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SYMMETRIZATION FOR MIXED OPERATORS

Sabri Bahrouni

Abstract. In this paper, we prove Talenti's comparison theorem for mixed local/nonlocal elliptic operators and derive the Faber–Krahn inequality for the first eigenvalue of the Dirichlet mixed local/nonlocal problem. Our findings are relevant to the fractional $p\&q-$Laplacian operator.

1. Introduction

1.1. Comparison results: an overview

In [19], Talenti states that if, for given $f \geq 0$, $f \in L^2(\Omega)$, $u \in H_0^1(\Omega)$ solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and if $v \in H_0^1\left(\Omega^\#\right)$ solves

$$\begin{cases} -\Delta v = f^\# & \text{in } \Omega^\#, \\ v = 0 & \text{on } \partial\Omega^\#, \end{cases}$$

then

$$u^\# \leq v \quad \text{a.e. in } \Omega^\#.$$

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Here $\Omega^\#$ is the ball centered at the origin such that $|\Omega^\#| = |\Omega|$ and $u^\#$ denotes the Schwarz symmetrization of u. There are two available proofs of this outcome. The first one, presented by Talenti [19], employs an isoperimetric inequality that concerns the De Giorgi perimeter of Ω. The second proof, on the other hand, was formulated by Lions [17] and does not rely on this particular inequality. Instead, Lions' approach hinges on a differential inequality that relates the distribution functions of u and v.

In [11], it is shown that, for $f \in L^p(\Omega)$ where p satisfies some suitable conditions, if u solves the nonlinear and nonlocal problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases}$$

and if v solves the symmetrized problem

$$\begin{cases} (-\Delta)^s v = f^\# & \text{in } \Omega^\#, \\ v = 0 & \text{on } \mathbb{R}^N \backslash \Omega^\#, \end{cases}$$

then

$$u^\# \prec v$$

where $\prec$ is the order relation in the form of mass concentration comparison (see Section 2 for precise definitions).

Can the comparison of mass concentration be refined to provide a pointwise estimate? To determine whether a local case result, such as the one proven by Talenti, could also apply to the non-local case, Section 4 in [11] examines certain special cases. The findings reveal that a pointwise estimate cannot be upheld, indicating that the result in [11] is optimal.

1.2. Main results

Mixed local and nonlocal problems have gained recent attention and are currently under intensive investigation. The main focus is on an elliptic operator that combines two different orders of differentiation, with the simplest model case being $\mathcal{L} := -\Delta + (-\Delta)^s$ for $s \in (0, 1)$.

Initial progress in this direction was achieved through probabilistic methods in [9, 10]. More recently, Biagi, Dipierro, Valdinoci, and Vecchi [2, 3, 4, 5] have undertaken a systematic investigation of problems involving mixed operators, with the publication of a number of results concerning regularity and qualitative behavior for solutions, maximum principles, and related variational principles.

In this note, we establish the comparison principle of Talenti in the context of a mixed local/nonlocal elliptic operator:

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, and $f \in L^2(\Omega)$ in order to guarantee the existence and the uniqueness of a weak solution.

Theorem 1.1. *Let $u \in \mathbb{X}(\Omega)$ be the weak solution of problem* (1.1). *Let $v \in H_0^s(\Omega^{\#})$ be the weak solution of the symmetrized problem*

$$\begin{cases} (-\Delta)^s v = f^{\#} & \text{in } \Omega^{\#}, \\ v = 0 & \text{on } \mathbb{R}^N \backslash \Omega^{\#}. \end{cases} \tag{1.2}$$

Then

$$u^{\#} \prec v.$$

As applications of Theorem 1.1 we will give the alternative proof of that Faber-Krahn inequality which was proved recently in [5, Theorem 1.1], and in [14, Theorem 4.1], [8, Corollary 1.2] for the local case.

Corollary 1.2 (A Faber-Krahn inequality). *Let Ω be a bounded open subset of $\mathbb{R}^n$ satisfying $|\Omega^{\#}| = |\Omega|$ and let $\lambda_1(\Omega)$ denotes the first eigenvalue of the Dirichlet problem*

$$\begin{cases} (-\Delta)^s u = \lambda_1(\Omega) u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases}$$

Then we have

$$\lambda_1(\Omega^{\#}) \leq \lambda_1(\Omega). \tag{1.3}$$

Moreover, if the equality holds in (1.3), *then Ω is a ball.*

Theorem 1.1 enables us to establish a priori estimates for solutions to problem (1.1), expressed in relation to the data f.

Corollary 1.3 (Some regularity). *Let u be the weak solution to problem* (1.1). *Then $u \in L^r(\Omega)$ with $r = \frac{2N}{N+4s}$ and there exists a constant C such that*

$$\|u\|_{L^r(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

This paper is structured as follows. In Section 2, we present preliminaries and useful results about the functional setting, rearrangements, and symmetrization. In Section 3, we provide the proofs for Theorem 1.1, Corollary 1.2, and Corollary 1.3. Additionally, in Section 4, we extend our findings to the fractional $p\&q-$Laplacian operator.

2. Preliminaries

2.1. Functional setting

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with continuous boundary. We then consider the space $\mathbb{X}(\Omega)$ defined as follows:

$$\mathbb{X}(\Omega) := \left\{ u \in H^1\left(\mathbb{R}^n\right) : u \equiv 0 \text{ a.e. on } \mathbb{R}^n \backslash \Omega \right\}.$$

We can establish that $\mathbb{X}(\Omega)$ is a real Hilbert space by using the scalar product defined as:

$$\langle u, v \rangle_{\mathbb{X}(\Omega)} := \int_\Omega \langle \nabla u, \nabla v \rangle dx.$$

The corresponding norm for this scalar product is given by:

$$\|u\|_{\mathbb{X}(\Omega)} := \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2}.$$

Additionally, the linear map $\mathcal{E}_0 \colon H_0^1(\Omega) \to \mathbb{X}(\Omega)$ defined by

$$\mathcal{E}_0(u) := u \cdot 1_\Omega$$

is a bijective isometry connecting $H_0^1(\Omega)$ and $\mathbb{X}(\Omega)$.

On the space $\mathbb{X}(\Omega)$, we consider the bilinear form

$$\mathcal{B}(u, v) := \int_\Omega \langle \nabla u, \nabla v \rangle dx + \iint_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy;$$

moreover, for every $u \in \mathbb{X}(\Omega)$ we define

$$\mathcal{D}(u) := \mathcal{B}(u, u).$$

Definition 2.1. Let $f \in L^2(\Omega)$. We say that a function $u\colon \mathbb{R}^n \to \mathbb{R}$ is a *weak solution* of problem (1.1), if it satisfies the following properties:

(i) $u \in \mathbb{X}(\Omega)$;

(ii) for every test function $\varphi \in \mathbb{X}(\Omega)$, one has

$$\mathcal{B}(u,\varphi) = \int_\Omega f\varphi dx. \tag{2.1}$$

Applying the Lax–Milgram Theorem to the bilinear form $\mathcal{B}$ yields the following existence result, see, [4, Theorem 1.1].

Theorem 2.2. *For every $f \in L^2(\Omega)$, there exists a unique weak solution $u \in \mathbb{X}(\Omega)$ of (1.1), further satisfying the 'a-priori' estimate*

$$\|u\|_{\mathbb{X}(\Omega)} \le c_0\|f\|_{L^2(\Omega)}.$$

Here, $c_0 > 0$ is a constant independent of f.

We also recall that the solution v to the symmetrized problem (1.2) is radial and radially decreasing, see for instance [2, Theorem 1.1].

2.2. Rearrangements and symmetrization

Definition 2.3. Let $h\colon \Omega \to [0,+\infty[$ be a measurable function, then the *decreasing rearrangement* h^* of h is defined as follows:

$$h^*(s) = \inf\{t \ge 0 : |\{x \in \Omega : |h(x)| > t\}| < s\}, \quad s \in [0,\Omega].$$

While the *Schwartz rearrangement* of h is defined as follows

$$h^\sharp(x) = h^*\left(\omega_n|x|^n\right), \quad x \in \Omega^\sharp.$$

We denote by ω_n the measure of the unit ball in $\mathbb{R}^n$, and $\Omega^{\#}$ the ball, centered at the origin, with the same measure as Ω.

It is easily checked that h, h^* and $h^\sharp$ are equi-distributed, i.e.

$$\begin{aligned}|\{x \in \Omega : |h(x)| > t\}| &= |\{s \in (0,|\Omega|) : h^*(s) > t\}| \\ &= \left|\left\{x \in \Omega^\sharp : h^\sharp(x) > t\right\}\right|, \quad t \ge 0,\end{aligned}$$

and then if $h \in L^P(\Omega),\ 1 \le p \le \infty$, then $h^* \in L^P(0,|\Omega|), h^\sharp \in L^p\left(\Omega^\sharp\right)$, and

$$\|h\|_{L^p(\Omega)} = \|h^*\|_{L^p(0,|\Omega|)} = \left\|h^\sharp\right\|_{L^p(\Omega^\sharp)}.$$

Moreover, the following inequality, known as Hardy–Littlewood inequality, holds true

$$\int_{\Omega} |h(x)g(x)|dx \leq \int_0^{|\Omega|} h^*(s)g^*(s)ds.$$

2.3. Mass concentration

Definition 2.4. Let $f, g \in L^1_{loc}(\mathbb{R}^n)$. We say that f is *less concentrated* than g, and we write $f \prec g$ if for all $r > 0$ we get

$$\int_{B_r(0)} f^\#(x)\mathrm{d}x \leq \int_{B_r(0)} g^\#(x)\mathrm{d}x.$$

The partial order relationship $\prec$ is called the *comparison of mass concentrations.*

Lemma 2.5 ([1, Corollary 2.1]). *Let $f, g \in L^1_+(\Omega)$. Then the following are equivalent:*

(i) *$f \prec g$;*
(ii) *for all $\phi \in L^\infty_+(\Omega)$,*

$$\int_{\Omega} f(x)\phi(x)dx \leq \int_{\Omega^\#} f^\#(x)\phi^\#(x)dx,$$

(iii) *for all convex, nonnegative functions $\Phi\colon [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ it holds that*

$$\int_{\Omega} \Phi(f(x))dx \leq \int_{\Omega} \Phi(g(x))dx.$$

3. Proofs

Proof of Theorem 1.1. Let $\mathcal{G}_{t,h}$, $t, h > 0$ be the truncation function

$$\mathcal{G}_{t,h}(\theta) = \begin{cases} h & \text{if } \theta > t+h, \\ \theta - t & \text{if } t < \theta \leq t+h, \\ 0 & \text{if } \theta \leq t. \end{cases}$$

Let us take $\varphi(x)=\mathcal{G}_{t,h}(u(x))$ as a test function in (2.1), we obtain

$$\int_\Omega \langle \nabla u, \nabla \mathcal{G}_{t,h}(u(x))\rangle dx + \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\mathcal{G}_{t,h}(u(x))-\mathcal{G}_{t,h}(u(y)))}{|x-y|^{n+2s}} dxdy \\ = \int_\Omega f(x)\mathcal{G}_{t,h}(u(x))\, dx.$$

It is proven in [11] that

$$\iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\mathcal{G}_{t,h}(u(x))-\mathcal{G}_{t,h}(u(y)))}{|x-y|^{n+2s}} dxdy \\ \geq \iint_{\mathbb{R}^{2n}} \frac{(u^\#(x)-u^\#(y))(\mathcal{G}_{t,h}(u^\#(x))-\mathcal{G}_{t,h}(u^\#(y)))}{|x-y|^{n+2s}} dxdy.$$

As a consequence we have

$$\int_\Omega \langle \nabla u, \nabla \mathcal{G}_{t,h}(u(x))\rangle dx \\ + \iint_{\mathbb{R}^{2n}} \frac{(u^\#(x)-u^\#(y))(\mathcal{G}_{t,h}(u^\#(x))-\mathcal{G}_{t,h}(u^\#(y)))}{|x-y|^{n+2s}} dxdy \\ \leq \int_\Omega f(x)\mathcal{G}_{t,h}(u(x))\, dx.$$

Letting h go to 0 yields

$$\begin{aligned}(3.1)\quad & -\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_t} |\nabla u|^2 \ \mathrm{d}x\right) \\ & + \int_0^r \left(\int_r^{+\infty} (\mu(\tau)-\mu(\rho))\Theta_{N,s}(\tau,\rho)\rho^{N-1}\ \mathrm{d}\rho\right)\tau^{N-1}\ \mathrm{d}\tau \\ & \leq \int_0^r f^*\left(\omega_N\rho^N\right)\rho^{N-1}\ \mathrm{d}\rho,\end{aligned}$$

where

$$\Omega_t = \{x\in\Omega:\ \ u(x)>t\},$$

and

$$\Theta_{N,s}(\tau,\rho) = \frac{1}{N\omega_N}\int_{|x'|=1}\left(\int_{|y'|=1} \frac{1}{|\tau x'-\rho y'|^{N+2s}}\ \mathrm{d}H^{N-1}(y')\right) dH^{N-1}(x').$$

From (3.1) we get

$$\int_0^r \left(\int_r^{+\infty} (\mu(\tau) - \mu(\rho))\Theta_{N,s}(\tau,\rho)\rho^{N-1}\,\mathrm{d}\rho \right) \tau^{N-1}\,\mathrm{d}\tau \\ \leq \int_0^r f^*\left(\omega_N \rho^N\right)\rho^{N-1}\,\mathrm{d}\rho.$$

The rest of the proof is the same as the proof of [11, Theorem 3.1] (Step 3 and Step 4). □

Proof of Corollary 1.2. Let $u \in \mathbb{X}(\Omega)$ be such that

$$\begin{cases} \mathcal{L}u = \lambda_1(\Omega)u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases}$$

Let $v \in \mathbb{X}(\Omega^{\#})$ be such that

$$\begin{cases} (-\Delta)^s v = \lambda_1(\Omega)u^{\#} & \text{in } \Omega^{\#}, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega^{\#}. \end{cases}$$

Then by Theorem 1.1, $u^{\#} \prec v$. If we take $\Phi(t) = t|v|$ in Lemma 2.5, we get

$$\int_{\Omega^{\#}} u^{\#}|v|\mathrm{d}x \leq \int_{\Omega^{\#}} |v|^2\mathrm{d}x.$$

But by the Rayleigh-quotient characterization of the first eigenvalue,

$$\lambda_1(\Omega) = \frac{\mathcal{D}_{\Omega^{\#}}(v)}{\int_{\Omega^{\#}} u^{\#}v} \geq \frac{\mathcal{D}_{\Omega^{\#}}(v)}{\int_{\Omega^{\#}} |v|^2\mathrm{d}x} \geq \lambda_1(\Omega^{\#}).$$

This gives the proof of (1.3). □

Proof of Corollary 1.3. We will use [7, Theorem 3.2] the integral form for the solution v to the symmetrized problem (1.2), namely

$$v(x) = \int_{\Omega^{\#}} \mathrm{G}_{\Omega^{\#}}(x,y)f^{\#}(y)\mathrm{d}y,$$

where $G_{\Omega^{\#}}$ is the Green function of the fractional Laplacian on the ball. From [15, Theorem 3.2], we have

$$\mathrm{G}_{\Omega^{\#}}(x,y) \leq \frac{C}{|x-y|^{N-2s}}$$

for any $x \neq y$ in $\Omega^{\#}$, then, extending f to 0 out of Ω, Hardy-Littlewood-Sobolev inequality [16, Theorem 4.3] implies,

$$\|u\|_{L^r(\Omega)} = \|u^{\#}\|_{L^r(\Omega^{\#})} \leq \|v\|_{L^q(\Omega^{\#})} = \|v\|_{L^q(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

where $r = \frac{2N}{N+4s}$. $\square$

4. Further extensions

In [12], it is proven that for $p \geq 2$ and $f \in L^m(\Omega)$ satisfying certain conditions, if u is a solution to the nonlinear and nonlocal problem

$$\begin{cases} (-\Delta)_p^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases}$$

and v is a solution to the symmetrized problem

$$\begin{cases} (-\Delta)^s v = g^{\#} & \text{in } \Omega^{\#}, \\ v = 0 & \text{on } \mathbb{R}^n \backslash \Omega^{\#}, \end{cases}$$

then

$$u^{\#} \prec v$$

where $g = g(|x|)$ is the radial function defined by

$$g(r) = \mathrm{H}(n,s,p) r^{\frac{(n-s)(p-2)}{p-1}} \Big[\frac{(n-s)(p-2)}{p-1} \frac{1}{r^n} \Big(\int_{B_r} f^{\#} dx \Big)^{\frac{1}{p-1}} + \frac{n\omega_n}{p-1} \Big(\int_{B_r} f^{\#} dx \Big)^{\frac{2-p}{p-1}} f^{\#}(x) \Big],$$

with

$$\mathrm{H}(n,s,p) = \frac{\gamma(n,s,2)}{n\omega_n} \frac{\left(\mathcal{P}_s\left(B_1\right)\right)^{\frac{p-2}{p-1}}}{\gamma(n,s,p)^{\frac{1}{p-1}}},$$

being

$$\mathcal{P}_s\left(B_1\right) = \int_{B_r} \int_{B_r^c} \frac{1}{|x-y|^{n+s}} dx dy.$$

We establish the comparison principle of Talenti for nonlocal, nonlinear, and nonhomogeneous elliptic problems of the form:

$$
\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u = f & \text{in } \Omega, \\ \qquad\qquad\qquad\quad u = 0 & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{4.1}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $2 \le q < p < \infty$, and f satisfies suitable conditions to ensure the existence and uniqueness of a weak solution.

In the case of the usual Sobolev spaces, for any $1 \le p < q \le \infty$, it is easy to see that $W_0^{1,q}(\Omega) \subset W_0^{1,p}(\Omega)$. In the fractional case, this kind of embedding is NOT TRUE. In fact, in [6, Lemma 2.6] it is proved that

$$
W_0^{s_1,p}(\Omega) \hookrightarrow W_0^{s_2,q}(\Omega) \quad \text{for any } 0 < s_1 < s_2 < 1 \le q < p < \infty,
$$

this also holds when $p = q$ (see [13, Theorem 2.2]). However, the embedding

$$
W_0^{s,p}(\Omega) \hookrightarrow W_0^{s,q}(\Omega) \quad \text{for any } 0 < s < 1 \le q < p < \infty
$$

is not true (see [18, Theorem 1.1]). So, in order to deal with our problem (4.1), we consider the space

$$
\mathcal{W}^s := W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)
$$

endowed with the norm $[\cdot]_s := [\cdot]_{s,p} + [\cdot]_{s,q}$.

We say that $u \in \mathcal{W}^s$ is a weak solution of problem (4.1) if

$$
\begin{aligned}
&\frac{K(n,s,p)}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\,dxdy \\
&\quad + \frac{K(n,s,q)}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sq}}\,dxdy \\
&\qquad\qquad = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in \mathcal{W}^s.
\end{aligned}
$$

THEOREM 4.1. *Let $u \in \mathcal{W}^s$ be the weak solution of problem* (4.1). *Let $v \in \mathcal{W}^s$ be the weak solution of the symmetrized problem*

$$
\begin{cases} (-\Delta)^s v = g & \textit{in } \Omega^\#, \\ \qquad\quad v = 0 & \textit{on } \mathbb{R}^n \backslash \Omega^\#. \end{cases}
$$

Then

$$
u^\# \prec v.
$$

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Mathematics Department
Faculty of Sciences
University of Monastir
5019 Monastir
Tunisia
e-mail: sabri.bahrouni@fsm.rnu.tn