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A NOTE ON ABELIAN GROUPS SUPPORTING UNITAL RINGS

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Abstract. Non-nil abelian groups are classified on which every ring, different from the zero-ring, is unital. It is shown that the assumption on the associativity of the considered rings does not influence the obtained classification. A significant mistake made by other authors studied this topic is corrected.

1. Introduction and motivations

One of the interesting issues studied by algebraists is how the additive structure of a ring influences its multiplication. It has a rich history in abstract algebra. Its starting point can be dated back to the end of the first half of the 20^{th} century when Beaumont published the paper on rings supported by direct sums of cyclic groups (see [4]). Shortly thereafter Fuchs, Redei, Szele, Zuckerman, Ree, and Wisner joined in the investigation of additive groups of rings which resulted in the next valuable papers. Many of their achievements can be found in, today already classic, two-volume monograph 'Infinite abelian groups' written by L. Fuchs (see, [10, Chapter XVII]). Subsequent results related to the topic were closely connected to the progress in abelian

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group theory. Further research, conducted with the momentous contribution of Feigelstock, led to the monograph [6] and its complement [7]. Currently, additive groups of rings are generating renewed interest, and consequently, papers concerning this subject appear quite regularly (see, e.g., [1–3, 11–16]). In particular, there is the problem of classifying abelian groups supporting rings belonging only to some fixed class. The class of unital rings is a very well motivated example of them. Probably the most classical results on the abelian groups supporting unital rings are presented as [6, Theorems 4.1.1 & 4.1.3] and [10, Proposition 120.8]. The research on such groups were continued in [5] by Breaz and Călugăreanu who investigated, among others, the structure of non-nil groups A such that every associative ring on A, different from the zero-ring A^0 on A, is unital. Abelian groups with this property were called S-identity groups. Recall that an abelian group is said to be non-nil if it supports any non-zero ring structure.

However, the mentioned authors did not avoid some essential gap in their reasoning related to switching from the ring-theoretical direct sums to the group-theoretical ones, and to the best of our knowledge, their paper is available only as a preprint on the website http://math.ubbcluj.ro/~calu/ident.pdf. More precisely, before [5, Theorem 1] they wrote: 'Since a finite direct (product) sum of rings has identity if and only if each component has identity (the finite decomposition of 1 into central idempotents), the ring obtained by any trivial extension has no identity. Hence S-identity groups are indecomposable'. But it is easily seen that the proposed arguments are insufficient to infer that the S-identity groups are indecomposable. For example, they do not work for decomposable non-nil abelian groups such that all their direct summands are nil groups. Indeed, let A and B be subgroups of the full rational group generated by the inverses of all primes and by the inverses of squares of all primes, respectively. Then A and B are torsion-free rank one abelian groups of non-idempotent types, so it follows from [10, Theorem 121.1] that A and B are both nil. Thus $A^0 \times B^0$ is the only ring with the additive group $A \oplus B$ that can be obtained by trivial extensions. Since $A \cdot A \subseteq B$, the multiplication $(a_1, b_1) * (a_2, b_2) = (0, a_1 \cdot a_2)$ defined for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, provides a non-zero associative ring structure on the group $A \oplus B$, which is clearly non-unital. This shows that the trivial extension argument is insufficient to infer that $A \oplus B$ is not an S-identity group; to deduce this, it is necessary to focus on the ring structure defined from the beginning on the entire group $A \oplus B$. In particular, this is closely related to the fact that neither A nor B can be an ideal I satisfying $I^2 \neq 0$ in any ring R with the additive group $A \oplus B$. Consequently, the proof of [5, Theorem 1] is incomplete and partially incorrect.

The aim of this note is to remove the mentioned inconsistency and prove [5, Theorem 1] correctly for the case of associative as well as not necessarily associative rings. We start with the second case to introduce the notion of

an SU-group understood as a non-nil group G such that G^0 is the only not unital ring on G. In particular, the prefix SU is an abbreviation from *strongly unital*. Since the concepts of SU-group and S-identity group will turn out to be equivalent (see Remark 2.3) as well as the definition of an SU-group is slightly less complicated than that of an S-identity group (due to the lack of the assumption on the associativity of rings) and the prefix SI naturally associated with the second of these notions has been previously assigned to the abelian groups supporting only rings whose subrings are ideals (see, e.g., [2,8]), this new name seems to be more appropriate.

The symbols $Z(p^\infty)$, $Z(n)$, $\mathbb{Z}_n$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N}$, and $\mathbb{P}$ stand for the quasi-cyclic p-group, finite cyclic group of order n, the rings of integers modulo n and integers, the field of rationals, and the sets of all positive integers and primes, respectively. The multiplication of the ring $\mathbb{Z}_n$ is denoted by $\odot_n$. Rings are not assumed to be associative, commutative, or unital. The additive group of a ring R is denoted by R^+. Throughout this paper all groups are abelian and written additively. For any abelian group A, the symbol $\mathbb{P}(A)$ stands for the set of all primes p for which the p-component A_p of A is non-trivial. In particular, the torsion part of A, denoted by $T(A)$, is the direct sum of p-components of A with p running over $\mathbb{P}(A)$. The notations $\mathcal{D}(A)$ and $\mathrm{Mult}(A)$ are used for the divisible hull of A (see [9, p. 107]) and the group of all ring multiplications on A, respectively. If a is an element of A, then the order of a is denoted by $o(a)$. All other designations are consistent with generally accepted standards (see, e.g., [9, 10]).

2. The classification of SU-groups and its consequences

In the classification of SU-groups, some specific ring multiplications constructed in [1, proofs of Theorems 3.1 & 3.6] will play an important role. Due to the quite technical character of these constructions and the need of some additional observations related to them, they will be repeated in a little bit simplified form and complemented in the proof of Theorem 2.2. For the transparency and completeness of this note, the following slight improvement of [1, Lemma 2.4] is also presented with the complete proof.

LEMMA 2.1. *Let p be a prime. If f is an epimorphism of $\mathbb{Q}^+$ onto $Z(p^\infty)$, and A is a non-trivial p-divisible subgroup of $\mathbb{Q}^+$, then $f(A) = Z(p^\infty)$.*

PROOF. Suppose, contrary to our claim, that $f(A) = \{0\}$. Take any $x \in \mathbb{Q}^+$, and define $m = \min\{n \in \mathbb{N}\colon nx \in A\}$. Then $mx = pa$ for some $a \in A$. If $p \mid m$, then there exists $r \in \mathbb{N}$ such that $mx = p(rx)$. Moreover, $T(A) = \{0\}$,

so $a = rx$, contrary to the minimality of m. Thus $p \nmid m$. Next, $mf(x) = f(mx) = 0$, whence $o(f(x)) \mid m$. But $f(A)$ is a p-group and $p \nmid m$, so $o\big(f(x)\big) = 1$. Therefore, $f(\mathbb{Q}^+) = \{0\}$, a contradiction. Hence $f(A)$ is a non-trivial subgroup of $Z(p^\infty)$. Furthermore, the group $f(A)$ is p-divisible, so it is divisible, and consequently, $f(A) = Z(p^\infty)$. □

THEOREM 2.2. *An abelian group A is an SU-group if and only if either $A \cong \mathbb{Q}^+$ or $A = Z(p)$ for some prime p.*

PROOF. Take any $p \in \mathbb{P}$. It is a well-known fact that the fields $\mathbb{Q}$ and $\mathbb{Z}_p$ are the only up to isomorphism rings on the groups $\mathbb{Q}^+$ and $Z(p)$ different from the zero-rings. Thus, $\mathbb{Q}^+$ and $Z(p)$ are both SU-groups. Conversely, suppose that A is an SU-group. We need to consider the following two cases:

(i) $T(A) = \{0\}$. Let $R = (A, \cdot)$ be any ring satisfying $R^2 \neq \{0\}$. Take any $n \in \mathbb{N}$, and define $\circ$ to be the n-th multiple of the multiplication of R in the group $\mathrm{Mult}(A)$. Then $(A, \circ)$ is a unital ring, so there exists $e \in A$ such that $x = x \circ e = n(x \cdot e)$ for every $x \in A$. Hence $A = nA$. Since n has been chosen arbitrarily, the non-trivial torsion-free group A is divisible. Combining this with [9, Theorem 23.1] leads to the conclusion that $A \cong \bigoplus_{i \in I} \mathbb{Q}^+$ with $I \neq \emptyset$, and consequently, there exists a subgroup H of A such that $A \cong \mathbb{Q}^+ \oplus H$. Moreover, $\mathbb{Q} \times H^0$ is a ring with the additive group isomorphic to the SU-group A, so the ring $\mathbb{Q} \times H^0$ is unital. Thus $H = \{0\}$, whence $A \cong \mathbb{Q}^+$.

(ii) $T(A) \neq \{0\}$. By way of contradiction, assume that the group $T(A)$ is divisible. Then it follows from [9, Theorem 24.5] that there exists a torsion-free subgroup B of A such that $A = T(A) \oplus B$. Since the group A is not nil, [6, Theorem 2.1.1] implies $B \neq \{0\}$.

First suppose $B \neq pB$ for some $p \in \mathbb{P}(A)$. Then $B/pB \cong \bigoplus_{i \in I} \mathbb{Z}_p^+$ with $I \neq \emptyset$. Hence there exists an epimorphism $\eta\colon A \to \mathbb{Z}_p^+$ satisfying $\eta\big(T(A)\big) = \{0\}$. As $p \in \mathbb{P}(A)$, there is $a_0 \in A$ such that $o(a_0) = p$. An easy computation shows that the function $*\colon A \times A \to A$ given by

$$a_1 * a_2 = \big(\eta(a_1) \odot_p \eta(a_2)\big)a_0 \quad \text{for all } a_1, a_2 \in A, \tag{1}$$

provides a ring structure on A. As $\eta(A) = \mathbb{Z}_p^+$ and $o(a_0) = p$, we have $A * A = \langle a_0 \rangle \cong \mathbb{Z}_p^+$, whence $A * A \subsetneq A$. Thus the ring $(A, *)$ is different from A^0 and not unital. Consequently, A is not an SU-group, a contradiction.

Now assume that $B = pB$ for each $p \in \mathbb{P}(A)$, and fix any $q \in \mathbb{P}(A)$. In view of [9, Theorems 23.1 & 24.1], there is no loss of generality in assuming that $\mathcal{D}(B) = Q \oplus E$, where $Q = \mathbb{Q}^+$ and E is some subgroup of $\mathcal{D}(B)$. Define $G = B \cap Q$. Then $G \neq \{0\}$ by [9, Lemma 24.3 & Theorem 24.4]. Moreover, $G = qG$, because B and Q are both torsion-free and q-divisible. Next, let $\psi\colon \mathcal{D}(B) \to Q$ be a natural epimorphism. Since $Q/\mathbb{Z}^+ \cong \bigoplus_{p \in \mathbb{P}} Z(p^\infty)$, there

exists an epimorphism $\xi\colon Q \to Z(q^\infty)$. A straightforward verification shows that the function $\vartheta\colon B \times B \to Z(q^\infty)$ given by

$$\vartheta(b_1, b_2) = \xi(\psi(b_1) \cdot \psi(b_2)) \quad \text{for all } b_1, b_2 \in B, \tag{2}$$

is bilinear. Take any $g \in G \setminus \{0\}$. As $G \subseteq \mathbb{Q}^+$, we get $g \cdot G \cong G$. Thus $g \cdot G$ is a non-trivial q-divisible subgroup of $\mathbb{Q}^+$. Combining this with Lemma 2.1 gives $\xi(g \cdot G) = Z(q^\infty)$. Moreover, $\psi(x) = x$ for each $x \in G$, so $\operatorname{im}\vartheta = Z(q^\infty)$. Consider the function $\circledast\colon \big(Z(q^\infty) \oplus B\big) \times \big(Z(q^\infty) \oplus B\big) \to \big(Z(q^\infty) \oplus B\big)$ given by

$$(d_1, b_1) \circledast (d_2, b_2) = \big(\vartheta(b_1, b_2), 0\big) \quad \text{for all } d_1, d_2 \in Z(q^\infty), b_1, b_2 \in B, \tag{3}$$

and define $S = \big(Z(q^\infty) \oplus B, \circledast\big)$. Since ϑ is a bilinear map with $\operatorname{im}\vartheta = Z(q^\infty)$, S is a ring satisfying $S^2 = Z(q^\infty) \oplus \{0\}$. Hence the ring S is not unital. As $q \in \mathbb{P}(A)$ and the group $T(A)$ is divisible, it follows from [9, Theorem 23.1] that $Z(q^\infty)$ is a direct summand in $T(A)$. Define $T_q = T(A)/Z(q^\infty)$ and $P = S \times T_q^0$. Then the ring P is not unital even if $T_q = \{0\}$. Moreover, $P^+ \cong A$, so A is not an SU-group, a contradiction.

Thus the group $T(A)$ is not divisible. Hence there exists $p \in \mathbb{P}(A)$ such that the group A_p is not divisible. In particular, A_p contains an element a of order p and of finite p-height (see [9, p. 98, (C)]). Of course, the p-height of a in A is the same as in A_p, so from [10, Corollary 27.2] we infer that there exist $m \in \mathbb{N}$ and a subgroup C of A such that $A = Z(p^m) \oplus C$. Since $\mathbb{Z}_{p^m} \times C^0$ is a ring with the additive group isomorphic to the SU-group A, we infer that $C = \{0\}$. Therefore, without loss of generality, we may assume that $A = \mathbb{Z}^+_{p^m}$. Suppose, contrary to our claim, that $m > 1$. Let $\star$ be the p-th multiple of $\odot_{p^m}$ in the group $\operatorname{Mult}\big(\mathbb{Z}^+_{p^m}\big)$. Then $a \star A = \{0\}$ and $A \star A \neq \{0\}$, so $(A, \star)$ is a non-zero ring without a unity, a contradiction. Hence $m = 1$, and finally, $A = Z(p)$. □

Remark 2.3. Notice that Theorem 2.2 remains true if we restrict the condition SU to the class of associative rings. Indeed, let us introduce the temporary notation SU_a for this modified condition. Retaining all the designations of the proof of Theorem 2.2 and assuming that the ring $R = (A, \cdot)$ is associative, we infer that so is the ring $(A, \circ)$. Thus if A is a torsion-free SU_a-group, then $A \cong \mathbb{Q}^+$. Next, in view of (1), the ring $(A, *)$ is commutative. Moreover, $(A * A) * A = \{0\}$ because of $a_0 \in T(A)$ and $\eta\big(T(A)\big) = \{0\}$. Consequently, $(A, *)$ is an associative ring. Since the function ϑ is symmetric, the ring S is commutative (see (2) and compare with (3)). Combining this with $\big(Z(q^\infty) \oplus \{0\}\big) \circledast S = \big\{(0, 0)\big\}$ leads to the conclusion that it is also associative, whence P is an associative ring. The associativity of the ring $(A, \star)$ follows immediately from the fact that $\mathbb{Z}_m$ is an associative ring. Therefore, if A is an SU_a-group which is not torsion-free, then $A = Z(p)$ for some $p \in \mathbb{P}$.

Thus if A is an SU_a-group, then either $A \cong \mathbb{Q}^+$ or $A = Z(p)$ for some $p \in \mathbb{P}$. The reverse implication is obvious.

Combining Theorem 2.2 with Remark 2.3 leads to the following.

COROLLARY 2.4. *For any abelian group A, the following conditions are equivalent:*

(i) *A is an SU-group;*
(ii) *either $A \cong \mathbb{Q}^+$ or $A = Z(p)$ for some prime p;*
(iii) *A satisfies the condition SU restricted to the class of associative rings;*
(iv) *A is not nil and every ring on A, different from A^0, is an integral domain;*
(v) *A is not nil and every ring on A, different from A^0, is a field.*

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