as well as others, looked deeper into the idea. The term "calculus without limits" is used to refer to quantum calculus, also referred to as q-calculus. The fundamental q-calculus formulae were discovered by Euler in the seventeenth century. However, Jackson [11] may have been the first to introduce the notion of the definite q-difference operator which is defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{x(q-1)}, \quad x \neq 0,$$

and $D_q f(0) = f'(0)$, where q is a fixed number, $q \in (0,1)$. The function f is defined on a q-geometric set $A \subseteq \mathbb{R}$ (or $\mathbb{C}$) that is $qx \in A$ whenever $x \in A$. Due to its use in a variety of mathematical fields, including orthogonal polynomials, fundamental hypergeometric functions, combinatorics, the calculus of variations, and the theory of relativity, quantum difference operators play an intriguing role in mathematics.

The term "Orlicz function" refers to a continuous, non-decreasing, convex function that has the following characteristics: $\mathcal{O}(0) = 0$, $\mathcal{O}(x) > 0$, for $x > 0$ and $\mathcal{O}(x) \to \infty$, as $x \to \infty$.

If there is a constant $K > 0$ such that $\mathcal{O}(2x) \leq K\mathcal{O}(x)$, for all values of $x \geq 0$, then an Orlicz function $\mathcal{O}$ is considered to satisfy the δ_2-condition for all values x.

Lindenstrauss and Tzafriri [12] constructed the sequence space

$$\ell_{\mathcal{O}} = \left\{ x \in \omega : \sum_{n=1}^{\infty} \mathcal{O}\Big(\frac{|x_n|}{\nu}\Big) < \infty, \text{ for some } \nu > 0 \right\}.$$

The norm using the idea of Orlicz function is as follows,

$$\|x\| = \inf \left\{ x \in \omega : \sum_{n=1}^{\infty} \left(\mathcal{O}\Big(\frac{|x_n|}{\nu}\Big) \right) \leq 1 \right\}.$$

Numerous authors, such as Bhardwaj and Singh [1], Bilgin [2], Gung et al. [10], Tripathy and Mahanta [23], Parashar and Choudhary [19] are worked on Orlicz space. These motivated others to study different types of new sequence spaces defined by the Orlicz function.

In 1960, Sargent [22] proposed the $m(\phi)$ space, which is closely connected to the ℓ_p space. He examined a few $m(\phi)$ space characteristics. Afterward, it was examined from the sequence space point of view, by Rath and Tripathy [20], Tripathy [24], and Tripathy and Sen [25] and others.

We establish some geometric properties on the convexity of the space ${}_{ru}m_{\phi}(\mathcal{O}, \nabla_q, p)$. Banach spaces, which are complete, normed, linear, metric spaces with an additional characteristic of convexity of the norm, are the spaces which will be discussed in the present research. Japanese mathematician Nakano [18] introduced the concept of modular spaces. Also, many other researchers worked on modular space such as Musielak and Orlicz ([15], [16]), Musielak [14], Musielak and Wasak [17], Yildiz [26] etc.

Throughout the article $\omega_f, \xi_s, P(f_n)$ represents the space of sequences of functions, the subset of natural numbers of cardinality not greater than s, the permutation of the sequence of functions (f_n), respectively.

2. Definitions and background

In this article, we shall use the following known sequence spaces defined by Orlicz functions, for $0 < p < 1$,

$$
\begin{aligned}
{}_{ru}\ell_p(\mathcal{O}, \nabla_q) &= \left\{ (f_n) \in \omega_f : \sum_{n=1}^{\infty} \left(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu} \Big) \right)^p < \varepsilon |\mu(x)|, \text{ for some } \nu > 0 \right\}, \\
{}_{ru}\ell_\infty(\mathcal{O}, \nabla_q) &= \left\{ (f_n) \in \omega_f : \sup_{n \ge 1} \mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu} \Big) < \varepsilon |\mu(x)|, \text{ for some } \nu > 0 \right\}, \\
{}_{ru}c_0(\mathcal{O}, \nabla_q) &= \left\{ (f_n) \in \omega_f : \lim_{n \to \infty} \mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu |\mu(x)|} \Big) = 0, \text{ for some } \nu > 0 \right\}, \\
{}_{ru}c(\mathcal{O}, \nabla_q) &= \left\{ (f_n) \in \omega_f : \lim_{n \to \infty} \mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu |\mu(x)|} \Big) = L, \text{ for some } \nu > 0 \right\}.
\end{aligned}
$$

In this article, we introduce the following spaces, for $0 < p < 1$,

$$
\begin{aligned}
{}_{ru}m_{\phi}(\mathcal{O}, \nabla_q, p) = \Bigg\{ & (f_n) \in \omega_f : \\
& \sup_{s \ge 1, \sigma \in \xi_s, x \in D} \frac{1}{\phi_s} \sum_{n \in \sigma} \mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu} \Big)^p < \varepsilon |\mu(x)|, \text{ for some } \nu > 0 \Bigg\}, \\
{}_{ru}n_{\phi}(\mathcal{O}, \nabla_q, p) = \Bigg\{ & (f_n) \in \omega_f : \\
& \sup_{u_n \in P(f_n)} \sum_{n=1}^{\infty} \left(\mathcal{O}\Big(\frac{|\nabla_q u_n(x)| \Delta \phi_n}{\nu} \Big) \right)^p < \varepsilon |\mu(x)|, \text{ for some } \nu > 0 \Bigg\},
\end{aligned}
$$

$$
{}_{ru}m_{\phi}(\mathcal{O},\nabla_q) = \Big\{(f_n)\in\omega_f : \sup_{s\ge 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu}\Big) < \varepsilon|\mu(x)|,\ \text{for some } \nu>0\Big\},
$$

$$
{}_{ru}n_{\phi}(\mathcal{O},\nabla_q) = \Big\{(f_n)\in\omega_f : \sup_{u_n\in P(f_n)}\sum_{n=1}^{\infty}\mathcal{O}\Big(\frac{|\nabla_q u_n(x)|\Delta\phi_n}{\nu}\Big) < \varepsilon|\mu(x)|,\ \text{for some } \nu>0\Big\}.
$$

Definition 2.1. A subset $Z_f \subset \omega_f$ is said to be *convergence free*, if $(f_n)\in Z_f$ implies $(g_n)\in Z_f$ for any sequence (g_n) such that $g_n(x)=0$ whenever $f_n(x)=0$ for $x\in D$.

Definition 2.2. A subset $Z_f \subset \omega_f$ is said to be *symmetric* if $(f_n)\in Z_f$ implies $(f_{\pi(n)})\in Z_f$, where π is a permutation of $\mathbb{N}$.

Definition 2.3. A subset $Z_f \subset \omega_f$ is said to be *solid* or *normal*, if $(f_n)\in Z_f$ implies $(g_n)\in Z_f$ for all sequence (g_n) such that $|g_n(x)|\le|f_n(x)|$ for every $n\in\mathbb{N}$ and for all $x\in D$.

Definition 2.4. Let $Z_f\subset\omega_f$ be a space of sequences of functions. Then Z_f is said to be *sequence algebra* if there is defined a product $*$ on Z_f such that

$$
(f_n),(g_n)\in Z_f \implies (f_n)*(g_n)\in Z_f.
$$

Lemma 2.5 (Tripathy and Sen [25, Theorem 7]).

$$
\ell_p \subseteq m(\phi,p)\subseteq \ell_\infty.
$$

In this section, we also give some known definitions of geometric terms related to normed spaces.

Definition 2.6. Let Z_f be a subset of a linear space. Then

(1) Z_f is called *convex* if and only if $(f_n),(g_n)\in Z_f$, $\lambda_1+\lambda_2=1$, $\lambda_1\ge 0$, $\lambda_2\ge 0$ implies $(\lambda_1 f_n+\lambda_2 g_n)\in Z_f$;
(2) Z_f is called *balanced* if and only if $(f_n)\in Z_f, |\lambda|\le 1 \implies (\lambda f_n)\in Z_f$;
(3) Z_f is a called *absolutely convex* if and only if $(f_n),(g_n)\in Z_f$, $|\lambda_1|+|\lambda_2|\le 1$ implies $(\lambda_1 f_n+\lambda_2 g_n)\in Z_f$.

Nakano [18] considered the definition of modular space as follows,

Definition 2.7. Let X be a linear space. A function $\zeta\colon X\to[0,\infty)$ is called *modular function* if

(1) $\zeta(x) = 0$ if and only if $x = \theta$,
(2) $\zeta(\alpha x) = \zeta(x)$ for all scalars α with $|\alpha| = 1$,
(3) $\zeta(\alpha x + \beta y) < \zeta(x) + \zeta(y)$, for all $x, y \in X$ and $\alpha, \beta \geq 0$, $|\alpha| = |\beta| = 1$.
Further, the modular ζ is called *convex* if
(4) $\zeta(\alpha x + \beta y) \leq \alpha\zeta(x) + \beta\zeta(y)$ holds for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For any modular ζ on X the space

$$X_\zeta = \Big\{x \in X : \lim_{\lambda \to 0^+} \zeta(\lambda x) = 0\Big\}$$

is called the *modular space.* A sequence (x_n) of elements of X_ζ is called *modular convergent* to $x \in X_\zeta$ if there exists a $\lambda > 0$ such that $\zeta\lambda(x_n - x) \to 0$, as $n \to \infty$. If ζ is convex modular, then we have the following formula,

$$\|x\|_L = \inf\Big\{\lambda > 0 : \zeta\Big(\frac{x}{\lambda}\Big) \leq 1\Big\}.$$

DEFINITION 2.8. A Banach space is said to be *uniformly convex* if, to each $\varepsilon > 0$, $0 < \varepsilon \leq 2$, there corresponds a $\zeta(\varepsilon) > 0$ such that the following condition holds:

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon \quad \Longrightarrow \quad \frac{1}{2}\|x + y\| \leq 1 - \zeta(\varepsilon).$$

3. Main results

THEOREM 3.1. *The space* ${}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$ *is a linear space, for* $p > 0$.

PROOF. Let $(f_n), (g_n) \in {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$ and α, β be two scalars. Then there exist positive numbers ν_1 and ν_2 such that

$$\sup_{s \geq 1, \sigma \in \xi_s, x \in D} \frac{1}{\phi_s} \sum_{n \in \sigma} \Big(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu_1}\Big)\Big)^p < \varepsilon_1 |\mu_1(x)|,$$

and

$$\sup_{s \geq 1, \sigma \in \xi_s, x \in D} \frac{1}{\phi_s} \sum_{n \in \sigma} \Big(\mathcal{O}\Big(\frac{|\nabla_q g_n(x)|}{\nu_2}\Big)\Big)^p < \varepsilon_2 |\mu_2(x)|.$$

Let $\nu_3 = \max\{|\alpha|\nu_1, |\beta|\nu_2\}$. Since $\mathcal{O}$ is a non-decreasing convex function,

$$\begin{aligned}
&\sup_{s\geq 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\left(\mathcal{O}\Big(\frac{|\alpha\nabla_q f_n(x)+\beta\nabla_q g_n(x)|}{\nu_3}\Big)\right)^p\\
&\qquad= \sup_{s\geq 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\left(\mathcal{O}\Big(\frac{|\alpha\nabla_q f_n(x)|}{\nu_3}+\frac{|\beta\nabla_q g_n(x)|}{\nu_3}\Big)\right)^p\\
&\qquad\leq \sup_{s\geq 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\left(\mathcal{O}\Big(\frac{|\alpha\nabla_q f_n(x)|}{\nu_3}\Big)\right)^p\\
&\qquad\quad+ \sup_{s\geq 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\left(\mathcal{O}\Big(\frac{|\beta\nabla_q g_n(x)|}{\nu_3}\Big)\right)^p\\
&\qquad\leq \sup_{s\geq 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\left(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu_1}\Big)\right)^p\\
&\qquad\quad+ \sup_{s\geq 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\left(\mathcal{O}\Big(\frac{|\nabla_q g_n(x)|}{\nu_2}\Big)\right)^p\\
&\qquad< \max\{\varepsilon_1,\varepsilon_2\}\max\{|\mu_1(x)|,|\mu_2(x)|\}<\infty.
\end{aligned}$$

Thus, $(\alpha\nabla_q f_n + \beta\nabla_q g_n) \in {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$. Hence, ${}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$ is a linear space. □

The following result is a consequence of the above Theorem 3.1

COROLLARY 3.2. *The classes of sequences of functions* ${}_{ru}n_\phi(\mathcal{O}, \nabla_q, p)$, ${}_{ru}m_\phi(\mathcal{O}, \nabla_q)$, ${}_{ru}n_\phi(\mathcal{O}, \nabla_q)$ *are linear spaces.*

THEOREM 3.3. *The space* ${}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$ *is a normed space, with the norm*

$$(3.1)\quad \|f\|_{{}_{ru}m_\phi(\mathcal{O},\nabla_q,p)} = \inf\Bigg\{\nu>0:$$

$$\sum_{n=1}^{\infty}\mathcal{O}\Big(\frac{|f_n(x)|}{\nu|\mu(x)|}\Big)+\sup_{s\geq 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\bigg(\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p\bigg)^{\frac{1}{p}}\leq 1\Bigg\},$$

for $1 \leq p < \infty$.

The space ${}_{ru}m_{\phi}(\mathcal{O}, \nabla_q, p)$ is a p-normed space, with the norm

$$(3.2)\quad \|f\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} = \inf\Bigg\{\nu > 0 : \sum_{n=1}^{\infty}\Big(\mathcal{O}\Big(\frac{|f_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p + \sup_{s\geq 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p \leq 1\Bigg\},$$

for $0 < p < 1$.

Proof. We check the conditions of norm:

(1) Clearly,

$$\|f\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} \geq 0, \quad \text{for all } (\nabla_q f_n) \in {}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p).$$

(2) Clearly, $\|f\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} = 0$, if and only if $f_n(x) = \overline{\theta}$, the null operator, for all $n \in \mathbb{N}$ and $x \in D$.

(3) We have, for $f = (f_n) \in {}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)$ and any scalar λ,

$$\|\lambda\nabla_q f\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} = \sup_{s\geq 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\Bigg(\mathcal{O}\Big(\frac{|\lambda\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Bigg)^p = |\lambda|^p\|\nabla_q f\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)}.$$

(4) Let $(f_n), (g_n) \in {}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)$. Then,

$$\sup_{s\geq 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\Bigg(\mathcal{O}\Big(\frac{|\nabla_q(f_n(x)+g_n(x))|}{\nu|\mu(x)|}\Big)\Bigg)^p \leq \sup_{s\geq 1,\sigma\in\xi_s,x\in D}\sum_{n\in\sigma}\Bigg(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Bigg)^p + \sup_{s\geq 1,\sigma\in\xi_s,x\in D}\sum_{n\in\sigma}\Bigg(\mathcal{O}\Big(\frac{|\nabla_q g_n(x)|}{\nu|\mu(x)|}\Big)\Bigg)^p.$$

Therefore,

$$\|f+g\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} \leq \|f\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} + \|g\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)}.$$

Thus, the space ${}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)$ is a p-normed space with the norm (3.2). Similarly, for $1 \leq p < \infty$, the space ${}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)$ is a norm space with the norm (3.1). □

Theorem 3.4. *If (Z, ru) is complete then the space ${}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)$ is complete with the norm* (3.1).

Proof. Let $(f_n) \in {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$ be a Cauchy sequence of functions, where $f_n = (f_n^i) = ((f_1^i), (f_2^i), \ldots) \in {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$, for each $i \in \mathbb{N}$. Let $\rho > 0$ and $f_0 > 0$ be fixed. Then, for each $\frac{\varepsilon}{\rho f_0} > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$(3.3) \qquad h(f^i - f^j) < \frac{\varepsilon}{\rho f_0}, \quad \text{for all } i, j \geq n_0,$$

$$\text{implies} \quad \inf\Big\{\nu > 0 : \sum_{n=1}^{\infty} \mathcal{O}\Big(\frac{|f_n^i(x) - f_n^j(x)|}{\nu|\mu(x)|}\Big) + \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q(f_n^i(x) - f_n^j(x))|}{\nu|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}} \leq 1\Big\} < \frac{\varepsilon}{\rho f_0},$$

for all $i, j \geq n_0$. Then we have for all $i, j \geq n_0$,

$$\sum_{n=1}^{\infty} \mathcal{O}\Big(\frac{|f_n^i(x) - f_n^j(x)|}{h(f^i - f^j)|\mu(x)|}\Big) + \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q(f_n^i(x) - f_n^j(x))|}{h(f^i - f^j)|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}} \leq 1$$

$$\text{or,} \quad \sum_{n=1}^{\infty} \mathcal{O}\Big(\frac{|f_n^i(x) - f_n^j(x)|}{h(f^i - f^j)|\mu(x)|}\Big) \leq 1 \quad \text{and}$$

$$\sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q(f_n^i(x) - f_n^j)|}{h(f^i - f^j)|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}} \leq 1.$$

At first, we consider

$$\mathcal{O}\Big(\frac{|f_n^i(x) - f_n^j(x)|}{h(f^i - f^j)|\mu(x)|}\Big) \leq 1, \quad \text{for all } i, j \geq n_0.$$

We can find $\rho > 0$ with $(\frac{\rho f_0}{2}) \geq \max(1, \phi_1)$, such that

$$\sum_{n=1}^{\infty} \mathcal{O}\Big(\frac{|f_n^i(x) - f_n^j(x)|}{h(f^i - f^j)|\mu(x)|}\Big) \leq \mathcal{O}\Big(\frac{\rho f_0}{2}\Big),$$

$$\frac{|f_n^i(x) - f_n^j(x)|}{|\mu(x)|} < \frac{\rho f_0}{2} \cdot h(f^i - f^j),$$

$$(3.4) \qquad \frac{|f_n^i(x) - f_n^j(x)|}{|\mu(x)|} < \frac{\rho f_0}{2} \cdot \frac{\varepsilon}{\rho f_0} < \frac{\varepsilon}{2}.$$

Hence, $(f_n^i(x))_{i=1}^{\infty}$, for all $n = 1, 2, 3, \ldots, n$ is a Cauchy sequence in D, w.r.t. the scale function $\mu(x)$, for $x \in D$.

Now, from the second part we get

$$\sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q(f_n^i(x)-f_n^j(x))|}{h(f^i-f^j)|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}} \leq 1,$$

for all $i, j \geq n_0$ and $n \in \mathbb{N}$,

$$\Big(\mathcal{O}\Big(\frac{|\nabla_q(f_n^i(x)-f_n^j(x))|}{h(f^i-f^j)|\mu(x)|}\Big)\Big)^p \leq \phi_1, \quad \text{for all } i, j \geq n_0 \text{ and } n \in \mathbb{N},$$

$$\Big(\mathcal{O}\Big(\frac{|\nabla_q(f_n^i(x)-f_n^j(x))|}{h(f^i-f^j)|\mu(x)|}\Big)\Big)^p \leq \mathcal{O}\Big(\frac{\rho f_0}{2}\Big), \quad \text{for all } i, j \geq n_0 \text{ and } n \in \mathbb{N},$$

$$|\nabla_q f_n^i(x) - \nabla_q f_n^j(x)| < \frac{\rho f_0}{2}\cdot\frac{\varepsilon}{\rho f_0} = \frac{\varepsilon}{2}. \tag{3.5}$$

Therefore, $\big(\nabla_q f_n^i\big)_{n=1}^{\infty}$ is a Cauchy sequence in D, w.r.t. the scale function $\mu(x)$, for $x \in D$. Hence, $\big(\nabla_q f_n^i\big)_{n=1}^{\infty}$ is convergent in D, w.r.t. the scale function $\mu(x), x \in D$, for each $n \in \mathbb{N}$. By equations (3.4) and (3.5), the sequence of the functions (f_n^i) is a Cauchy sequence in D, w.r.t. the scale function $\mu(x)$, for $x \in D$, for all $n \in \mathbb{N}$ and it is convergent in D. Therefore for each $n \in \mathbb{N}$, there exists $(f_n) \in (Z, ru)$ such that $\Big(\frac{f_n^i - f_n}{|\mu(x)|}\Big) \to 0$, as $i \to \infty$ for each $n \in \mathbb{N}$.

Using the continuity of $\mathcal{O}$ we have

$$\sum_{n=1}^{\infty}\mathcal{O}\Big(\frac{|f_n^i(x)-\lim_{j\to\infty} f_n^j(x)|}{\nu|\mu(x)|}\Big) + \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q(f_n^i(x)-\lim_{j\to\infty} f_n^j(x))|}{\nu|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}} \leq 1,$$

for some $\nu > 0$,

$$\sum_{n=1}^{\infty}\mathcal{O}\Big(\frac{|f_n^i(x)-f_n(x)|}{\nu|\mu(x)|}\Big) + \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q(f_n^i(x)-f_n(x))|}{\nu|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}} \leq 1,$$

for some $\nu > 0$ and $j \to \infty$. Taking the infimum value of $\nu's$ in the above and making use of (3.3) we get,

$$\inf\left\{\nu > 0 : \sum_{n=1}^{\infty} \mathcal{O}\Big(\frac{|f_n^i(x) - f_n(x)|}{\nu|\mu(x)|}\Big) + \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\bigg(\sum_{n\in\sigma}\bigg(\mathcal{O}\Big(\frac{|\nabla_q(f_n^i(x) - f_n(x))|}{\nu|\mu(x)|}\Big)\bigg)^p\bigg)^{\frac{1}{p}} \leq 1\right\} < \frac{\varepsilon}{\rho f_0},$$

for all $i \geq n_0$. Therefore, $(f_n^i(x) - f_n(x)) \in {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$, for all $n \geq n_0$. Let $i \geq n_0$ and since ${}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$ is a linear space, therefore

$$(f_n(x)) = (f_n^i(x)) + (f_n(x) - f_n^i(x)) \in {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p).$$

Hence, the space ${}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$ is complete. □

In view of Theorem 3.4, we formulate the following result without proof.

THEOREM 3.5. *The space ${}_{ru}n_\phi(\mathcal{O}, \nabla_q, p)$ is complete.*

THEOREM 3.6. ${}_{ru}\ell_p(\mathcal{O}, \nabla_q) \subseteq {}_{ru}c_0(\mathcal{O}, \nabla_q) \subseteq {}_{ru}c(\mathcal{O}, \nabla_q) \subseteq {}_{ru}\ell_\infty(\mathcal{O}, \nabla_q)$.

The above theorem is easy to prove. That is why we avoid the proof of the above theorem.

PROPOSITION 3.7. *Let $0 < p < 1$. Then the inclusion relation ${}_{ru}\ell_p(\mathcal{O}, \nabla_q) \subset {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$ holds and it is strict.*

PROOF. Taking $g_n(x) = \mathcal{O}\big(\frac{|\nabla_q f_n(x)|}{\nu}\big)$, for all $n \in \mathbb{N}$, the inclusion follows from Lemma 2.5. □

The inclusion is strict following the example below.

EXAMPLE 3.8. Consider the sequence of functions $f_n\colon [0,1] \to \mathbb{R}$, defined by

$$f_n(x) = nx, \quad \text{for all } x \in [0,1].$$

Consider another non-decreasing sequence $\phi_n = n$, for all $n \in \mathbb{N}$ and also consider an Orlicz function $\mathcal{O}\colon [0,\infty) \to [0,\infty)$ such that, $\mathcal{O}(x) = x^2$. Thus,

$(f_n) \in {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$ w.r.t. the scale function $\mu(x) = x^2$, for all $x \in [0,1]$. But $(f_n) \notin {}_{ru}\ell_p(\mathcal{O}, \nabla_q)$. Hence,

$$ {}_{ru}\ell_p(\mathcal{O}, \nabla_q) \not\supseteq {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p). $$

THEOREM 3.9. *Let $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2$ be Orlicz functions satisfying Δ_2 condition. Then*

(1) ${}_{ru}m_\phi(\mathcal{O}_1, \nabla_q, p) \subseteq {}_{ru}m_\phi(\mathcal{O} \circ \mathcal{O}_1, \nabla_q, p)$,
(2) ${}_{ru}m_\phi(\mathcal{O}_1, \nabla_q, p) \cap {}_{ru}m_\phi(\mathcal{O}_2, \nabla_q, p) \subseteq {}_{ru}m_\phi(\mathcal{O}_1 + \mathcal{O}_2, \nabla_q, p)$.

PROOF. (1) Let $(f_n) \in {}_{ru}m_\phi(\mathcal{O}_1, \nabla_q, p)$. Then there exists $\nu > 0$ such that,

$$ \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q(f_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}} < \varepsilon. $$

Let η be such that $\mathcal{O}(t) < \eta$, for all $0 \leq t < \eta < 1$. Note that

$$ \sum_{n\in\sigma}\mathcal{O}\Big(\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big) = \sum_1 \mathcal{O}\Big(\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big) + \sum_2 \mathcal{O}\Big(\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big), $$

where the first summation is over $\mathcal{O}_1\Big(\frac{|f_n(x)|}{\nu|\mu(x)|}\Big) \leq \eta$, and the second summation is over $\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big) > \eta$. Since $\mathcal{O}$ is continuous, by the remark we have

$$ \sum_1 \mathcal{O}\Big(\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big) \leq \mathcal{O}(1)\sum_1 \mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big). \tag{3.6} $$

For $\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big) > \eta$, we use the fact that

$$ \mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big) < \mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\eta^{-1} \leq 1 + \mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\eta^{-1}. $$

Because $\mathcal{O}$ is convex and non-decreasing,

$$\begin{aligned}\mathcal{O}\Big(\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big) &< \mathcal{O}\Big(1+\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\frac{1}{\eta}\Big)\\ &\le \frac{1}{2}\mathcal{O}(2)+\frac{1}{2}\mathcal{O}\Big(2\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\frac{1}{\eta}\Big)\\ &= \frac{1}{2}\mathcal{O}(2)+\frac{1}{2}\mathcal{O}(2)\Big(\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\frac{1}{\eta}\Big).\end{aligned}$$

There exists $K>0$ such that, given that $\mathcal{O}$ meets the requirement of Δ_2,

$$\begin{aligned}\mathcal{O}\Big(\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big) &\le \frac{1}{2}K\mathcal{O}(2)\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\frac{1}{\eta}+\frac{1}{2}K\mathcal{O}(2)\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\frac{1}{\eta}\\ &= K\mathcal{O}(2)\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\frac{1}{\eta}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_1 \mathcal{O}\Big(\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big) &\le \max\{1,(K\mathcal{O}(2)\eta^{-1})\}\frac{1}{\phi_s}\sum_2 \mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\\ &\le \max\{1,(K\mathcal{O}(2)\eta^{-1})\}\frac{1}{\phi_s}\sum_{n\in\sigma} \mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big).\end{aligned} \tag{3.7}$$

From inequalities (3.6) and (3.7) it follows that

$$\begin{aligned}&\sup_{s\ge 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\\ &\qquad\le \max\{1,(K\mathcal{O}(2)\eta^{-1})\}\frac{1}{\phi_s}\sum_2 \mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big),\\ &\sup_{s\ge 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big(\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}}\\ &\qquad\le \max\{1,K\mathcal{O}(2)\eta^{-1}\}\frac{1}{\phi_s}\Big(\sum_2\Big(\mathcal{O}_1\Big(\frac{|\nabla_p f_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}}<\varepsilon.\end{aligned}$$

Hence, $(f_n)\in {}_{ru}m_\phi(\mathcal{O}_1\circ\mathcal{O}_2,\nabla_q,p)$.

(2) Let $(f_n) \in {}_{ru}m_\phi(\mathcal{O}_1, \nabla_q, p) \cap {}_{ru}m_\phi(\mathcal{O}_2, \nabla_q, p)$. Therefore, $(f_n) \in {}_{ru}m_\phi(\mathcal{O}_1, \nabla_q, p)$ and $(f_n) \in {}_{ru}m_\phi(\mathcal{O}_2, \nabla_q, p)$. Consequently, there are $\nu_1 > 0$ and $\nu_2 > 0$ such that

$$\sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big(\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu_1|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}} < \frac{\varepsilon}{2},$$

and

$$\sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big(\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu_2|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}} < \frac{\varepsilon}{2}.$$

Let $\nu_3 = \max\{\nu_1, \nu_2\}$. Then

$$\begin{aligned}
&\sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big((\mathcal{O}_1+\mathcal{O}_2)\Big(\frac{|\nabla_q f_n(x)|}{\nu_1|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}} \\
&\quad\leq \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big(\mathcal{O}_1\Big(\frac{|\nabla_q f_n(x)|}{\nu_1|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}} \\
&\quad\quad+ \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\Big(\sum_{n\in\sigma}\Big(\mathcal{O}_2\Big(\frac{|\nabla_q f_n(x)|}{\nu_1|\mu(x)|}\Big)\Big)^p\Big)^{\frac{1}{p}} \\
&\quad< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore, $(f_n) \in {}_{ru}m_\phi(\mathcal{O}_1 + \mathcal{O}_2, \nabla_q, p)$. Hence,

$${}_{ru}m_\phi(\mathcal{O}_1, \nabla_q, p) \cap {}_{ru}m_\phi(\mathcal{O}_2, \nabla_q, p) \subseteq {}_{ru}m_\phi(\mathcal{O}_1 + \mathcal{O}_2, \nabla_q, p).$$

□

LEMMA 3.10. ${}_{ru}m_\phi(\mathcal{O}, \nabla_q) \subseteq {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$.

PROOF. Let $(f_n) \in {}_{ru}m_\phi(\mathcal{O}, \nabla_q)$. Then there exists $\nu > 0$ such that

$$\sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu}\Big) < \varepsilon|\mu(x)|.$$

Hence, for each fixed s,

$$\sum_{n=1}^{\infty}\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu}\Big) < \varepsilon|\mu(x)|\phi_s, \quad \sigma\in\xi_s,$$

$$\sup_{s\geq 1,\sigma\in\xi_s,x\in D}\left(\sum_{n\in\sigma}\left(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\right)^p\right)^{\frac{1}{p}}<\varepsilon\phi_s,\quad \sigma\in\xi_s,$$

$$\sup_{s\geq 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\left(\sum_{n\in\sigma}\left(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\right)^p\right)^{\frac{1}{p}}<\varepsilon,\quad \sigma\in\xi_s,$$

$$(f_n)\in {}_{ru}m_\phi(\mathcal{O},\nabla_q,p).$$

Hence, ${}_{ru}m_\phi(\mathcal{O},\nabla_q)\subseteq {}_{ru}m_\phi(\mathcal{O},\nabla_q,p)$. □

REMARK 3.11. The class of sequences of functions ${}_{ru}m_\phi(\mathcal{O},\nabla_q,p)$ is not convergence free.

This result follows from the following example.

EXAMPLE 3.12. Consider an Orlicz function $\mathcal{O}\colon[0,\infty)\to[0,\infty)$ such that $\mathcal{O}(x)=x,$ for all $x\in[0,\infty),\ \phi_n=n,$ for all $n\in\mathbb{N}$, and we take a sequence of functions (f_n) defined as follows

$$f_n(x)=\begin{cases}\frac{1}{n^2x}, & \text{for } x\neq 0,\\ 0, & \text{for } x=0,\end{cases}$$

with respect to the scale function defined as

$$\mu(x)=\begin{cases}\frac{1}{x}, & \text{for } x\neq 0,\\ 0, & \text{for } x=0.\end{cases}$$

Thus, $(f_n)\in\ {}_{ru}m_\phi(\mathcal{O},\nabla_q,p)$. Now we consider another sequence of functions such as

$$g_n(x)=\begin{cases}n/x, & \text{for } x\neq 0,\\ 0, & \text{for } x=0.\end{cases}$$

But $(g_n)\notin\ {}_{ru}m_\phi(\mathcal{O},\nabla_q,p)$.

Hence, the class of sequences of functions ${}_{ru}m_\phi(\mathcal{O},\nabla_q,p)$ is not convergence free.

REMARK 3.13. The class of sequences of functions ${}_{ru}m_\phi(\mathcal{O},\nabla_q,p)$ is not symmetric.

This result follows from the following example.

EXAMPLE 3.14. Let $\phi_n = n$, for all $n \in \mathbb{N}$ and also consider an Orlicz function $\mathcal{O}\colon [0,\infty) \to [0,\infty)$ such that $\mathcal{O}(x) = x$, for all $x \in [0,\infty)$. Now we consider the sequence of function (f_n) defined by

$$f_n(x) = \begin{cases} nx, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0, \end{cases}$$

then $(f_n) \in {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$ with respect to the scale function $\mu(x)$ such that,

$$\mu(x) = \begin{cases} x, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

Now, take the rearrangement (g_n) of (f_n) defined as follows

$$(g_n) = (f_1, f_3, f_5, f_7, f_2, \ldots).$$

Then $(g_n) \notin {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$. Hence, ${}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$ is not symmetric.

THEOREM 3.15. *The class of sequences of functions* ${}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$ *is solid.*

PROOF. Let $(f_n) \in {}_{ru}m_\phi(\mathcal{O}, \nabla_q, p)$. Then

$$\sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} \left(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\right)^p < \varepsilon.$$

Let (λ_n) be any sequence of scalars with $|\lambda_n| \leq 1$, for all $n \in \mathbb{N}$. Then

$$\begin{aligned} &\sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} \left(\mathcal{O}\Big(\frac{|\lambda_n \nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\right)^p \\ &\qquad \leq \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} |\lambda_n|^p \left(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\right)^p \\ &\qquad \leq \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} \left(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\right)^p < \varepsilon. \end{aligned}$$

□

THEOREM 3.16. ${}_{ru}\ell_1(\mathcal{O}, \nabla_q) \subseteq {}_{ru}m_\phi(\mathcal{O}, \nabla_q) \subseteq {}_{ru}\ell_\infty(\mathcal{O}, \nabla_q)$.

PROOF. Let $(f_n) \in {}_{ru}\ell_1(\mathcal{O}, \nabla_q)$. Then we have

$$\sum_{n=1}^{\infty} \mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu}\Big) < \varepsilon|\mu(x)|, \quad \text{for all } \nu > 0.$$

Since (ϕ_n) is monotonically increasing, so we have

$$\frac{1}{\phi_s}\sum_{n\in\sigma} \mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu}\Big) \leq \frac{1}{\phi_1}\sum_{n=1}^{\infty} \mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu}\Big) < \varepsilon|\mu(x)|.$$

Hence,

$$\sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma} \mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu}\Big) < \varepsilon|\mu(x)|.$$

Thus, $(f_n) \in {}_{ru}m_\phi(\mathcal{O}, \nabla_q)$. Therefore, ${}_{ru}\ell_1(\mathcal{O}, \nabla_q) \subseteq {}_{ru}m_\phi(\mathcal{O}, \nabla_q)$.

Let $(f_n) \in {}_{ru}m_\phi(\mathcal{O}, \nabla_q)$. Then we have

$$\sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma} \mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu}\Big) < \varepsilon|\mu(x)|, \quad \text{for some } \nu > 0$$

(on taking cardinally of σ to be 1). Therefore, $(f_n) \in {}_{ru}\ell_\infty(\mathcal{O}, \nabla_q)$ and thus, ${}_{ru}m_\phi(\mathcal{O}, \nabla_q) \subseteq {}_{ru}\ell_\infty(\mathcal{O}, \nabla_q)$. □

4. Geometric properties

We introduced the space

$${}_{ru}m_\phi(\mathcal{O}, \nabla_q) = \{(f_n) \in \omega_f : \zeta(\lambda f_n) < \varepsilon \text{ for some } \lambda > 0\},$$

where

$$\zeta(f_n(x)) = \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma} \Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)^p,$$

$\nabla_q f_n(x) = f_n(x) - q f_{n-1}(x)$ and $p > 0$. It is equipped with the norm defined by

$$\|f_n\|_{{}_{ru}m_\phi(\mathcal{O},\nabla_q)} = \inf\Big\{\nu > 0 : \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma} \mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)^p \leq 1\Big\}.$$

THEOREM 4.1. *The space ${}_{ru}m_{\phi}(\mathcal{O}, \nabla_q, p)$ is a convex modular space with the modular*

$$\zeta(f_n) = \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p.$$

PROOF. (1) Clearly, $\zeta(f_n) = 0,$ if and only if $f_n = \overline{\theta}$, where $\overline{\theta}$ represents the null sequence of functions.

(2) Now for any scalar α with $|\alpha| = 1$,

$$\begin{aligned}\zeta(\alpha f_n) &= \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\alpha\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p \\ &= |\alpha|^p \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p \\ &= \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p \\ &= \zeta(f_n).\end{aligned}$$

(3) Let (f_n) and (g_n) be in ${}_{ru}m_{\phi}(\mathcal{O}, \nabla_q, p)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. To prove the convexity of the function note that

$$\begin{aligned}\zeta(\alpha f_n + \beta g_n) &= \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\alpha\nabla_q f_n(x) + \beta\nabla_q g_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p \\ &\leq \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}|\alpha|^p\Big(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p \\ &\quad + \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}|\beta|^p\Big(\mathcal{O}\Big(\frac{|\nabla_q g_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p \\ &\leq \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p \\ &\quad + \sup_{s\geq 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\Big(\mathcal{O}\Big(\frac{|\nabla_q g_n(x)|}{\nu|\mu(x)|}\Big)\Big)^p \\ &= \zeta(f_n) + \zeta(g_n).\end{aligned}$$

Hence, ${}_{ru}m_{\phi}(\mathcal{O}, \nabla_q, p)$ is a modular space. Also, it can be proven that $\zeta(\alpha f_n + \beta g_n) \leq \alpha\zeta(f_n) + \beta\zeta(g_n)$, for $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Thus, ${}_{ru}m_{\phi}(\mathcal{O}, \nabla_q, p)$ is a convex modular space. □

THEOREM 4.2. *The space* ${}_{ru}m_{\phi}(\mathcal{O}, \nabla_q, p)$ *is uniform convex.*

PROOF. Fix $0 < \varepsilon \leq 2$. Let us consider two sequences $(f_n), (g_n)$ of functions from the Banach space ${}_{ru}m_{\phi}(\mathcal{O}, \nabla_q)$ and assume that

$$\|f_n\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} = \|g_n\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} = 1, \quad \|f_n - g_n\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} \geq \varepsilon. \tag{4.1}$$

Note that

$$\|f_n + g_n\|^2_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} + \|f_n - g_n\|^2_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} = 2(\|f_n\|^2_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} + \|g_n\|^2_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)}),$$

and from (4.1) we get

$$\begin{aligned}\|f_n + g_n\|^2_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} &= 4 - \|f_n - g_n\|^2_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} \\ &\leq 4 - \varepsilon^2.\end{aligned}$$

Consequently,

$$\frac{1}{2}\|f_n + g_n\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} \leq \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

So, there corresponds a $\vartheta(\varepsilon) > 0$ such that

$$\frac{1}{2}\|f_n + g_n\|_{{}_{ru}m_{\phi}(\mathcal{O},\nabla_q,p)} \leq 1 - \vartheta(\varepsilon).$$

Hence, the space ${}_{ru}m_{\phi}(\mathcal{O}, \nabla_q, p)$ is uniform convex. □

THEOREM 4.3. *The space* ${}_{ru}m_{\phi}(\mathcal{O}, \nabla_q, p)$ *is convex.*

PROOF. Let $(f_n), (g_n) \in {}_{ru}m_{\phi}(\mathcal{O}, \nabla_q, p)$. By the definition we have

$$\sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma} \mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu_1}\Big)^p < \varepsilon_1|\mu_1(x)|, \quad \text{for some } \nu_1 > 0,$$

and

$$\sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma} \mathcal{O}\Big(\frac{|\nabla_q g_n(x)|}{\nu_2}\Big)^p < \varepsilon_2|\mu_2(x)|, \quad \text{for some } \nu_2 > 0.$$

Take $\nu = \max\{|\lambda_1|\nu_1, |\lambda_2|\nu_2\}$, where $\lambda_1,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1$. Since $\lambda_1,\ \lambda_2,\ \nu_1,\ \nu_2 > 0$, then $|\lambda_1|\nu_1 > 0$, $|\lambda_2|\nu_2 > 0$. Now,

$$\begin{aligned}
&\sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\Big(\frac{|\lambda_1\nabla_q f_n(x) + \lambda_2\nabla_q g_n(x)|}{\nu}\Big)^p \\
&\qquad = \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\Big(\frac{|\lambda_1\nabla_q f_n(x) + (1-\lambda_1)\nabla_q g_n(x)|}{\nu}\Big)^p \\
&\qquad \leq \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\Big(\frac{|\lambda_1\nabla_q f_n(x)|}{\nu}\Big)^p \\
&\qquad\quad + \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\Big(\frac{|(1-\lambda_1)\nabla_q g_n(x)|}{\nu}\Big)^p \\
&\qquad \leq |\lambda_1|^p \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu_1}\Big)^p \\
&\qquad\quad + |1-\lambda_1|^p \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\Big(\frac{|\nabla_q g_n(x)|}{\nu_2}\Big)^p \\
&\qquad \leq |\lambda_1|^p \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\Big(\frac{|\nabla_q f_n(x)|}{\nu_1}\Big)^p \\
&\qquad\quad + (1+|\lambda_1|^p) \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\Big(\frac{|\nabla_q g_n(x)|}{\nu_2}\Big)^p \\
&\qquad \leq |\lambda_1|^p\varepsilon_1|\mu_1(x)| + (1+|\lambda_1|^p)\varepsilon_2|\mu_2(x)| \\
&\qquad \leq |\lambda_1|^p(\varepsilon_1|\mu_1(x)| + \varepsilon_2|\mu_2|) + \varepsilon_2|\mu_2(x)| \\
&\qquad \leq \varepsilon|\mu(x)|,
\end{aligned}$$

where

$$\mu(x) = |\lambda_1|^p(\varepsilon_1|\mu_1(x)| + \varepsilon_2|\mu_2|) + \varepsilon_2|\mu_2(x)|.$$

□

THEOREM 4.4. *The space ${}_{ru}m_\phi(\mathcal{O},\nabla_q,p)$ is absolutely convex.*

PROOF. Let $0 < r < 1$. $V = \big\{f = (f_n) \in \omega_f : \|f\|_{{}_{ru}m_\phi(\mathcal{O},\nabla_q,p)} \leq r\big\}$ is absolutely convex because if $(f_n), (g_n) \in V$ and $|\lambda_1| + |\lambda_2| \leq 1$, then

$$\|\lambda_1 f_n + \lambda_2 g_n\|_{{}_{ru}m_\phi(\mathcal{O},\nabla_q,p)} \leq (|\lambda_1| + |\lambda_2|)r \leq r.$$

□

In view of Theorem 4.4, we formulate the following without proof.

COROLLARY 4.5. *The space* ${}_{ru}m_{\phi}(\mathcal{O}, \nabla_q, p)$ *is balanced.*

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