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ON MERSENNE NUMBERS AND THEIR BIHYPERBOLIC GENERALIZATIONS

Dorota Bród, Anetta Szynal-Liana

Abstract. In this paper, we introduce Mersenne and Mersenne–Lucas bihyperbolic numbers, i.e. bihyperbolic numbers whose coefficients are consecutive Mersenne and Mersenne–Lucas numbers. Moreover, we study one parameter generalizations of Mersenne and Mersenne–Lucas bihyperbolic numbers. We present some properties of these numbers and relations between them.

1. Introduction and preliminary results

Let $n \geq 0$ be an integer. The nth Mersenne number M_n and the nth Mersenne–Lucas number H_n are defined recursively by

$$M_n = 3M_{n-1} - 2M_{n-2}, \text{ for } n \geq 2 \text{ with } M_0 = 0, \; M_1 = 1$$

and

$$H_n = 3H_{n-1} - 2H_{n-2}, \text{ for } n \geq 2 \text{ with } H_0 = 2, \; H_1 = 3,$$

respectively. Note that Mersenne–Lucas numbers are also called as Fermat numbers. The Binet type formulas of these sequences have the form $M_n = 2^n - 1$ and $H_n = 2^n + 1$, so $H_n = M_n + 2$.

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Mersenne sequence has been studied in many papers, see for example [2, 3, 6, 7, 9]. In the literature, we can find some generalizations of Mersenne numbers, see [4, 10]. In [8], Ochalik and Włoch introduced the generalized Mersenne numbers as follows. Let $k \geq 3$ be a fixed integer. For any integer $n \geq 0$ let $M(k,n)$ be the nth generalized Mersenne number defined by the second order linear recurrence relation of the form

$$M(k,n) = kM(k,n-1) - (k-1)M(k,n-2) \tag{1.1}$$

for $n \geq 2$ with $M(k,0) = 0$ and $M(k,1) = 1$.

For $n = 0,1,2,3,4,\dots$ the generalized Mersenne numbers are $0,1,k$, $k^2-k+1, k^3-2k^2+2k,\dots$. Moreover, $M(3,n) = M_n$.

By analogy, we define the generalized Mersenne–Lucas numbers in the following way. Let $k \geq 3$ be a fixed integer. For any integer $n \geq 0$ let $H(k,n)$ be the nth generalized Mersenne–Lucas number defined by

$$H(k,n) = kH(k,n-1) - (k-1)H(k,n-2) \tag{1.2}$$

for $n \geq 2$ with $H(k,0) = 2$ and $H(k,1) = 3$.

Then the first few terms of the generalized Mersenne–Lucas sequence are $2,3,k+2,k^2-k+3,k^3-2k^2+2k+2,\dots$. It is easily seen that $H(3,n) = H_n$.

Proposition 1.1. *Let $k \geq 3$ be a fixed integer. For any integer $n \geq 0$ we have $H(k,n) = M(k,n) + 2$.*

Proof. (By induction on n.) If $n = 0$ then $M_0 = 0$, $H_0 = 2$. If $n = 1$ then $M_1 = 1$, $H_1 = 3$. Now assume that for any $n \geq 0$, we have $H(k,n) = M(k,n)+2$ and $H(k,n+1) = M(k,n+1)+2$. We shall show that $H(k,n+2) = M(k,n+2)+2$. Applying the induction's hypothesis we obtain

$$\begin{aligned} H(k,n+2) &= kH(k,n+1) - (k-1)H(k,n) \\ &= k\left(M(k,n+1)+2\right) - (k-1)\left(M(k,n)+2\right) \\ &= kM(k,n+1) - (k-1)M(k,n) + 2 \\ &= M(k,n+2) + 2, \end{aligned}$$

and by the induction's rule the formula follows. □

Some identities, properties, combinatorial interpretations and matrix generators of $M(k,n)$ were given in [8] and [11]. In the next part of the paper we use the following results.

THEOREM 1.2 ([8]). *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$M(k,n) = \frac{1}{k-2}\left((k-1)^n - 1\right). \tag{1.3}$$

THEOREM 1.3 ([8]). *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$M(k,n+1) - M(k,n) = (k-1)^n. \tag{1.4}$$

THEOREM 1.4 ([11]). *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$M(k,n+1) = (k-1)M(k,n) + 1. \tag{1.5}$$

Using the fact that $H(k,n) = M(k,n) + 2$, we can give some properties of generalized Mersenne–Lucas numbers.

COROLLARY 1.5. *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$H(k,n) = \frac{(k-1)^n + 2k - 5}{k-2}, \tag{1.6}$$

$$H(k,n+1) - H(k,n) = (k-1)^n \tag{1.7}$$

and

$$H(k,n+1) = (k-1)H(k,n) - 2k + 5. \tag{1.8}$$

The Mersenne numbers and their generalizations have applications also in the theory of hypercomplex numbers. In [5], Daşdemir and Bilgici introduced and studied Mersenne quaternions, Gaussian Mersenne numbers and generalized Mersenne quaternions. In [11], the authors considered the Mersenne hybrid numbers and generalized Mersenne hybrid numbers. In this paper, we use the Mersenne, Mersenne–Lucas numbers and their generalizations in the theory of bihyperbolic numbers.

Hyperbolic numbers are two dimensional number system. Hyperbolic imaginary unit, so-called *unipotent*, is an element $\mathbf{h} \neq \pm 1$ such that $\mathbf{h}^2 = 1$. Bihyperbolic numbers are a generalization of hyperbolic numbers. Let $\mathbb{H}_2$ be the set of bihyperbolic numbers ζ of the form

$$\zeta = x_0 + x_1 j_1 + x_2 j_2 + x_3 j_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and j_1, j_2, $j_3 \notin \mathbb{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1,\ j_1 j_2 = j_2 j_1 = j_3,\ j_1 j_3 = j_3 j_1 = j_2,\ j_2 j_3 = j_3 j_2 = j_1. \tag{1.9}$$

From the above rules the multiplication of bihyperbolic numbers can be made analogously to the multiplication of algebraic expressions. The addition and the subtraction of bihyperbolic numbers is done by adding and subtracting corresponding terms and hence their coefficients. The addition and multiplication on $\mathbb{H}_2$ are commutative and associative, $(\mathbb{H}_2, +, \cdot)$ is a commutative ring. For the algebraic properties of bihyperbolic numbers, see [1].

Let $n \geq 0$ be an integer. The nth bihyperbolic Mersenne number BhM_n and the nth bihyperbolic Mersenne–Lucas number BhH_n are defined by

$$BhM_n = M_n + M_{n+1}j_1 + M_{n+2}j_2 + M_{n+3}j_3,$$

$$BhH_n = H_n + H_{n+1}j_1 + H_{n+2}j_2 + H_{n+3}j_3,$$

respectively, where M_n is the nth Mersenne number, H_n is the nth Mersenne–Lucas number and j_1, j_2, j_3 are units which satisfy (1.9).

The nth generalized bihyperbolic Mersenne number BhM_n^k we define in the following way

$$BhM_n^k = M(k,n) + M(k,n+1)j_1 + M(k,n+2)j_2 + M(k,n+3)j_3, \tag{1.10}$$

where $M(k,n)$ denotes the nth generalized Mersenne number, defined by (1.1). By analogy, the nth bihyperbolic Mersenne–Lucas number BhH_n^k is defined by

$$BhH_n^k = H(k,n) + H(k,n+1)j_1 + H(k,n+2)j_2 + H(k,n+3)j_3, \tag{1.11}$$

where $H(k,n)$ denotes the nth generalized Mersenne–Lucas number, defined by (1.2). For $k = 3$ we have $BhM_n^3 = BhM_n$ and $BhH_n^3 = BhH_n$.

Using the above definitions, we can write initial generalized bihyperbolic Mersenne numbers

$$\begin{aligned} BhM_0^k &= j_1 + kj_2 + (k^2 - k + 1)j_3, \\ BhM_1^k &= 1 + kj_1 + (k^2 - k + 1)j_2 + (k^3 - 2k^2 + 2k)j_3, \end{aligned} \tag{1.12}$$

generalized bihyperbolic Mersenne–Lucas numbers

$$\begin{aligned} BhH_0^k &= 2 + 3j_1 + (k+2)j_2 + (k^2 - k + 3)j_3, \\ BhH_1^k &= 3 + (k+2)j_1 + (k^2 - k + 3)j_2 + (k^3 - 2k^2 + 2k + 2)j_3, \end{aligned} \tag{1.13}$$

bihyperbolic Mersenne numbers

$$BhM_0 = j_1 + 3j_2 + 7j_3,$$

$$BhM_1 = 1 + 3j_1 + 7j_2 + 15j_3,$$

and bihyperbolic Mersenne–Lucas numbers

$$BhH_0 = 2 + 3j_1 + 5j_2 + 9j_3,$$

$$BhH_1 = 3 + 5j_1 + 9j_2 + 17j_3.$$

2. Main results

In this section, we present some properties of the generalized bihyperbolic Mersenne and Mersenne–Lucas numbers.

THEOREM 2.1. *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$BhM_{n+2}^k = kBhM_{n+1}^k - (k-1)BhM_n^k,$$

where BhM_0^k and BhM_1^k are defined by (1.12).

PROOF. By formulas (1.10) and (1.1) we get

$$\begin{aligned}
&kBhM_{n+1}^k - (k-1)BhM_n^k \\
&= k\left(M(k,n+1) + M(k,n+2)j_1 + M(k,n+3)j_2 + M(k,n+4)j_3\right) \\
&\quad - (k-1)\left(M(k,n) + M(k,n+1)j_1 + M(k,n+2)j_2 + M(k,n+3)j_3\right) \\
&= kM(k,n+1) - (k-1)M(k,n) \\
&\quad + \left(kM(k,n+2) - (k-1)M(k,n+1)\right) j_1 \\
&\quad + \left(kM(k,n+3) - (k-1)M(k,n+2)\right) j_2 \\
&\quad + \left(kM(k,n+4) - (k-1)M(k,n+3)\right) j_3 \\
&= M(k,n+2) + M(k,n+3)j_1 + M(k,n+4)j_2 + M(k,n+5)j_3 \\
&= BhM_{n+2}^k.
\end{aligned}$$

□

In the same way, using (1.11) and (1.2), we can prove the next theorem.

THEOREM 2.2. *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$BhH_{n+2}^k = kBhH_{n+1}^k - (k-1)BhH_n^k,$$

where BhH_0^k and BhH_1^k are defined by (1.13).

THEOREM 2.3. *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$BhM_{n+1}^k = (k-1)BhM_n^k + 1 + j_1 + j_2 + j_3,$$

where BhM_0^k is defined by (1.12).

PROOF. Using (1.10) and (1.5), we have

$$\begin{aligned}
&BhM_{n+1}^k - (k-1)BhM_n^k \\
&= M(k,n+1) + M(k,n+2)j_1 + M(k,n+3)j_2 + M(k,n+4)j_3 \\
&\quad - (k-1)\left(M(k,n) + M(k,n+1)j_1 + M(k,n+2)j_2 + M(k,n+3)j_3\right) \\
&= M(k,n+1) - (k-1)M(k,n) + \left(M(k,n+2) - (k-1)M(k,n+1)\right)j_1 \\
&\quad + \left(M(k,n+3) - (k-1)M(k,n+2)\right)j_2 \\
&\quad + \left(M(k,n+4) - (k-1)M(k,n+3)\right)j_3 \\
&= 1 + j_1 + j_2 + j_3.
\end{aligned}$$

□

THEOREM 2.4. *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$BhH_{n+1}^k = (k-1)BhH_n^k + (-2k+5)(1 + j_1 + j_2 + j_3) - 2j_1 - 4j_2 - 6j_3,$$

where BhH_0^k is defined by (1.13).

PROOF. Using (1.11) and (1.8), we have

$$\begin{aligned}
&BhH_{n+1}^k - (k-1)BhH_n^k \\
&= H(k,n+1) + H(k,n+2)j_1 + H(k,n+3)j_2 + H(k,n+4)j_3 \\
&\quad - (k-1)\left(H(k,n) + H(k,n+1)j_1 + H(k,n+2)j_2 + H(k,n+3)j_3\right) \\
&= H(k,n+1) - (k-1)H(k,n) + \left(H(k,n+2) - (k-1)H(k,n+1)\right)j_1 \\
&\quad + \left(H(k,n+3) - (k-1)H(k,n+2)\right)j_2 \\
&\quad + \left(H(k,n+4) - (k-1)H(k,n+3)\right)j_3 \\
&= -2k + 5 + (-2k+3)\,j_1 + (-2k+1)\,j_2 + (-2k-1)\,j_3 \\
&= (-2k+5)(1 + j_1 + j_2 + j_3) - 2j_1 - 4j_2 - 6j_3.
\end{aligned}$$

□

Next theorems give the Binet formulas for the generalized bihyperbolic Mersenne and Mersenne–Lucas numbers.

Theorem 2.5. *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$(2.1)\quad BhM_n^k = \frac{(k-1)^n - 1}{k-2}(1 + j_1 + j_2 + j_3) + (k-1)^n \left(j_1 + kj_2 + (k^2 - k + 1)j_3\right).$$

Proof. Using (1.4), we have $M(k, n+1) = M(k, n) + (k-1)^n$, hence $M(k, n+2) = M(k, n+1) + (k-1)^{n+1} = M(k, n) + (k-1)^n + (k-1)^{n+1}$ and $M(k, n+3) = M(k, n) + (k-1)^n + (k-1)^{n+1} + (k-1)^{n+2}$. Thus

$$\begin{aligned} BhM_n^k &= M(k,n) + M(k,n+1)j_1 + M(k,n+2)j_2 + M(k,n+3)j_3 \\ &= M(k,n)\,(1 + j_1 + j_2 + j_3) \\ &\quad + (k-1)^n j_1 + \left((k-1)^n + (k-1)^{n+1}\right) j_2 \\ &\quad + \left((k-1)^n + (k-1)^{n+1} + (k-1)^{n+2}\right) j_3 \\ &= M(k,n)(1 + j_1 + j_2 + j_3) + (k-1)^n \left(j_1 + kj_2 + (k^2 - k + 1)j_3\right). \end{aligned}$$

Putting $M(k, n) = \frac{1}{k-2}\left((k-1)^n - 1\right)$ (see (1.3)), we obtain the desired formula. □

Theorem 2.6. *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$(2.2)\quad BhH_n^k = \frac{(k-1)^n + 2k - 5}{k-2}(1 + j_1 + j_2 + j_3) + (k-1)^n \left(j_1 + kj_2 + (k^2 - k + 1)j_3\right).$$

Proof. Using (1.6), (1.7) and proceeding analogously as in the proof of the previous theorem we obtain the desired formula. □

Corollary 2.7. *Let $n \geq 0$ be an integer. For $k = 3$ we have*

$$\begin{aligned} BhM_n &= (2^n - 1)(1 + j_1 + j_2 + j_3) + 2^n\,(j_1 + 3j_2 + 7j_3) \\ &= 2^n\,(1 + 2j_1 + 4j_2 + 8j_3) - (1 + j_1 + j_2 + j_3) \end{aligned}$$

and

$$\begin{aligned} BhH_n &= (2^n + 1)\,(1 + j_1 + j_2 + j_3) + 2^n\,(j_1 + 3j_2 + 7j_3) \\ &= 2^n\,(1 + 2j_1 + 4j_2 + 8j_3) + (1 + j_1 + j_2 + j_3). \end{aligned}$$

For simplicity of notation let $A = 1 + j_1 + j_2 + j_3$. Using (1.3), (1.6) and (1.12), we can write (2.1) and (2.2) as

$$BhM_n^k = A \cdot M(k,n) + (k-1)^n BhM_0^k \tag{2.3}$$

and

$$BhH_n^k = A \cdot H(k,n) + (k-1)^n BhM_0^k, \tag{2.4}$$

respectively.

Using the Binet formula (2.3) and identity (1.3), we can derive the Catalan identity for the generalized bihyperbolic Mersenne numbers.

THEOREM 2.8. *Let $n \geq 0$, $r \geq 0$, $k \geq 3$ be integers such that $n \geq r$. Then*

$$\begin{aligned} &BhM_{n+r}^k \cdot BhM_{n-r}^k - \left(BhM_n^k\right)^2 \\ &\quad = \frac{(k-1)^n - 1}{k-2}(k^2+2)(k-1)^{n-r}\left(1-(k-1)^r\right)^2(1+j_1+j_2+j_3). \end{aligned}$$

PROOF. By formula (2.3) we get

$$\begin{aligned} &BhM_{n+r}^k \cdot BhM_{n-r}^k - \left(BhM_n^k\right)^2 \\ &= \left(A \cdot M(k,n) + (k-1)^{n+r} BhM_0^k\right)\left(A \cdot M(k,n) + (k-1)^{n-r} BhM_0^k\right) \\ &\quad - \left(A \cdot M(k,n) + (k-1)^n BhM_0^k\right)\left(A \cdot M(k,n) + (k-1)^n BhM_0^k\right) \\ &= A \cdot M(k,n) \cdot BhM_0^k \cdot (k-1)^{n-r} + A \cdot M(k,n) \cdot BhM_0^k \cdot (k-1)^{n+r} \\ &\quad - 2A \cdot M(k,n) \cdot BhM_0^k \cdot (k-1)^n \\ &= A \cdot M(k,n) \cdot BhM_0^k \cdot (k-1)^{n-r}\left(1 + (k-1)^{2r} - 2(k-1)^r\right) \\ &= M(k,n) \cdot A \cdot BhM_0^k \cdot (k-1)^{n-r}\left(1-(k-1)^r\right)^2. \end{aligned}$$

Moreover,

$$\begin{aligned} A \cdot BhM_0^k &= (1+j_1+j_2+j_3)\left(j_1 + kj_2 + (k^2-k+1)j_3\right) \\ &= j_1 + kj_2 + (k^2-k+1)j_3 + 1 + kj_3 + (k^2-k+1)j_2 \\ &\quad + j_3 + k + (k^2-k+1)j_1 + j_2 + kj_1 + (k^2-k+1) \\ &= (k^2+2)(1+j_1+j_2+j_3). \end{aligned}$$

Hence we get

$$\begin{aligned}&BhM^k_{n+r}\cdot BhM^k_{n-r}-\left(BhM^k_n\right)^2\\&=\frac{(k-1)^n-1}{k-2}(k^2+2)(k-1)^{n-r}\left(1-(k-1)^r\right)^2(1+j_1+j_2+j_3),\end{aligned}$$

which completes the proof. □

In the same way, using (2.4) and (1.6), we obtain the Catalan identity for the generalized bihyperbolic Mersenne–Lucas numbers.

Theorem 2.9. *Let $n\geq 0$, $r\geq 0$, $k\geq 3$ be integers such that $n\geq r$. Then*

$$\begin{aligned}&BhH^k_{n+r}\cdot BhH^k_{n-r}-\left(BhH^k_n\right)^2\\&=\frac{(k-1)^n+2k-5}{k-2}(k^2+2)(k-1)^{n-r}\left(1-(k-1)^r\right)^2(1+j_1+j_2+j_3).\end{aligned}$$

For $r=1$ we obtain Cassini identities for the generalized bihyperbolic Mersenne and Mersenne–Lucas numbers.

Corollary 2.10. *Let $n\geq 1$, $k\geq 3$ be integers. Then*

$$\begin{aligned}&BhM^k_{n+1}\cdot BhM^k_{n-1}-\left(BhM^k_n\right)^2\\&\qquad=\left((k-1)^n-1\right)(k-1)^{n-1}(k-2)(k^2+2)(1+j_1+j_2+j_3).\end{aligned}$$

Corollary 2.11. *Let $n\geq 1$, $k\geq 3$ be integers. Then*

$$\begin{aligned}&BhH^k_{n+1}\cdot BhH^k_{n-1}-\left(BhH^k_n\right)^2\\&\qquad=\left((k-1)^n+2k-5\right)(k-1)^{n-1}(k-2)(k^2+2)(1+j_1+j_2+j_3).\end{aligned}$$

For $k=3$ we obtain Catalan and Cassini identities for the bihyperbolic Mersenne and Mersenne–Lucas numbers.

Corollary 2.12. *Let $n\geq 1$ be an integer. Then*

$$\begin{aligned}&BhM_{n+r}\cdot BhM_{n-r}-\left(BhM_n\right)^2\\&\qquad=11(2^n-1)\left(2^{n-r}-2^{n+1}+2^{n+r}\right)(1+j_1+j_2+j_3).\end{aligned}$$

COROLLARY 2.13. *Let $n \geq 1$ be an integer. Then*

$$BhH_{n+r} \cdot BhH_{n-r} - (BhH_n)^2 = 11(2^n+1)\left(2^{n-r} - 2^{n+1} + 2^{n+r}\right)(1 + j_1 + j_2 + j_3).$$

COROLLARY 2.14. *Let $n \geq 1$ be an integer. Then*

$$BhM_{n+1} \cdot BhM_{n-1} - (BhM_n)^2 = 11(2^n-1)2^{n-1}(1 + j_1 + j_2 + j_3).$$

COROLLARY 2.15. *Let $n \geq 1$ be an integer. Then*

$$BhH_{n+1} \cdot BhH_{n-1} - (BhH_n)^2 = 11(2^n+1)2^{n-1}(1 + j_1 + j_2 + j_3).$$

Now we give ordinary generating functions for the generalized bihyperbolic Mersenne and Mersenne–Lucas numbers.

THEOREM 2.16. *The generating function for the generalized bihyperbolic Mersenne number sequence $\{BhM_n^k\}$ is*

$$G(t) = \frac{BhM_0^k + (BhM_1^k - kBhM_0^k)t}{1 - kt + (k-1)t^2}.$$

PROOF. Assume that the generating function of the generalized bihyperbolic Mersenne number sequence $\{BhM_n^k\}$ has the form $G(t) = \sum_{n=0}^{\infty} BhM_n^k t^n$. Then

$$\begin{aligned}(1 - kt+&(k-1)t^2)G(t)\\ &= (1 - kt + (k-1)t^2)(BhM_0^k + BhM_1^k t + BhM_2^k t^2 + \ldots)\\ &= BhM_0^k + BhM_1^k t + BhM_2^k t^2 + \ldots\\ &\quad - kBhM_0^k t - kBhM_1^k t^2 - kBhM_2^k t^3 - \ldots\\ &\quad + (k-1)BhM_0^k t^2 + (k-1)BhM_1^k t^3 + (k-1)BhM_2^k t^4 + \ldots\\ &= BhM_0^k + (BhM_1^k - kBhM_0^k)t,\end{aligned}$$

since $BhM_n^k = kBhM_{n-1}^k - (k-1)BhM_{n-2}^k$ and the coefficients of t^n for $n \geq 2$ are equal to zero. Moreover, $BhM_0^k = j_1 + kj_2 + (k^2 - k + 1)j_3$, $BhM_1^k - kBhM_0^k = 1 + (-k+1)j_2 + (-k^2+k)j_3$. □

THEOREM 2.17. *The generating function for the generalized bihyperbolic Mersenne-Lucas number sequence* $\{BhH_n^k\}$ *is*

$$g(t) = \frac{BhH_0^k + (BhH_1^k - kBhH_0^k)t}{1 - kt + (k-1)t^2}.$$

PROOF. The proof of this theorem is similar to the proof of the previous theorem. Note only that $BhH_0^k = 2 + 3j_1 + (k+2)j_2 + (k^2 - k + 3)j_3$ and $BhH_1^k - kBhH_0^k = (3 - 2k) + (2 - 2k)j_1 + (3 - 3k)j_2 + (2 - k - k^2)j_3$. □

REMARK 2.18. The generating function $\gamma(t)$ for the bihyperbolic Mersenne number sequence $\{BhM_n\}$ is

$$\gamma(t) = \frac{BhM_0 + (BhM_1 - 3BhM_0)t}{1 - 3t + 2t^2},$$

where $BhM_0 = j_1 + 3j_2 + 7j_3$ and $BhM_1 - 3BhM_0 = 1 - 2j_2 - 6j_3$.

REMARK 2.19. The generating function $\eta(t)$ for the bihyperbolic Mersenne–Lucas number sequence $\{BhH_n\}$ is

$$\eta(t) = \frac{BhH_0 + (BhH_1 - 3BhH_0)t}{1 - 3t + 2t^2},$$

where $BhH_0 = 2+3j_1+5j_2+9j_3$ and $BhH_1-3BhH_0 = -3-4j_1-6j_2-10j_3$.

At the end, we give the matrix representations of the defined bihyperbolic numbers.

THEOREM 2.20. *Let* $n \geq 0$, $k \geq 3$ *be integers. Then*

$$\begin{bmatrix} BhM_{n+2}^k & BhM_{n+1}^k \\ BhM_{n+1}^k & BhM_n^k \end{bmatrix} = \begin{bmatrix} BhM_2^k & BhM_1^k \\ BhM_1^k & BhM_0^k \end{bmatrix} \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix}^n.$$

PROOF. (By induction on n.) If $n = 0$ then assuming that the matrix to the power 0 is the identity matrix the result is obvious. Now suppose that for any $n \geq 0$ holds

$$\begin{bmatrix} BhM_{n+2}^k & BhM_{n+1}^k \\ BhM_{n+1}^k & BhM_n^k \end{bmatrix} = \begin{bmatrix} BhM_2^k & BhM_1^k \\ BhM_1^k & BhM_0^k \end{bmatrix} \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix}^n.$$

We shall show that

$$\begin{bmatrix} BhM_{n+3}^k & BhM_{n+2}^k \\ BhM_{n+2}^k & BhM_{n+1}^k \end{bmatrix} = \begin{bmatrix} BhM_2^k & BhM_1^k \\ BhM_1^k & BhM_0^k \end{bmatrix} \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix}^{n+1}.$$

By simple calculations, using induction's hypothesis we have

$$\begin{aligned}
&\begin{bmatrix} BhM_2^k & BhM_1^k \\ BhM_1^k & BhM_0^k \end{bmatrix} \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix}^n \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix} \\
&\qquad = \begin{bmatrix} BhM_{n+2}^k & BhM_{n+1}^k \\ BhM_{n+1}^k & BhM_n^k \end{bmatrix} \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix} \\
&\qquad = \begin{bmatrix} k \cdot BhM_{n+2}^k - (k-1) \cdot BhM_{n+1}^k & BhM_{n+2}^k \\ k \cdot BhM_{n+1}^k - (k-1) \cdot BhM_n^k & BhM_{n+1}^k \end{bmatrix} \\
&\qquad = \begin{bmatrix} BhM_{n+3}^k & BhM_{n+2}^k \\ BhM_{n+2}^k & BhM_{n+1}^k \end{bmatrix},
\end{aligned}$$

which completes the proof. □

THEOREM 2.21. *Let $n \geq 0$, $k \geq 3$ be integers. Then*

$$\begin{bmatrix} BhH_{n+2}^k & BhH_{n+1}^k \\ BhH_{n+1}^k & BhH_n^k \end{bmatrix} = \begin{bmatrix} BhH_2^k & BhH_1^k \\ BhH_1^k & BhH_0^k \end{bmatrix} \cdot \begin{bmatrix} k & 1 \\ -(k-1) & 0 \end{bmatrix}^n .$$

COROLLARY 2.22. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} BhM_{n+2} & BhM_{n+1} \\ BhM_{n+1} & BhM_n \end{bmatrix} = \begin{bmatrix} BhM_2 & BhM_1 \\ BhM_1 & BhM_0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}^n .$$

COROLLARY 2.23. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} BhH_{n+2} & BhH_{n+1} \\ BhH_{n+1} & BhH_n \end{bmatrix} = \begin{bmatrix} BhH_2 & BhH_1 \\ BhH_1 & BhH_0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}^n .$$

Note that multiplication of bihyperbolic numbers is commutative and determinant properties can be used. For example, calculating determinants in Theorems 2.20–2.21 and Corollaries 2.22–2.23, we can also obtain Cassini identities. Using algebraic operations and matrix algebra could give many other interesting properties of these numbers.

3. Declarations

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References

[1] M. Bilgin and S. Ersoy, *Algebraic properties of bihyperbolic numbers*, Adv. Appl. Clifford Algebr. **30** (2020), no. 1, Paper No. 13, 17 pp. DOI: 10.1007/s00006-019-1036-2
[2] A. Boussayoud, M. Chelgham, and S. Boughaba, *On some identities and generating functions for Mersenne numbers and polynomials*, Turkish Journal of Analysis and Number Theory **6** (2018), no. 3, 93–97.
[3] P. Catarino, H. Campos, and P. Vasco, *On the Mersenne sequence*, Ann. Math. Inform. **46** (2016), 37–53.
[4] M. Chelgham and A. Boussayoud, *On the k-Mersenne–Lucas numbers*, Notes Number Theory Discrete Math. **27** (2021), no. 1, 7–13. DOI: 10.7546/nntdm.2021.27.1.7-13
[5] A. Daşdemir and G. Bilgici, *Gaussian Mersenne numbers and generalized Mersenne quaternions*, Notes Number Theory Discrete Math. **25** (2019), no. 3, 87–96. DOI: 10.7546/nntdm.2019.25.3.87-96
[6] R. Frontczak and T. Goy, *Mersenne–Horadam identities using generating functions*, Carpathian Math. Publ. **12** (2020), no. 1, 34–45. DOI: 10.15330/cmp.12.1.34-45
[7] T. Goy, *On new identities for Mersenne numbers*, Appl. Math. E-Notes **18** (2018), 100–105.
[8] P. Ochalik and A. Włoch, *On generalized Mersenne numbers their interpretations and matrix generators*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **72** (2018), no. 1, 69–76. DOI: 10.17951/a.2018.72.1.69-76
[9] N. Saba, A. Boussayoud, and K.V.V. Kanuri, *Mersenne Lucas numbers and complete homogeneous symmetric functions*, J. Math. Computer Sci. **24** (2022), no. 2, 127–139. DOI: 10.22436/jmcs.024.02.04
[10] Y. Soykan, *A study on generalized Mersenne numbers*, Journal of Progressive Research in Mathematics **18** (2021), no. 3, 90–108.
[11] A. Szynal-Liana and I. Włoch, *On generalized Mersenne hybrid numbers*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **74** (2020), no. 1, 77–84. DOI: 10.17951/a.2020.74.1.77-84

Rzeszow University of Technology
Faculty of Mathematics and Applied Physics
al. Powstańców Warszawy 12
35-959 Rzeszów
Poland
e-mail: dorotab@prz.edu.pl
e-mail: aszynal@prz.edu.pl