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ALMOST EVERYWHERE CONVERGENCE OF VARYING PARAMETER SETTING CESÀRO MEANS OF FOURIER SERIES WITH RESPECT TO WALSH–KACZMARZ SYSTEM

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Abstract. In this paper, the almost everywhere convergence of Cesàro means of Walsh–Kaczmarz–Fourier series in a varying parameter setting is investigated. In particular, we define subsequence $\mathbb{N}_{\alpha_n,q}$ of natural numbers and prove that the maximal operator

$$\sup_{n\in\mathbb{N}_{\alpha_n,q}} |\sigma_n^{\alpha_n} f|$$

is of strong type (H^1, L^1), where H^1 is a Hardy space.

1. Introduction

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$ and $\mathbb{R}$ denote the set of real numbers. In this paper, C denote absolute positive constants and C_q denote positive constants depending at most on q although not always the same in different occurrences.

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The Walsh–Paley system (the detail briefs can be obtained in the books of [17] and [19]) is a special product generated by the so-called Rademacher functions r_n $(n \in \mathbb{N})$. For the definition let r be the function given on the interval $[0, 1)$ by

$$r(x) = \begin{cases} 1, & \text{if } 0 \leq x < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \leq x < 1, \end{cases}$$

and extended to the whole real line $\mathbb{R}$ periodically by 1.

Now, define $r_n(x) := r(2^n x)$ $(x \in [0, 1), n \in \mathbb{N})$. Then the usual product system $(w_n, n \in \mathbb{N})$ of $r_n's$ is obtained in the following way:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}, \quad n \in \mathbb{N},$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$ is the binary decomposition of n, i.e. $n_k \in \{0, 1\}$ $(k \in \mathbb{N})$. It is well-known (for details see the book [19]) that $(w_n,\, n \in \mathbb{N})$ is a complete orthonormal system with respect to the Lebesgue measure of $[0, 1)$.

Then a basic property of the Walsh–Dirichlet Kernel is

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } 0 \leq x < 2^{-n}, \\ 0, & \text{if } 2^{-n} \leq x < 1. \end{cases} \tag{1.1}$$

This interval $[0, 1)$ can be treated as the so called dyadic group, i.e. the set of all sequences $(x_k, k \in \mathbb{N})$ where $x_k = 0 \vee 1$. The group operation $\dot{+}$ is the coordinate-wise addition modulo 2, i.e. if $x = (x_k, k \in \mathbb{N})$, $y = (y_k, k \in \mathbb{N})$ then $x \dot{+} y := x_k \oplus y_k, k \in \mathbb{N}$, where $a \oplus b$ denotes the addition modulo 2 of a, $b \in \mathbb{N}$. For example the Rademacher functions can be computed in this sense $r_n(x) = (-1)^{x_n}$ $(x \in [0, 1), n \in \mathbb{N})$. Furthermore, $D_{2^n} = 2^n \chi I_n$ $(n \in \mathbb{N})$ where I_n is the set of all $(x_k, k \in \mathbb{N})$ such that $x_0 = x_1 = \cdots = x_{n-1} = 0$ and χI_n is its characteristic function.

In this work, we focus on summability methods of Walsh–Kaczmarz–Fourier series. For any $n = 2^s + \sum_{k=0}^{s-1} n_k 2^k$, where $0 < n \in \mathbb{N}$, $s \in \mathbb{N}$, the so-called Kaczmarz rearrangement $(\psi_n, n \in \mathbb{N})$ (called Walsh–Kaczmarz system) of Walsh–Paley system is defined in the following way

$$\psi_n := r_s \prod_{k=0}^{s-1} r_{s-k-1}^{n_k} \quad \text{and} \quad \psi_0 := w_0,$$

and is called Walsh–Kaczmarz system. We commonly use the following notations. Let $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $2^{|n|} \leq n < 2^{|n|+1}$) and $n^{(s)} := \sum_{k=0}^{s-1} n_k 2^k$.

If $f\in L^1[0,1)$, then we can define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Walsh–Kaczmarz system in the usual manner:

$$\widehat{f}(k) := \int_{[0,1)} f\psi_k d\mu, \quad k\in\mathbb{N},$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\,\psi_k, \quad n\in\mathbb{N}_+, \quad S_0 f := 0,$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad n\in\mathbb{N}_+.$$

It is known that (for details see [21]) ψ is a complete orthonormal system,

$$\psi_{2^m} = w_{2^m} = r_m$$

and

$$\left\{\psi_k : k=2^m,...,2^{m+1}-1\right\} = \left\{w_k : k=2^m,...,2^{m+1}-1\right\}, \quad m\in\mathbb{N}.$$

Moreover, if we define

$$\tau_s(x) := (x_{s-1},x_{s-2},...,x_1,x_0,x_s,x_{s+1},...), \quad x\in[0,1),$$

then

$$\psi_n(x) = w_n(\tau_s(x)) = r_s(x) w_{n-2^s}(\tau_s(x)) \tag{1.2}$$

and

$$D_{2^j}(\tau_j(x)) = D_{2^j}(x), \quad j\in\mathbb{N}, x\in[0,1).$$

The Fejér means and kernels with respect to the Walsh–Kaczmarz system are defined in the usual manner:

$$\sigma_n^1 f := \frac{1}{n}\sum_{k=1}^{n} S_k f, \quad n\in\mathbb{N}_+,$$

$$K_n := \frac{1}{n}\sum_{k=1}^{n} D_k = \sum_{k=0}^{n-1}\Big(1-\frac{k}{n}\Big) w_k, \quad n\in\mathbb{N}_+.$$

Let $K_o := 0$. The next estimation with respect to K_n (see [21]) will be used often in this work: if $x \in [0,1)$, $0 < n \in \mathbb{N}$ then

$$(1.3) \quad |K_n(x)| \leq \sum_{j=0}^{s} 2^{j-s-1} \sum_{i=j}^{s} (D_{2^i}(x) + D_{2^i}(x \dot{+} 2^{-j-1})), \quad 2^s \leq n < 2^{s+1}.$$

From this it follows by (1.1) the uniform L_1- boundedness of K_n in which

$$(1.4) \qquad \sup_n \|K_n\|_1 \leq \infty.$$

Let $0 < \alpha \leq 1$, $k \in \mathbb{N}$, and $f \in L^1[0,1)$. Then, the n^{th} (C,α) Walsh–Kaczmarz Kernels and (C,α) Walsh–Kaczmarz means with respect to ψ will be defined respectively as follows

$$\Theta_n^\alpha := \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-k-1}^\alpha \psi_k,$$

$$\sigma_n^\alpha f(x) := \int_0^1 f(t)\Theta_k^\alpha(x+t)dt, \quad x \in [0,1), n \in \mathbb{N},$$

where

$$A_k^\alpha := \prod_{i=1}^{k} \frac{\alpha+i}{i}.$$

It is well-known that (see [24])

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}$$

and

$$A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1} \quad \text{and} \quad A_n^\alpha \sim n^\alpha.$$

α may also be a sequence $\alpha = (\alpha_n)$. In this case we have sequence of (C,α_n).

The maximal operator of (C,α_n) means is defined as

$$\sigma_{*,n}^\alpha f := \sup_n |\sigma_n^\alpha f|.$$

Here, we give also the most important concepts with respect to the dyadic Hardy spaces. Let the maximal function of $f \in L^1[0,1)$ be given by

$$f^*(x) = \sup_n 2^n \Big| \int_{x \dot{+} I_n} f(t) d\mu(t) \Big|, \quad x \in [0,1).$$

Then, Hardy space on $[0,1)$ is defined as

$$H^1[0,1) := \left\{ f : \|f\|_{H_1} := \|f^*\|_1 < \infty \right\}.$$

A function $a \in L^\infty[0,1)$ is called a 1-*atom* if either a is identically equal to 1 or there exists a dyadic interval $I = x \dot{+} I_N$ for some $N \in \mathbb{N}$, $x \in [0,1)$ such that

$$\operatorname{supp} a \subset I, \quad \|a\|_\infty \leq 2^N$$

and $\int_0^1 a = 0$. We shall say that a is supported on I.

Definition 1.1 ([19]). A sublinear operator T which maps $H^1[0,1)$ into the collection of measurable functions defined on $[0,1)$ is called 1-*quasi-local* if there exists a constant C such that

$$\int_{[0,1)\setminus I} |Ta| \leq C$$

for every p-atom a supported on I.

Lemma 1.2. *Let* 1*-quasi-local operator* T *is* L^∞*-bounded, i.e.,*

$$\|Tf\|_\infty \leq C \|f\|_\infty .$$

Then T *is bounded from* $H^1[0,1)$ *to* $L^1[0,1)$.

Definition 1.3. It is already defined in [2] that

$$P(n,\alpha) := \sum_{i=0}^{\infty} n_i 2^{i\alpha} \quad \text{for } n \in \mathbb{N}, \alpha \in \mathbb{R}.$$

For example $P(n,1) = n$.

Moreover, for the set of sequences $\alpha = (\alpha_n)$ and positive real number q, we consider the following subset of natural numbers:

$$\mathbb{N}_{\alpha_n, q} := \Big\{ n \in \mathbb{N} : \frac{P(n,\alpha_n)}{n^{\alpha_n}} \leq q \Big\}. \tag{1.5}$$

The first result on the a.e. convergence of the $(C,1)$ means of Walsh–Fourier series is due to Fine [8] and Schipp [18], if the Walsh functions are considered by Paley's ordering. The analogical result in the case of Walsh–Kaczmarz system was also investigated by many authors. One of the Kaczmarz analogue of Schipp's [18] results was given by Gát [10]. Besides, he proved also an (H^1, L^1)-like inequality for the maximal operator of Fejér means with respect to Walsh–Kaczmarz system

$$\left\| \sup_{k\in\mathbb{N}} |\sigma_k^1 f| \right\|_1 \le c \, \|f\|_{H^1}, \quad f \in H^1.$$

Convergence and summability of Cesàro means of the one and two dimensional cases in Lebesgue and martingale Hardy spaces were studied by a lot of authors. We mention Akhobadze [3], Blahota, Persson and Tephnadze [5], Blahota, Tephnadze and Toledo [7], Blahota, Tephnadze [6], Fridli [9], Gát [12], Nagy [15, 16], Simon [20], Weisz [23].

In 2007, Akhobadze [4] introduced the notion of Cesàro means of trigonometric Fourier series with variable parameter setting. The varying parameter settings of the (C,α) means of the Walsh–Paley–Fourier series for different situation were investigated in [1], [2], [13] and with respect to the character systems of the group of 2-adic integers in [22] (for the more general orthonormal system, i.e., with respect to Vilenkin system, in [14]). However, these problems with respect to Walsh–Kaczmarz orthonormal system have not been investigated yet.

Thus, in this paper, it is going to be proved that the maximal operator of Cesàro means of Walsh–Kaczmarz–Fourier series is of weak type (L^1, L^1). Moreover, the almost everywhere convergence of Cesàro means with varying parameter setting of integrable functions (i.e. $\sigma_n^{\alpha_n} f \to f$, as $n \to \infty$) is proved, for $f \in L^1$, for every sequence $\alpha = (\alpha_n, n \in \mathbb{N})$ where $0 < \alpha_n < 1$.

2. Main results

Lemma 2.1. *Let* $0 < \alpha_n < 1$, $n \in \mathbb{N}$. *Then,*

$$\Theta_n^{\alpha_n} = \sum_{t=1}^{6} \beta_t,$$

where

$$\beta_1 := 1 + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} A_{n-2^j-1}^{\alpha_n} \Big(D_{2^{j+1}}(x) - D_{2^j}(x) \Big),$$

$$\beta_2 := -\frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x))(2^j-1)A_{n-2^j-1}^{\alpha-1}K_{2^j-1}(\tau_j(x)),$$

$$\beta_3 := \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=1}^{2^j-2} kA_{n-2^{j+1}+k+1}^{\alpha_n-2}K_k(\tau_j(x)),$$

$$\beta_4 := \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x))A_{n^{(k-1)}-1}^{\alpha_n}D_{2^{n_k}}(\tau_{n_1}(x)),$$

$$\beta_5 := -\frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x))A_{n^{(k-1)}-1}^{\alpha_n-1}(2^{n_k}-1)K_{2^{n_k}-1}(\tau_{n_1}(x)),$$

$$\beta_6 := \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \sum_{j=1}^{2^{n_k}-2} A_{n^{(k)}+j+1}^{\alpha_n-2}jK_j(\tau_{n_1}(x)).$$

Proof. Consider the binary expansion of $0 < n \in \mathbb{N}$, where $n_k \in \mathbb{N}$, $k = 1, ..., q$ and $n_k \geq n_{k+1}$, $k = 1, ..., q-1$. Then,

$$\Theta_n^\alpha = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=0}^{n-1} A_{n-k-1}^{\alpha_n}\psi_k = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=0}^{2^{n_1}-1} A_{n-k-1}^{\alpha_n}\psi_k + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2^{n_1}}^{n-1} A_{n-k-1}^{\alpha_n}\psi_k$$
$$=: \Theta_{n_1}^{\alpha_n} + \Theta_{n_2}^{\alpha_n}.$$

Let $x \in [0,1)$, thus by applying (1.2) we get

$$\begin{aligned}
\Theta_{n_1}^{\alpha_n}(x) &= 1 + \frac{1}{A_{n-1}^{\alpha}} \sum_{j=0}^{n_1-1} \sum_{k=0}^{2^j-1} A_{n-1-(2^{j+1}-1-k)}^{\alpha}\psi_{2^{j+1}-1-k}(x) \\
&= 1 + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} \sum_{k=0}^{2^j-1} A_{n-2^{j+1}+k}^{\alpha_n}w_{2^{j+1}-1-k}(\tau_j(x)) \\
&= 1 + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} \sum_{k=0}^{2^j-1} A_{n-2^{j+1}+k}^{\alpha_n}w_{2^{j+1}-1}(\tau_j(x))w_k(\tau_j(x)) \\
&= 1 + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \\
&\quad \times \Big(\sum_{k=0}^{2^j-1} A_{n-2^{j+1}+k}^{\alpha_n}(D_{k+1}(\tau_j(x)) - D_k(\tau_j(x))\Big).
\end{aligned}$$

Applying Abel's transformation, we get the following

$$\Theta_{n_1}^{\alpha_n} = 1 + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x))$$
$$\times \Big(\sum_{k=1}^{2^j} A_{n-2^{j+1}+k-1}^{\alpha_n} D_k(\tau_j(x)) - \sum_{k=0}^{2^j-1} A_{n-2^{j+1}+k}^{\alpha_n} D_k(\tau_j(x)) \Big)$$
$$= 1 + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) A_{n-2^j-1}^{\alpha_n} D_{2^j}(\tau_j(x))$$
$$- \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) (\sum_{k=1}^{2^j-1} A_{n-2^{j+1}+k-1}^{\alpha_n} - A_{n-2^{j+1}+k}^{\alpha_n}) D_k(\tau_j(x))$$
$$= 1 + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) A_{n-2^j-1}^{\alpha_n} D_{2^j}(\tau_j(x))$$
$$- \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=1}^{2^j-1} A_{n-2^{j+1}+k}^{\alpha_n} D_k(\tau_j(x))$$
$$:= \Theta_{n_1}^{\alpha_n,1} + \Theta_{n_1}^{\alpha_n,2}.$$

By considering

$$D_k = kK_k - (k-1)K_{k-1}, \quad 0 < k \in \mathbb{N},$$

we can transform $\Theta_{n_1}^{\alpha_n,2}$ as follows:

$$\Theta_{n_1}^{\alpha_n,2} = -\frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=1}^{2^j-1} A_{n-2^{j+1}+k}^{\alpha_n-1}$$
$$\times \Big(kK_k(\tau_j(x)) - (k-1)K_{k-1}(\tau_j(x)) \Big)$$
$$= -\frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=1}^{2^j-1} A_{n-2^{j+1}+k}^{\alpha_n-1} kK_k(\tau_j(x))$$
$$+ \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=1}^{2^j-1} A_{n-2^{j+1}+k}^{\alpha_n-2} (k-1)K_{k-1}(\tau_j(x))$$

$$
\begin{aligned}
&= -\frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=1}^{2^j-1} A_{n-2^{j+1}+k}^{\alpha_n-1} k K_k(\tau_j(x)) \\
&\quad + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=0}^{2^j-2} A_{n-2^{j+1}+k}^{\alpha_n-2} k K_k(\tau_j(x)) \\
&= -\frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x))(2^j-1) A_{n-2^j-1}^{\alpha_n-1} K_{2^j-1}(\tau_j(x)) \\
&\quad + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} w_{2^{j+1}-1}(\tau_j(x)) \sum_{k=1}^{2^j-2} k A_{n-2^{j+1}+k+1}^{\alpha_n-2} K_k(\tau_j(x)) \\
&=: \beta_2 + \beta_3.
\end{aligned}
$$

If $x_0 = ... = x_{j-1} = 0$, note that $w_{2^{j+1}-1}(\tau_j(x)) = r_j(x)$, then by (1.1) we get

$$
w_{2^{j+1}-1}(\tau_j(x)) D_{2^j}(x) = r_j(x) D_{2^j}(x) = D_{2^{j+1}}(x) - D_{2^j}(x).
$$

Thus,

$$
\Theta_{n_1}^{\alpha_n,1} = 1 + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} A_{n-2^j-1}^{\alpha_n} \Big(D_{2^{j+1}}(x) - D_{2^j}(x) \Big) =: \beta_1.
$$

For $x \in [0,1)$, the situation for $\Theta_{n_2}^{\alpha_n}(x)$ becomes

$$
\begin{aligned}
\Theta_{n_2}^{\alpha_n} &= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2^{n_1}}^{n-1} A_{n-k-1}^{\alpha_n} \psi_k(x) = \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=1}^{q-1} \sum_{j=2^{n_1}+\ldots+2^{n_k}}^{2^{n_1}+\ldots+2^{n_{k+1}}-1} A_{n-j-1}^{\alpha_n} \psi_j(x) \\
&= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=1}^{q-1} \sum_{j=0}^{2^{n_{k+1}}-1} A_{n-1-(2^{n_1}+\ldots+2^{n_{k+1}}-1-j)}^{\alpha_n} \psi_{2^{n_1}+\ldots+2^{n_{k+1}}-1-j}(x) \\
&= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=1}^{q-1} w_{2^{n_1}+\ldots+2^{n_{k+1}}-1}(\tau_{n_1}(x)) \sum_{j=0}^{2^{n_{k+1}}-1} A_{n-(2^{n_1}+\ldots+2^{n_{k+1}})+j}^{\alpha_n} w_j(\tau_{n_1}(x)) \\
&= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q} w_{2^{n_1}+\ldots+2^{n_k}-1}(\tau_{n_1}(x)) \sum_{j=0}^{2^{n_k}-1} A_{n-(2^{n_1}+\ldots+2^{n_k})+j}^{\alpha_n} w_j(\tau_{n_1}(x)) \\
&= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \sum_{j=0}^{2^{n_k}-1} A_{n^{(k)}+j}^{\alpha_n} w_j(\tau_{n_1}(x)).
\end{aligned}
$$

Using Abel's transformation, where $n^{(k)} := \sum_{i=k+1}^{\infty} 2^{n_i}$, $k = 1, \dots, q$, we get

$$
\begin{aligned}
\Theta_{n_2}^{\alpha_n}(x) &= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \sum_{j=0}^{2^{n_k}-1} A_{n^{(k)}+j}^{\alpha_n} \big(D_{j+1}(\tau_{n_1}(x)) - D_j(\tau_{n_1}(x))\big) \\
&= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \\
&\quad \times \Big[\sum_{j=1}^{2^{n_k}} A_{n^{(k)}+j-1}^{\alpha_n} D_j(\tau_{n_1}(x)) - \sum_{j=0}^{2^{n_k}-1} A_{n^{(k)}+j}^{\alpha_n} D_j(\tau_{n_1}(x)) \Big] \\
&= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \Big(A_{n^{(k)}+2^{n_k}-1}^{\alpha_n} D_{2^{n_k}}(\tau_{n_1}(x)) \\
&\quad - \sum_{j=1}^{2^{n_k}-1} A_{n^{(k)}+j}^{\alpha_n-1} D_j(\tau_{n_1}(x)) \Big) \\
&= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) A_{n^{(k-1)}-1}^{\alpha_n} D_{2^{n_k}}(\tau_{n_1}(x)) \\
&- \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q-1} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \sum_{j=1}^{2^{n_k}-1} A_{n^{(k)}+j}^{\alpha_n-1} \big(jK_j(\tau_{n_1}(x)) - (j-1)K_{j-1}(\tau_{n_1}(x))\big) \\
&= \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) A_{n^{(k-1)}-1}^{\alpha_n} D_{2^{n_k}}(\tau_{n_1}(x)) \\
&\quad - \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) A_{n^{(k-1)}-1}^{\alpha_n-1} (2^{n_k} - 1) K_{2^{n_k}-1}(\tau_{n_1}(x)) \\
&\quad + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=2}^{q} w_{n-n^{(k)}-1}(\tau_{n_1}(x)) \sum_{j=1}^{2^{n_k}-2} A_{n^{(k)}+j+1}^{\alpha_n-2} jK_j(\tau_{n_1}(x)) \\
&=: \beta_4 + \beta_5 + \beta_6.
\end{aligned}
$$

Hence, the theorem follows. □

Define the maximal operator

$$
\sigma_{*,n}^{\alpha_n} f := \sup_{n \in \mathbb{N}_{\alpha_n,\, q}} |\sigma_n^{\alpha_n} f| = \sup_{n \in \mathbb{N}_{\alpha_n,\, q}} \Big| \int_I \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=0}^{n-1} A_{n-k-1}^{\alpha_n} \psi_k * f \Big|.
$$

Lemma 2.2. *Let $\alpha = (\alpha_n,\, n \in \mathbb{N})$ where $0 < \alpha_n < 1$. Then, the maximal operator $\sigma_{*,n}^{\alpha_n} f$ is quasi-local.*

Proof. By the definition of quasi-locality, let $f \in L^1[0,1)$ be such that

$$\operatorname{supp} f \subset I_N(u), \qquad \int_{I_N(u)} f d\mu = 0$$

for some dyadic interval $I_N(u)$. Then,

$$\begin{aligned}
&\int_{[0,1)\setminus I_N(u)} \sup_{n\in\mathbb{N}} \left| \int_{I_N(u)} \frac{1}{A_{n-1}^{\alpha_n}} \sum_{k=0}^{n-1} A_{n-k-1}^{\alpha_n} \psi_k(x \dotplus y) f(x) d\mu(y) \right| d\mu(x) \\
&\quad \le C \int_{[0,1)\setminus I_N(u)} \sup_{n\in\mathbb{N}} \int_{I_N(u)} \left|\Theta_{n_1}^{\alpha_n}(x \dotplus y)\right| |f(x)|\, d\mu(y) d\mu(x) \\
&\qquad + C \int_{[0,1)\setminus I_N(u)} \sup_{n\in\mathbb{N}} \int_{I_N(u)} \left|\Theta_{n_2}^{\alpha_n}(x \dotplus y)\right| |f(x)|\, d\mu(y) d\mu(x) \\
&\quad := \alpha_1 + \alpha_2.
\end{aligned}$$

Since for $n \in \mathbb{N}$, $n \le 2^N$ and $x \in I_N(u)$ we have $\sigma_n^{\alpha_n} f = 0$, thus

$$\sigma_{*,n}^{\alpha_n} f = \sup_{n>2^N,\, n\in\mathbb{N}_{\alpha_n,q}} \left|\sigma_n^{\alpha_n} f\right|.$$

From the proof of Lemma 2.1, we have the decomposition

$$\sigma_n^{\alpha_n} f = \tilde{\sigma}_n^{\alpha_n} f + \bar{\sigma}_n^{\alpha_n} f,$$

where

$$\begin{aligned}
\tilde{\sigma}_n^{\alpha_n} f(y) &:= \int_0^1 \Theta_{n_1}^{\alpha_n}(x \dotplus y) f(x) d\mu(x), \\
\tilde{\sigma}_n^{\alpha_n} f(y) &:= \int_0^1 \Theta_{n_2}^{\alpha_n}(x \dotplus y) f(x) d\mu(x).
\end{aligned}$$

Again, from the proof of Lemma 2.1, we have

$$\begin{aligned}
\tilde{\sigma}_n^{\alpha_n} f(y) &= \int_0^1 \beta_1(x \dotplus y) f(x) d\mu(x) \\
&\quad + \int_0^1 \beta_2(x \dotplus y) f(x) d\mu(x) + \int_0^1 \beta_3(x \dotplus y) f(x) d\mu(x) \\
&=: I + II + III.
\end{aligned}$$

If $x \in [0,1) \setminus I_N$ then by (1.1), we get

$$\int_0^1 f(x) D_t(y \dotplus x) d\mu(x) = 0$$

for all $t = 0, \ldots, 2^{n_1}$. From the proof of Lemma 2.1, we have $I = 0$. By Lemma 2.1 in [11], $II = 0$. The situation for III: with respect to x and for any $0 \le j < 2^N$, we have that the Fejér kernel $K_j(y \dotplus x)$ depends only on the coordinates x_0, x_1, …, x_{N-1}. This implies that,

$$\int_{I_N} f(x) \left| K_j(y \dotplus x) \right| d\mu(x) = \left| K_j(y) \right| \int_{I_N} f(x) d\mu(x) = 0.$$

Thus, we can re-write

$$\begin{aligned}\int_{[0,1)\setminus I_N(u)} \sup_{k\in\mathbb{N}} \Big| \int_{I_N(u)} \beta_3(y \dotplus x) f(x) d\mu(x) \Big| d\mu(y) \\ = \int_{[0,1)\setminus I_N(u)} \sup_{k\ge 2^N,\, k\in\mathbb{N}} \Big| \int_{I_N(u)} \beta_3(y \dotplus x) f(x) d\mu(x) \Big| d\mu(y).\end{aligned}$$

So, using Lemma 3 in [11], we get

$$\begin{aligned}&\int_{[0,1)\setminus I_N(u)} \sup_{k\ge 2^N,\, k\in\mathbb{N}} \Big| \int_{I_N(u)} \beta_3(y \dotplus x) f(x) d\mu(x) \Big| d\mu(y) \\ &\le C \int_{I_N(u)} |f(x)| \int_{[0,1)\setminus I_N(u)} \sum_{j=0}^{n_1} \sum_{k=2^N}^{2^j-2} \sup_{k\ge 2^N} k \Big| K_k(\tau_j(y \dotplus x)) \Big| d\mu(x) \\ &\le C \int_{I_N(u)} |f(x)| d\mu(x) \le C \left\| f \right\|_1 .\end{aligned}$$

Hence,

$$\int_{[0,1)\setminus I_N(u)} \sup_{k\in\mathbb{N}} \Big| \int_{I_N(u)} (\beta_3 + \beta_2 + \beta_1)(y \dotplus x) f(x) d\mu(x) \Big| d\mu(y) \le C \left\| f \right\|_1 .$$

Note that

$$\int_{[0,1)\setminus I_N(u)} \sup_{k\ge 2^N,\, k\in\mathbb{N}} \Big| \int_{I_N(u)} \beta_4(y \dotplus x) f(x) d\mu(x) \Big| d\mu(y) = 0,$$

since $f * D_{2^{n_k}} = 0$ for $n_l < n_s \le n_k$ because of the A_{n_k} measurablity of $D_{2^{n_k}}$ and $\int f = 0$. Moreover, $D_{2^{n_k}}(y \dotplus x) = 0$ for $n_s > n_k$, $y \dotplus x \notin I_N$.

From Lemma 1.1 of [14] (see also [4]), we have

$$\frac{A_{n^{(k)}+j+1}^{\alpha_n-1}}{A_{n-1}^{\alpha_n}} \leq C\frac{(n^{(k)}+j)^{\alpha_n}}{(n)^{\alpha_n}}, \quad j=1,\dots,2^{n_k}-1,\ k=2,\dots,q-1.$$

Thus, by the fact that $n \in \mathbb{N}_{\alpha_n,\,q}$, we have (see(1.5))

$$\sum_{k=2}^{q-1}\sum_{j=1}^{2^{n_k}-1}\frac{A_{n^{(k)}+j+1}^{\alpha-2}}{A_{n-1}^{\alpha}}j \leq C\sum_{k=2}^{q-1}\sum_{t=0}^{n_k-1}\sum_{j=2^l}^{2^{l+1}-1}\frac{(n^{(k)}+j)^{\alpha_n}}{(n)^{\alpha_n}}j$$

$$\leq C\sum_{k=2}^{q-1}\sum_{t=0}^{n_k-1}\frac{(n^{(k)}+2^l)^{\alpha_n}}{(n)^{\alpha_n}}\sum_{j=2^l}^{2^{l+1}-1}j \leq C\sum_{k=2}^{q-1}\frac{2^{k\alpha_n}}{n^{\alpha_n}} \leq C_q.$$

Consequently, using (1.4), we can estimate

$$\int_{[0,1)\setminus I_N(u)}\sup_n\left|\int_{I_N(u)}\left[\beta_5(y \dotplus x)+\beta_6(y \dotplus x)f(x)\right]d\mu(x)\right|d\mu(y) \leq C_q\left\|f\right\|_1.$$

Hence, the lemma is proved. □

Lemma 2.3. *Let* $\alpha=(\alpha_n, n\in\mathbb{N})$*, where* $0<\alpha_n<1$ *satisfy condition* (1.5)*. Then:*

(I) $\|\Theta_n^{\alpha_n}\|_1 \leq C_q$,
(II) *there exists an absolute constant* C_q *such that* $\|\sigma_n^{\alpha_n}f\|_1 \leq C_q\left\|f\right\|_1$,
(III) *the maximal operator* $\sigma_{*,n}^{\alpha_n}$ *is of type* (L^∞, L^∞).

Proof. To prove (I) we use Lemma 2.1 and estimation (1.3). That is,

$$|\beta_3| \leq Cn^{-\alpha_n}\frac{1}{A_{n-1}^{\alpha_n}-1}\sum_{j=1}^{n_1}\sum_{s=1}^{j-1}\sum_{l=2^{s-1}}^{2^s-1}\left|A_{n-2^j+l+1}^{\alpha_n-2}\right|$$

$$\times\sum_{i=0}^{s-1}\sum_{m=0}^{i}2^m(D_{2^i}(\tau_{j-1})(x)+D_{2^i}(\tau_{j-1})(x \dotplus e_m))$$

$$\leq Cn^{-\alpha_n}\frac{1}{A_{n-1}^{\alpha_n}-1}\sum_{j=1}^{n_1}\sum_{i=0}^{j-2}\sum_{i=i+1}^{j-1}\sum_{l=2^{s-1}}^{2^s-1}(n-2^j+l)^{\alpha_n-2}$$

$$\times\sum_{m=0}^{i}2^m(D_{2^i}(\tau_{j-1})(x)+D_{2^i}(\tau_{j-1})(x \dotplus e_m))$$

$$= Cn^{-\alpha_n} \frac{1}{A_{n-1}^{\alpha_n} - 1} \sum_{j=1}^{n_1} \sum_{i=0}^{j-2} \gamma_{ij} \sum_{m=0}^{i} 2^m (D_{2^i}(\tau_{j-1})(x) + D_{2^i}(\tau_{j-1})(x \dotplus e_m))$$

$$\leq Cn^{-\alpha_n} \frac{1}{A_{n-1}^{\alpha_n} - 1} \sum_{j=1}^{n_1} \sum_{i=0}^{j-2} \Big(\gamma_{ij} 2^i D_{2^i}(\tau_{j-1})(x) + \sum_{m=0}^{i} 2^m D_{2^i}(\tau_{j-1})(x \dotplus e_m) \Big),$$

where $e_m := 2^{-m-1} = (0, \dots, 0, 1, 0, \dots)$ and

$$\gamma_{ij} = \sum_{i=i+1}^{j-1} \sum_{l=2^{s-1}}^{2^s-1} (n - 2^j + l)^{\alpha_n - 2} \leq C \int_{2^i}^{j-1} (n - 2^j + x)^{\alpha_n - 2} d\mu(x) \leq C 2^{i(\alpha_n - 1)}.$$

With a similar computation we show that the same estimation can be obtained for β_2. Thus, $\Theta_{n_1}^{\alpha_n,2}(x)$ can be estimated as

$$\begin{aligned} &\Theta_{n_1}^{\alpha_n,2}(x) = \beta_2 + \beta_3 \\ &\leq Cn^{-\alpha_n} \sum_{j=1}^{n_1} \sum_{i=0}^{j-2} 2^{i(\alpha_n - 1)} \Big(2^i D_{2^i}(\tau_{j-1})(x) + \sum_{m=0}^{i} 2^m D_{2^i}(\tau_{j-1})(x \dotplus e_m) \Big). \end{aligned}$$

Applying (1.1), the previous estimation implies for $\|\beta_2 + \beta_3\|$ that

$$\|\beta_2 + \beta_3\|_1 \leq Cn^{-\alpha_n} \sum_{j=1}^{n_1} \sum_{i=0}^{j-2} 2^{i(\alpha_n - 1)} 2^i \leq Cn^{-\alpha_n} \sum_{j=1}^{n_1} \sum_{i=0}^{j-2} 2^{i\alpha_n} \leq C_q.$$

Analogically, it can also be obtained for the L^1-norm estimation of β_1. Consider that $w_{2^{j+1}-1}(\tau_j(x)) = r_j(x)$ when $x_0 = \ldots = x_{j-1} = 0$. Then by (1.1) we get

$$w_{2^{j+1}-1}(\tau_j(x)) D_{2^j}(x) = r_j(x) D_{2^j}(x) = D_{2^{j+1}}(x) - D_{2^j}(x),$$

that is

$$\begin{aligned} \beta_1 &= 1 + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=0}^{n_1-1} A_{n-2^j-1}^{\alpha_n} \Big(D_{2^{j+1}}(x) - D_{2^j}(x) \Big) \\ &= 1 + \frac{1}{A_{n-1}^{\alpha_n}} \Big(\sum_{j=1}^{n_1} A_{n-2^j-1}^{\alpha_n} D_{2^j}(x) - \sum_{j=0}^{n_1-1} A_{n-2^j-1}^{\alpha_n} D_{2^j}(x) \Big) \\ &= 1 + \frac{1}{A_{n-1}^{\alpha_n}} A_{n-2^{n_1}-1}^{\alpha_n} D_{2^{n_1}}(x) - \frac{1}{A_{n-1}^{\alpha_n}} A_{n-2}^{\alpha_n} \\ &\quad + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{n_1-1} \Big(A_{n-2^{j-1}-1}^{\alpha_n} - A_{n-2^j-1}^{\alpha_n} \Big) D_{2^j}(x). \end{aligned}$$

From this and (1.1) we get

$$\begin{aligned}\|\beta_1\| &\le C_q + \frac{1}{A_{n-1}^{\alpha_n}} \sum_{j=1}^{n_1-1} \left(A_{n-2^{j-1}-1}^{\alpha_n} - A_{n-2^j-1}^{\alpha_n} \right) \\ &\le C_q + \frac{1}{A_{n-1}^{\alpha_n}} \left(A_{n-2^{j-1}-1}^{\alpha_n} - A_{n-2^j-1}^{\alpha_n} \right) \le C_q.\end{aligned}$$

Let us deal now with the situation $\|\beta_4\|_1$, $\|\beta_5\|_1$ and $\|\beta_6\|_1$ as follows:

$$\begin{aligned}\|\beta_4\|_1 &\le C_q n^{-\alpha_n} \sum_{k=2}^{q} A_{n^{(k-1)}-1}^{\alpha_n} \le C_q n^{-\alpha_n} \sum_{k=2}^{q} (n^{(k-1)})^{\alpha_n} \\ &\le C_q n^{-\alpha_n} \sum_{k=2}^{q} 2^{n_k \alpha_n} \le C_q.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\beta_5\| &\le C_q n^{-\alpha_n} \sum_{k=2}^{q} 2^{n_k} (n^{(k-1)})^{\alpha_n - 1} \|K_{2^{n_k}-1}\|_1 \\ &\le C_q n^{-\alpha_n} \sum_{k=2}^{q} 2^{n_k} (2^{(\alpha_n - 1)n_k}) \le C_q.\end{aligned}$$

From (1.4), $\|\beta_6\|_1$ can be estimated as follows:

$$\begin{aligned}\|\beta_6\|_1 &\le C n^{-\alpha_n} \sum_{k=2}^{r} \sum_{j=1}^{2^{n_k}-2} A_{n^{(k)}+j+1}^{\alpha_n - 1} j \|K_j\|_1 \\ &\le C n^{-\alpha_n} \frac{1}{A_{n-1}^{\alpha}} \sum_{k=2}^{r} \sum_{l=0}^{n_k-1} \sum_{j=2^l}^{2^{l+1}-1} (n^{(k)} + j)^{\alpha_n - 2} j \\ &\le C n^{-\alpha_n} \sum_{k=2}^{r} \sum_{l=0}^{n_k-1} (n^{(k)} + 2^l)^{\alpha_n - 2} \sum_{j=2^l}^{2^{l+1}-1} j \\ &\le C n^{-\alpha_n} \sum_{k=2}^{r} \sum_{l=0}^{n_k-1} (n^{(k)} + 2^l)^{\alpha_n - 2} 2^{2l} \\ &\le C n^{-\alpha_n} \sum_{k=2}^{r} \sum_{l=0}^{n_k-1} 2^{l(\alpha_n - 2)} 2^{2l} \le C n^{-\alpha_n} \sum_{k=2}^{r} 2^{\alpha n_k} \le C_q.\end{aligned}$$

Thus, (I) follows. The results in (II) and (III) are a direct consequence of (I). Hence, the theorem follows. □

THEOREM 2.4. *Let* $\alpha = (\alpha_n,\, n \in \mathbb{N})$, *where* $0 < \alpha_n < 1$ *and* $f \in L^1[0,1)$. *Then:*

(I) *the maximal operator* $\sup_{n\in\mathbb{N}_{\alpha_n,q}} |\sigma_n^{\alpha_n} f|$ *is of weak type* (L^1, L^1),

(II) $\mu\{|\sigma_n^{\alpha_n} f - f| > 0\} = 0$ *as* $n \to \infty$ *where* $n \in \mathbb{N}_{\alpha_n,q}$,

where constant C_q *depends on* q *indicated in equation* (1.5) *above.*

PROOF. To prove (I) of this theorem, we apply the Calderon–Zygmund decomposition Lemma [11]. That is, let $f \in L^1[0,1)$ and $\|f\|_1 < \delta$. Then there is a decomposition:

$$f = f_0 + \sum_{j=1}^{\infty} f_j$$

such that

$$\|f_0\|_\infty \le C\delta, \quad \|f_0\|_1 \le C\|f\|_1$$

and $[0,1)^j = I_{k_j}(u^j)$ are disjoint intervals for which

$$\operatorname{supp} f_j \subset I_{k_j}(u^j), \quad \int_{I_{k_j}(u^j)} f_j d\mu = 0, \quad u^j \in [0,1),\, k_j \in \mathbb{N},\, j \in \mathbb{N}_+,$$

and

$$|F| \le \frac{C\|f\|_1}{\delta}, \quad \text{where } F = \bigcup_{i=1}^{\infty} I_{k_j}(u^j).$$

By the σ-sublinearity of the maximal operator with an appropriate constant C_q we have

$$\mu(\sup_{n\in\mathbb{N}_{\alpha,q}} |\sigma_n^{\alpha_n} f| > 2C_q\delta) \le \mu(\sup_{n\in\mathbb{N}_{\alpha_n,q}} |\sigma_n^{\alpha_n} f_0| > C_q\delta)$$
$$+ \mu(\sup_{n\in\mathbb{N}_{\alpha_n,q}} |\sigma_n^{\alpha_n} f_j| > C_q\delta) =: A + B.$$

Since $\sup_{n\in\mathbb{N}_{\alpha_n,q}} |\sigma_n^{\alpha_n}|$ is of type $(L^\infty,\, L^\infty)$, we have

$$\Big\| \sup_{n\in\mathbb{N}_{\alpha_n,q}} |\sigma_n^{\alpha_n} f_0| \Big\|_\infty \le C_q\|f_0\|_\infty \le C_q\delta.$$

Then we have $A = 0$. The case for B becomes,

$$B = \mu(\sup_{n\in\mathbb{N}_{\alpha_n,\,q}} |\sigma_n^{\alpha_n} f_j| > C_q\delta) \le |F| + \mu(\bar{F} \cap [\sup_{n\in\mathbb{N}_{\alpha_n,\,q}} \Big|\sigma_n^{\alpha_n} \sum_{j=1}^{\infty} f_j\Big| > C_q\delta])$$

$$\leq \frac{C\|f\|_1}{\delta} + \frac{C_q}{\delta} \sum_{j=1}^{\infty} \int_{[0,1)\setminus I_{k_j}(u^j)} \sup_{n\in\mathbb{N}_{\alpha_n,\,q}} |\sigma_n^{\alpha_n} f_j| \, d\mu =: \frac{C\|f\|_1}{\delta} + \frac{C_q}{\delta} \sum_{j=1}^{\infty} N_j,$$

where

$$N_j = \int_{[0,1)\setminus I_{k_j}(u^j)} \sup_{n\in\mathbb{N}_{\alpha_n,\,q}} |\sigma_n^{\alpha_n} f_j| \, d\mu.$$

From Lemma 2.2 we get

$$\begin{aligned} N_j &\leq \int_{[0,1)\setminus I_{k_j}(u^j)} \sup_{n\in\mathbb{N}_{\alpha_n,\,q}} \Big| \int_{I_{k_j}(u^j)} f_j(x) \Theta_n^{\alpha_n}(y+x) d\mu(x) \Big| d\mu(y) \\ &\leq C_q \|f_j\|_1. \end{aligned}$$

Finally, we have

$$\mu(\sup_{n\in\mathbb{N}_{\alpha_n,\,q}} |\sigma_n^{\alpha_n} f| > 2C_q\delta) \leq C_q \frac{\|f\|_1}{\delta}.$$

This shows that the maximal operator $\sup_{n\in\mathbb{N}_{\alpha_n,\,q}} |\sigma_n^{\alpha_n}|$ is of weak type $(L^1,\, L^1)$.

Now, we prove (II). Let $t \geq 2^k$. Then we have $S_t p \equiv p$, where p is a Walsh–Kaczmarz polynomial which can be given by

$$p(x) = \sum_{i=0}^{2^k-1} C_i \psi_i(x).$$

This implies the statement $\sigma_n^{\alpha_n} p \to p$ holds everywhere not only for $n \in \mathbb{N}_{\alpha_n,q}$.

Now, fix $\eta, \epsilon > 0$, $f \in L^1[0,1)$. Let p be a one dimensional Walsh–Kaczmarz polynomial such that

$$\|f-p\|_1 < \eta.$$

Since from (I) the maximal operator $\sup_{n\in\mathbb{N}_{\alpha,q}} |\sigma_n^{\alpha_n}|$ is of weak type (L^1, L^1), we get

$$\begin{aligned} &\mu\big(\overline{\lim}_{n\in\mathbb{N}_{\alpha_n,q}} |\sigma_n^{\alpha_n} f - f| > \epsilon\big) \leq \mu\Big(\overline{\lim}_{n\in\mathbb{N}_{\alpha_n,q}} |\sigma_n^{\alpha_n}(f-p)| > \frac{\epsilon}{3}\Big) \\ &\qquad + \mu\Big(\overline{\lim}_{n\in\mathbb{N}_{\alpha_n,q}} |\sigma_n^{\alpha_n} p - p| > \frac{\epsilon}{3}\Big) + \mu\Big(\overline{\lim}_{n\in\mathbb{N}_{\alpha,q}} |p-f| > \frac{\epsilon}{3}\Big) \\ &\leq \mu\Big(\sup_{n\in\mathbb{N}_{\alpha_n,q}} |\sigma_n^{\alpha_n}(f-p)| > \frac{\epsilon}{3}\Big) + 0 + \frac{3}{\epsilon}\|p-f\|_1 \leq C_q \|p-f\|_1 \frac{3}{\epsilon} \leq \frac{C_q}{\epsilon}\eta. \end{aligned}$$

This is true for all $\eta > 0$.

Thus, we get

$$\mu(\overline{\lim}_{n\in\mathbb{N}_{\alpha_n,q}} |\sigma_n^{\alpha_n} f - f| > \epsilon) = 0,$$

for an arbitrary $\epsilon > 0$. As a result, we have

$$\mu(\overline{\lim}_{n\in\mathbb{N}_{\alpha_n,q}} |\sigma_n^{\alpha_n} f - f| > 0) = 0.$$

Finally, for all $n \in \mathbb{N}_{\alpha_n,\, q}$,

$$\mu\left\{|\sigma_n^{\alpha_n} f - f| > 0\right\} = 0.$$

Hence, the theorem follows. □

THEOREM 2.5. *The maximal operator* $\sup_{n\, \in\mathbb{N}_{\alpha_n,\, q}} |\sigma_n^{\alpha_n} f|$ *is of strong type* (H^1, L^1) *and* (L^p, L^p), *for all* $1 < p \leq \infty$.

PROOF. By combining Lemma 2.4 and Marcinkiewicz interpolation theorem of [13], it is possible to get that operator $\sup_{n\, \in\mathbb{N}_{\alpha_n,\, q}} |\sigma_n^{\alpha_n} f|$ is of type (L^p, L^p) for all $1 < p \leq \infty$. Moreover, by the σ-sublinearity of $\sup_{n\, \in\mathbb{N}_{\alpha_n,\, q}} \left|\sigma_n^{\alpha_n}\right|$ and since $\sigma_n^{\alpha_n}$ is A_k measurable for $n < 2^k$, we prove that it is of type (H^1, L^1). □

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