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GLOBAL CENTRAL LIMIT THEOREMS FOR STATIONARY MARKOV CHAINS

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Abstract. Let P be a Markov operator on a general state space (S, Σ) with an invariant probability measure m, assumed to be ergodic. We study conditions which yield that for every centered non-zero $f \in L^2(m)$ a non-degenerate annealed CLT and an L^2-normalized CLT hold.

1. Introduction

Let $P = P(x, A)$ be a Markov transition probability function on a general state space (S, Σ), with invariant probability measure m (i.e. $m(\cdot) = \int_S P(x, \cdot)dm(x)$). Let $\Omega := S^{\mathbb{N}}$ be the space of trajectories with σ-algebra $\mathcal{A} := \Sigma^{\otimes\mathbb{N}}$, and let $\mathbb{P}_x$ be the probability measure on $\mathcal{A}$ governing the chain with transition probability function P and initial distribution δ_x. The probability of the chain with initial distribution m is then $\mathbb{P}_m = \int_S \mathbb{P}_x dm(x)$. By invariance of m, $\mathbb{P}_m$ is shift invariant on $(\Omega, \mathcal{A})$. Let X_n be the projection of Ω on the nth coordinate. Then (X_n) on $(\Omega, \mathcal{A}, \mathbb{P}_m)$ is a stationary Markov chain with state space S.

For $1 \le p < \infty$ we denote by $L^p(m)$ the Banach space $\{f \colon S \to \mathbb{R} : \int_S |f|^p dm < \infty\}$, and put $L_0^p(m) = \{f \in L^p(m) : \int_S f\, dm = 0\}$.

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We assume m ergodic for P, which means (by one of the equivalent definitions) that if $f \in L^2(m)$ satisfies $f(x) = \int_S f(y)P(x,dy)$ m-a.e., then f is constant a.e. Then the chain is ergodic too, i.e. the shift θ on $(\Omega, \mathcal{A}, \mathbb{P}_m)$, defined by $\theta(X_n)_{n\in\mathbb{N}} = (X_{n+1})_{n\in\mathbb{N}}$, is ergodic.

We say that a real centered $f \in L^2_0(m)$ satisfies the *annealed* CLT if in $(\Omega, \mathbb{P}_m)$ we have

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} f(X_k) \overset{\mathcal{D}}{\to} \mathcal{N}(0,\sigma^2), \quad \text{where} \quad \mathcal{N}(0,0) := \delta_0.$$

We say that a real centered $f \in L^2_0(m)$ satisfies the L^2-*normalized* CLT if

$$\frac{1}{\sigma_n(f)}\sum_{k=1}^{n} f(X_k) \overset{\mathcal{D}}{\to} \mathcal{N}(0,1),$$

provided $\sigma_n(f) := \|\sum_{k=1}^n f(X_k)\|_{L^2(\mathbb{P}_m)} > 0$ for sufficiently large $n \in \mathbb{N}$.

We denote by P also the Markov operator defined as

$$Pf(x) := \int_S f(y)P(x,dy)$$

for every bounded measurable f and every $x \in S$. By invariance of m, P extends to all $L^1(m)$ functions, and is a contraction of all $L^p(m)$ spaces, $1 \le p \le \infty$, meaning that it does not increase the norm of functions in these spaces. As previously mentioned, ergodicity implies that $Pf = f \in L^p$ holds only for f constant. We denote by P^n the n-fold composition of the operator P, and by $Ef := \int_S f\,dm$ the expectation (with respect to the probability measure m) of $f \in L^p(m)$, $p \ge 1$.

Following the early work of Doeblin, many efforts were made to identify conditions on an ergodic Markov operator P with invariant measure m which would ensure that *every* centered $f \in L^2(m)$ satisfies the annealed CLT – an L^2-*global* annealed CLT for the chain.

2. History

Nagaev ([21]) used the following condition of Dobrushin: there exist $k \in \mathbb{N}$ and $\delta < 1$ such that

$$\sup_{x,y\in S} |P^k(x,A) - P^k(y,A)| < \delta, \quad \forall A \in \Sigma.$$

This condition implies *uniform geometric ergodicity*: $\sup_x \|P^n(x,\cdot) - m\|_{TV} \le M\rho^n$ for some $M > 0$ and $0 < \rho < 1$. But the latter condition implies $\|P^n - E\|_\infty \to 0$, which turns out to be equivalent to Doeblin's condition;

see [24, p. 213]. Ibragimov ([18]) used a strong mixing condition (φ-mixing), which also turns out to imply Doeblin's condition. Davydov ([9], [10]) constructed a positive recurrent aperiodic chain with countable state space such that the CLT fails for some centered $f \in L^2(m)$.

THEOREM 1 (M. Rosenblatt, [24]). *If* $\|P^n - E\|_2 \to 0$, *then every centered* $f \in L^2(m)$ *satisfies the annealed CLT.*

Rosenblatt proved that his condition is equivalent to ρ-mixing of the chain, and gave examples that it yields neither $\|P^n - E\|_\infty \to 0$ nor $\|P^n - E\|_1 \to 0$, although each of these conditions implies it; but $\|P^n - E\|_2 \to 0$ *if and only if* $\|P^n - E\|_p \to 0$ *for some (every)* $1 < p < \infty$. Importantly, Rosenblatt's condition does not necessarily imply Harris recurrence, see an example below.

EXAMPLE (Random walks on the unit circle $\mathbb{T}$). Let μ be a probability measure on $\mathbb{T}$, and define the convolution operator $Pf = \mu * f$, $f \in L^1(\mathbb{T}, m)$, m the normalized Haar (Lebesgue) measure. It is shown in [11] that if $\lim_{|k|\to\infty} \hat{\mu}(k) = 0$, that is, the Fourier transform of μ vanishes at infinity (i.e. μ is Rajchman), then $\|P^n - E\|_2 \to 0$. When μ is Rajchman with all its powers singular with respect to Lebesgue measure, P is not Harris recurrent.

A contraction T on a Banach space $\mathcal{X}$ is called *uniformly ergodic* if $\frac{1}{n}\sum_{k=1}^n T^k$ converges in the operator norm. The limit is a projection onto $Fix(T) := \{f \in \mathcal{X} : Tf = f\}$ corresponding to the decomposition $\mathcal{X} = Fix(T) \oplus \overline{(I-T)\mathcal{X}}$. A contraction T is uniformly ergodic if and only if $(I-T)\mathcal{X}$ is closed in $\mathcal{X}$ ([20]).

When P is uniformly ergodic in $L^2(m)$, we have $L^2_0(m) = (I-P)L^2(m) = (I-P)L^2_0(m)$. (Recall that $L^2_0(m) := \{f \in L^2 : Ef = 0\}$). If $\|P^n - E\|_2 \to 0$, then P is uniformly ergodic on $L^2(m)$; moreover, the spectral radius $r(P_{|L^2_0(m)}) < 1$, meaning P has a *spectral gap* in the complex $L^2_0(m)$.

THEOREM 2 (Gordin-Lifshits, [15]). *Let P be a Markov operator with invariant probability measure m, and assume that P is ergodic.*

If $f \in (I-P)L^2(m)$, then f satisfies the annealed CLT, with

$$\sigma^2 = \sigma_f^2 := \lim_{n\to\infty} \frac{1}{n} \|\sum_{k=1}^n f(X_k)\|_2^2 = \|g\|^2 - \|Pg\|^2,$$

where $f = (I-P)g$ with $g \in L^2_0(m)$.

When $\sigma_f^2 > 0$ (which is the case when P^*P is ergodic), f satisfies also the L^2-normalized CLT, which follows from a theorem of Slutsky ([25]) (see [8, p. 254]).

By [7], $f \in (I-P)L^2(m)$ if and only if $\sup_n \|\sum_{k=1}^n P^k f\|_2 < \infty$.

Theorem 1 now follows from Corollary 3 below.

Corollary 3. *Let P be a Markov operator with invariant probability measure m, and assume that P is uniformly ergodic in $L^2(m)$ with limit equal to E. Then every $f \in L^2_0(m)$ satisfies the annealed CLT.*

Note that uniform ergodicity does not necessarily imply Harris recurrence.

Problem 1. *Let P be a Markov operator with invariant probability measure m, and assume that P is ergodic. If every $f \in L^2_0(m)$ satisfies the annealed CLT, does it follow that P is uniformly ergodic in $L^2(m)$?*

3. Some ergodic properties

Theorem 4 (Derriennic-Lin, [11]). *Let P be a Markov operator with invariant probability measure m, and assume P is ergodic. Then the following conditions are equivalent:*

(i) *P is uniformly ergodic in $L^2(m)$.*
(ii) *For every $f \in L^2_0(m)$ we have $\sup_{n\ge 1} \|\frac{1}{n}\sum_{k=1}^n f(X_k)\|^2_{L^2(\mathbb{P}_m)} < \infty$.*
(iii) *For every $f \in L^2_0(m)$ we have $\sup_{n\ge 1} \|\frac{1}{\sqrt{n}}\sum_{k=1}^n P^k f\|_2 < \infty$.*
(iv) *For every $f \in L^2_0(m)$ we have $\sup_{n\ge 1} |\sum_{k=1}^n \langle P^k f, f\rangle| < \infty$.*

Note that P is a contraction also of each complex $L^p(m)$ space, $1 \le p \le \infty$, and it is uniformly ergodic in the complex $L^p(m)$ iff it is uniformly ergodic in the real $L^p(m)$. A similar statement holds also for norm convergence of P^n.

Theorem 5. *Let P be a Markov operator with invariant probability measure m. If P is uniformly ergodic on $L^p(m)$, $1 \le p < \infty$, and is weakly mixing on the complex $L^p(m)$ (the only unimodular eigenvalue of P is 1), then $\|P^n - E\|_p \to 0$.*

The proof primarily relies on positivity and ergodicity.

Lemma 6. *If P^*P is ergodic, then for every $f \in L^2_0(m)$ we have $P^n f \to 0$ weakly in $L^2(m)$; thus the shift θ on $(\Omega, \mathcal{A}, \mathbb{P}_m)$ is weakly mixing, hence totally ergodic (all powers θ^k are ergodic). Moreover, $\|(P^*P)^n f\|_2 \to 0$ for every $f \in L^2_0(m)$ if and only if P^*P is ergodic.*

Proof. We assume that P^*P is ergodic. Let $\mathcal{K}$ be the unitary space of P:

$$\mathcal{K} := \{g \in L^2(m) : \|P^n g\|_2 = \|P^{*n} g\|_2 = \|g\|_2 \quad \text{for every} \quad n \ge 1\}.$$

Clearly $\|Pg\|_2^2 = \|g\|_2^2$ if and only if $\langle P^*Pg, g\rangle = \|g\|_2^2$. Hence, by the Cauchy-Schwarz inequality, $g \in \mathcal{K}$ implies $P^*Pg = g$, and the ergodicity of P^*P implies that $\mathcal{K}$ contains only the constant functions. Any f centered is therefore orthogonal to $\mathcal{K}$, and by [13] both $P^n f \to 0$ and $P^{*n} f \to 0$ weakly in $L^2(m)$. Thus P is weakly mixing.

The weak mixing of P implies that the shift θ is weakly mixing; see [1, Section 2].

The operator P^*P is symmetric positive semi-definite in the complex $L^2(m)$, so its spectrum is a subset of $[0,1]$. If P^*P is ergodic, then for centered $f \in L^2(m)$ we have $\|(P^*P)^n f\|_2 \to 0$ by the spectral theorem.

Conversely, if $\|(P^*P)^n f\|_2 \to 0$ for every centered $f \in L^2(m)$, then obviously P^*P is ergodic. □

LEMMA 7. *Let the shift θ be totally ergodic on $(\Omega, \mathcal{A}, \mathbb{P}_m)$, which is the case when P^*P is ergodic. If $f \neq 0$ belongs to $L^2_0(m)$, then $\sigma_n(f) > 0$ for every $n \geq 1$.*

PROOF. By stationarity of the chain (X_n), $\sigma_n(f) = 0$ implies

$$\|\sum_{k=0}^{n-1} f(X_k)\|_{L^2(\mathbb{P}_m)} = 0,$$

so

$$\begin{aligned} f(X_0) \circ \theta^n - f(X_0) &= f(X_n) + \Big[\sum_{k=0}^{n-1} f(X_k)\Big] - f(X_0) \\ &= \sum_{k=1}^{n} f(X_k) = \Big[\sum_{k=0}^{n-1} f(X_k)\Big] \circ \theta = 0. \end{aligned}$$

By ergodicity of θ^n, $f(X_0)$ is a constant, which is zero since f is centered. □

4. Global central limit theorems

THEOREM 8. *Let P be a Markov operator with invariant probability measure m. If P^*P is ergodic and P is uniformly ergodic, then $\|P^n - E\|_2 \to 0$, and every centered $0 \neq f \in L^2(m)$ satisfies a non-degenerate annealed CLT and the L^2-normalized CLT.*

Moreover, if $0 \neq f \in L^3(m)$ is centered, then

$$\sup_{t\in\mathbb{R}} \Big|\mathbb{P}_m\Big\{\frac{\sum_{k=1}^n f(X_k)}{\sigma_f\sqrt{n}} \leq t\Big\} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^t e^{-x^2/2}dx\Big| = O\Big(\frac{1}{\sqrt{n}}\Big). \tag{1}$$

PROOF. Ergodicity of P^*P implies ergodicity of P, by Lemma 6. The assumption of uniform ergodicity implies that every $f \in L^2_0(m)$ is of the form $f = (I - P)g$ with $g \in L^2(m)$ centered.

Fix $0 \neq f = (I-P)g$ with $g \in L^2(m)$ centered. By the Gordin-Lifshits CLT, the annealed CLT holds for f, with variance of the limit expressed as

$$\sigma^2 = \sigma_f^2 = \lim_{n\to\infty} \Big\| \frac{1}{\sqrt{n}} \sum_{k=1}^n f(X_k) \Big\|^2_{L^2(\mathbb{P}_m)} = \|g\|_2^2 - \|Pg\|_2^2, \quad g \in L_0^2(m).$$

Hence $\sigma_f = 0$ if and only if $P^*Pg = g$. If $\sigma_f = 0$, then g is constant by the ergodicity of P^*P. Since g is centered, $\sigma_f = 0$ implies $g = 0$, so $f = 0$.

By Lemma 6 the shift is totally ergodic, so Lemma 7 yields $\sigma_n(f) > 0$ for $n \geq 1$. Thus, for centered $f \neq 0$ we have $n^{-1/2}\sigma_n(f) \to \sigma_f > 0$, so the annealed CLT implies the L^2-normalized CLT, by Slutsky's theorem [25].

Ergodicity of P^*P implies weak mixing of P (Lemma 6), so uniform ergodicity yields $\|P^n - E\|_2 \to 0$ (by Theorem 5). For $0 \neq f \in L^3(m)$ centered $\sigma_f > 0$ as shown above, and (1) holds by [17]. □

Corollary 9. *Let P be a Markov operator with invariant probability measure m, and assume that P is ergodic and uniformly ergodic. Every centered $0 \neq f \in L^2(m)$ satisfies a non-degenerate annealed CLT if and only if P^*P is ergodic.*

Proof. When P^*P is ergodic Theorem 8 applies. For the converse, if $P^*Pg = g$ for non-constant $g \in L^2(m)$, then $P^*P(g - Eg) = g - Eg$, and $f = (I-P)(g-Eg) \neq 0$ satisfies the CLT with $\sigma_f = 0$. □

Proposition 10. *Let P be a Markov operator with invariant probability measure m, and assume that P is normal in $L^2(m)$, i.e. $P^*P = PP^*$. If $\|P^n - E\|_2 \to 0$, then P^*P is ergodic, and Theorem 8 applies.*

Proof. Let $P^*Pg = g \in L^2(m)$. Since $P^*E = E$, normality yields $\|g - Eg\|_2 = \|(P^*P)^n g - Eg\|_2 = \|P^{*n}P^n g - P^{*n}Eg\|_2 \leq \|P^n g - Eg\|_2 \to 0$. □

Example. In general, $\|P^n - E\|_2 \to 0$ does not imply that P^*P is ergodic.

Let us define P on $S := \{1,2,3\}$ by the matrix $\left[\frac{1}{2}\ \frac{1}{2}\ 0 \parallel 0\ 0\ 1 \parallel \frac{1}{2}\ \frac{1}{2}\ 0\right]$. The invariant probability vector is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and P^* is given by the adjoint matrix. P has no non-trivial invariant sets, its only unimodular eigenvalue is 1, but P^*P is not ergodic.

Problem 2. *If a Markov operator P is ergodic, and every centered nonzero $f \in L^2(m)$ satisfies a non-degenerate annealed CLT, does $\|P^n - E\|_2 \to 0$?*

Note that P^*P is ergodic (proof of Corollary 9), so P is weakly mixing.

Below we present a sufficient "moment improving" condition for uniform ergodicity (called *hyperboundedness*); this condition is sometimes easy to check.

THEOREM 11 (Glück, [14]). *Let P be a Markov operator with invariant probability measure m, assumed to be ergodic. Assume that for some $1 \leq s < r < \infty$ we have $P L^s(m) \subset L^r(m)$. Then P is uniformly ergodic in all $L^p(m)$ spaces, $1 < p < \infty$ (i.e. $\left\|\frac{1}{n}\sum_{k=1}^{n} P^k - E\right\|_p \to 0$); hence (by Corollary 3) every centered $f \in L^2(m)$ satisfies the annealed CLT.*

EXAMPLE (A hyperbounded Markov operator). Let (S, m) be the unit circle with normalized Lebesgue measure. Let $0 \leq g \in L^2(m)$ with $\int g\,dm = 1$, and define P by $Pf = g * f$. Then m is invariant, P is ergodic and normal in $L^2(m)$. Since $\|Pf\|_2 = \|g * f\|_2 \leq \|g\|_2\|f\|_1$ for $f \in L^1(m)$, P maps $L^1(m)$ into $L^2(m)$.

PROPOSITION 12 (Becker, [2])). *A power-bounded operator T (i.e. $\sup_{n\geq 0}\|T^n\| < \infty$) on a Banach space $\mathcal{X}$ is uniformly ergodic if and only if for every $f \in \overline{(I-T)\mathcal{X}}$ the series $\sum_{n\geq 1} n^{-1}T^n f$ converges in $\mathcal{X}$.*

PROPOSITION 13. *Let P be a Markov operator with invariant probability measure m, assumed to be ergodic. Then the following conditions are equivalent:*

(i) *The Markov chain is ρ-mixing*[1].
(ii) $\|P^n - E\|_2 \to 0$.
(iii) *For every $f \in L_0^2(m)$ the series $\sum_{k=1}^{\infty}\langle P^k f, f\rangle$ converges.*
(iv) *For every $f \in L_0^2(m)$ we have $\sum_{n=1}^{\infty}\|P^n f\|_2^2 < \infty$.*
(v) *There exists $1 \leq p < \infty$ such that for every $f \in L_0^p(m)$ there exists $r > 1$ with $\sum_{n=1}^{\infty}\|P^n f\|_p^r < \infty$.*

If either of the above conditions holds, then the annealed CLT holds for every $f \in L_0^2(m)$. The variance of the limiting normal distribution is

$$\sigma_f^2 = \|f\|_2^2 + 2\sum_{k=1}^{\infty}\langle P^k f, f\rangle.$$

PROOF. The equivalence of (i) and (ii) is by [24, p. 207].

By [11, Proposition 3.1], condition (ii) is equivalent to the existence of $\rho < 1$ and $M > 0$ such that $\|P^n - E\|_2 \leq M\rho^n$ for $n \geq 1$. This yields (iii) and (iv).

(iii) implies uniform ergodicity, by Theorem 4. By [13, Lemma 2.1], (iii) implies $P^n f \to 0$ weakly in $L^2(m)$ for every $f \in L_0^2(m)$; hence P is weakly mixing. Now (ii) holds by Theorem 5.

Obviously (iv) implies (v) with $p = 2$.

[1] See definition, as "asymptotically uncorrelated", in [24, pp. 206–207].

If (v) holds, then for every centered $f \in L^p(m)$, Hölder's inequality, applied with $s = r/(r-1)$, yields

$$\sum_{n=1}^{\infty} \frac{\|P^n f\|_p}{n} \le \Big(\sum_{n=1}^{\infty} \frac{1}{n^s}\Big)^{\frac{1}{s}} \Big(\sum_{n=1}^{\infty} \|P^n f\|_p^r\Big)^{\frac{1}{r}} < \infty.$$

Hence the series $\sum_{n=1}^{\infty} \frac{P^n f}{n}$ is convergent in L^p-norm when $f \in L^p(m)$ is centered. By Becker's Proposition 12, P is then uniformly ergodic in $L^p(m)$. Since condition (v) implies that P has no unimodular eigenvalues, we have $\|P^n - E\|_p \to 0$ (by Theorem 5), and by [24, Theorem VII.4.1] (ii) holds.

Finally, (ii) implies the CLT statement by Theorem 1. By Theorem 2 the variance of the limit is $\lim_{n\to\infty} \sigma_n(f)^2/n$. □

Proposition 14. *Let P be a Markov operator with invariant probability measure m. If every $0 \ne f \in L^2_0(m)$ satisfies the L^2-normalized CLT, then P^*P is ergodic. Consequently (Lemma* 6 *and Theorem* 5*), if P is uniformly ergodic, then $\|P^n - E\|_2 \to 0$.*

5. α-mixing

Rosenblatt in [24] introduced a certain "strong mixing" condition, now called *α-mixing*, and proved that for the stationary chain generated by P with invariant probability measure m, α-mixing is equivalent to

$$4\alpha(n) := \sup_{\int f\,dm = 0} \frac{\|P^n f\|_1}{\|f\|_\infty} \to 0 \quad \text{as} \quad n \to \infty.$$

The above supremum is bounded by $\|P^n - E\|_2$, so ρ-mixing implies α-mixing. Clearly α-mixing implies $\|P^n g - Eg\|_2 \to 0$ for every $g \in L^2(m)$, hence total ergodicity of the shift θ.

A stationary Markov chain which is Harris recurrent and aperiodic is α-mixing; see [4, Section 3.2].

Theorem 15. *Let P be a Markov operator with invariant probability measure m, and assume that the chain is α-mixing. If every $0 \ne f \in L^2_0(m)$ satisfies the L^2-normalized CLT, then P^*P is ergodic, every $0 \ne f \in L^2_0(m)$ satisfies a non-degenerate annealed CLT, and $\|P^n - E\|_2 \to 0$.*

Proof. By Proposition 14 P^*P is ergodic, so the shift is totally ergodic. Hence for $0 \ne f \in L^2_0(m)$, $\sigma_n(f) > 0$ for every $n \ge 1$, by Lemma 7.

Let $\gamma \in (0,1)$ be fixed. Fix $0 \ne f \in L^2_0(m)$, and put $\sigma_n = \sigma_n(f)$. Since the chain is α-mixing, the stationary sequence $\{f(X_j)\}$ is also α-mixing. By a result in [19], the L^2-normalized CLT implies that there exists a function

$L(t)$, $t > 0$, slowly varying at ∞, such that $\sigma_n^2 = nL(n)$. By a property of slowly varying functions, we obtain $n^{-(\gamma+1)}\sigma_n^2 = n^{-\gamma}L(n) \to 0$. Then

$$\frac{1}{n^{(\gamma+1)/2}}\Big\|\sum_{k=1}^n P^k f\Big\|_2 \le \frac{1}{n^{(\gamma+1)/2}}\Big\|\sum_{k=1}^n f(X_k)\Big\|_{L^2(\mathbb{P}_m)} = n^{-(\gamma+1)/2}\sigma_n \to 0.$$

The above convergence holds for every $f \in L_0^2(m)$. Denoting $\epsilon = (1-\gamma)/2$, we apply it to $f = g - Eg$, $g \in L^2(m)$, to obtain

$$n^\epsilon \Big\|\frac{1}{n}\sum_{k=1}^n P^k g - Eg\Big\|_2 = \frac{1}{n^{(\gamma+1)/2}}\Big\|\sum_{k=1}^n P^k(g - Eg)\Big\|_2 \le C_g \quad \forall n \ (g \in L^2(m)).$$

By the Banach-Steinhaus theorem, the norms $\big\{n^\epsilon\big\|\frac{1}{n}\sum_{k=1}^n P^k - E\big\|_2\big\}$ are bounded, so $\big\|\frac{1}{n}\sum_{k=1}^n P^k - E\big\|_2 \le \frac{K}{n^\epsilon} \to 0$. Thus P is uniformly ergodic. Theorem 8 yields $\|P^n - E\|_2 \to 0$ and the non-degenerate annealed CLT for every $0 \ne f \in L_0^2(m)$. □

Theorem 16. *Let P be an ergodic Markov operator with invariant probability measure m. Then the following conditions are equivalent:*

(i) $\|P^n - E\|_2 \to 0$ *and P^*P is ergodic.*

(ii) *The chain is α-mixing and every $0 \ne f \in L_0^2(m)$ satisfies the L^2-normalized CLT.*

(iii) *Every $0 \ne f \in L_0^2(m)$ satisfies a non-degenerate annealed CLT and the L^2-normalized CLT.*

Proof. (i) implies (ii) follows from Theorem 8 and the fact that ρ-mixing implies α-mixing (combined with Proposition 13).

(ii) implies (i): Indeed, P^*P is ergodic by Proposition 14, and $\|P^n - E\|_2 \to 0$ by Theorem 15.

(i) implies (iii) by Theorem 8.

(iii) implies (i): First of all, P^*P is ergodic by Proposition 14. Further, fix $0 \ne f \in L_0^2(m)$. We shall prove that $\{\sigma_n(f)/\sqrt{n}\}$ is bounded. For the sake of contradiction, suppose it is not bounded. Then there is an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\sqrt{n_k}/\sigma_{n_k}(f)$ converges to zero, whence

$$\frac{1}{\sigma_{n_k}(f)}\sum_{j=1}^{n_k} f(X_j) = \frac{\sqrt{n_k}}{\sigma_{n_k}(f)} \cdot \frac{1}{\sqrt{n_k}}\sum_{j=1}^{n_k} f(X_j). \tag{2}$$

The left-hand side of (2) converges in distribution to $\mathcal{N}(0,1)$ by the assumption of the L^2-normalized CLT for f; the right-hand side converges to $\mathcal{N}(0,0)$, by the assumed annealed CLT for f and Slutsky's theorem, leading to a contradiction. Hence $\{\sigma_n(f)/\sqrt{n}\}$ is bounded for every $f \in L_0^2(m)$. By Theorem 4, P is uniformly ergodic. By Proposition 14, P^*P is ergodic, so P is weakly mixing by Lemma 6, and then $\|P^n - E\|_2 \to 0$ by Theorem 5. □

Problem 3. *Assume that P is a Makov operator with invariant probability measure m such that*

$$\lim_{n\to\infty} \|P^n g - Eg\|_2 = \lim_n \|P^{*n} g - Eg\|_2 = 0 \qquad \text{for every} \quad g \in L^2(m),$$

and assume that every non-zero $f \in L^2_0(m)$ satisfies the L^2-normalized CLT. Does it follow that P is uniformly ergodic in $L^2(m)$?

If yes, then $\|P^n - E\|_2 \to 0$ by Theorem 5, since P is weakly mixing by the strong convergence of P^n. Note that the assumption implies that P^*P is ergodic, by Proposition 14.

By Theorem 15, the answer is *yes* for P which is Harris recurrent and aperiodic.

Example (P not uniformly ergodic with (P^*P) ergodic). Let Q be ergodic with invariant probability measure m which is not uniformly ergodic. For $\varepsilon \in (0,1)$ define $P = P_\varepsilon := \varepsilon I + (1-\varepsilon)Q$. We shall prove that P^*P is ergodic. Clearly m is invariant also for P and for P^*P. For $A \in \Sigma$ we have

$$P^*P1_A = \varepsilon^2 1_A + \varepsilon(1-\varepsilon)(Q^*1_A + Q1_A) + (1-\varepsilon)^2 Q^*Q1_A.$$

If $P^*P1_A = 1_A$ a.e., then for almost every $x \notin A$ the above summands are zero, so in particular $Q1_A \le 1_A$ a.e. Since m is invariant, $Q1_A = 1_A$, and A is trivial by the ergodicity of Q. By definition $(I-P)L^2(m) = (I-Q)L^2(m)$, so when Q is not uniformly ergodic $(I-P)L^2(m)$ is not closed; hence P is not uniformly ergodic.

6. Geometric ergodicity

Definition. A Markov operator P with invariant probability measure m is called *geometrically ergodic* if, for some $\rho < 1$,

$$M_x := \sup_n \rho^{-n} \|P^n(x,\cdot) - m\|_{TV} < \infty \quad \text{a.e.}$$

Geometric ergodicity implies aperiodic Harris recurrence and α-mixing, with the α-mixing coefficients $\alpha(n)$ converging to 0 exponentially fast; see [4, Section 3.2].

Theorem 17 (Doukhan-Massart-Rio, [12]). *Let Σ be countably generated and let P be a geometrically ergodic Markov operator. Then any centered f with $\int |f|^2 \log^+ |f|\, dm < \infty$ satisfies the annealed CLT.*

Theorem 18 (Roberts-Tweedie, [23]). *Let Σ be countably generated, and let P be a Harris positive recurrent Markov chain. If $\|P^n - E\|_2 \to 0$, then P is geometrically ergodic.*

Note that $\|P^n - E\|_2 \to 0$ does not necessarily imply Harris recurrence; therefore Harris recurrrence must be assumed.

NOTE. The converse may fail – in [3] and [16] are examples of P geometrically ergodic with some centered $f \in L^2(m)$ which does not satisfy the annealed CLT, so $\lim_{n\to\infty} \|P^n - E\|_2 > 0$.

THEOREM 19. *Let P be a Markov operator with invariant probability measure m, and assume that P is normal in $L^2(m)$. Then $\|P^n - E\|_2 \to 0$ if (and only if) the α-mixing coefficients converge to zero (at least) exponentially fast.*

Bradley ([5]) proved the theorem when P is symmetric.

In general, if P is geometrically ergodic, then P is Harris aperiodic and the α-mixing coefficients converge to zero exponentially fast. We do not know if a Harris aperiodic P whose α–mixing coefficients converge to zero exponentially fast is geometrically ergodic.

COROLLARY 20. *Let Σ be countably generated. If a Markov operator P is geometrically ergodic, and is additionally normal in $L^2(m)$, then*

$$\|P^n - E\|_2 \to 0.$$

The symmetric case is in [22]. For S countable Corollary 20 is established in [26].

REMARKS.

1. P in Theorem 19 need not be Harris recurrent.
2. When Σ is countably generated and P is Harris recurrent and normal in $L^2(m)$, Theorems 18 and 19 yield that exponential decay to 0 of $\alpha(n)$, geometric ergodicity and ρ-mixing are equivalent.

In Bradley's and Häggström's examples P is geometrically ergodic, and every centered $f \in L^p(m)$, $p > 2$, satisfies the CLT, by Theorem 17; however, P does not have a spectral gap in $L^p(m)$, i.e. $\lim_{n\to\infty} \|P^n - E\|_p > 0$, since otherwise it would imply $\lim_{n\to\infty} \|P^n - E\|_2 = 0$ ([24]), and so the CLT for every centered $f \in L^2(m)$. By Corollary 20, P in such examples cannot be normal in $L^2(m)$.

The examples of Bradley and Häggström show that without normality Theorem 19 fails, although we have geometric ergodicity.

PROBLEM 4. *Let P be a Harris aperiodic Markov chain, and suppose that every centered f such that $\int |f|^2 \log^+ |f|\, dm < \infty$ satisfies the annealed CLT. Does this imply that P is geometrically ergodic? (Is a converse of Theorem 17 true?).*

Dedecker informed the author that an example of Bradley ([6]) exhibits P Harris recurrent which is *not* geometrically ergodic, such that every

$f \in L_0^p(m)$, $p > 2$, satisfies the annealed CLT. In Problem 4 we (necessarily) assume more, i.e. that the annealed CLT is satisfied by a strictly larger subset of $L_0^2(m)$.

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