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# FIXED POINT AND BEST PROXIMITY POINT RESULTS IN PIV-$S$-METRIC SPACES

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**Abstract.** This paper presents the concept of a partial idempotent valued $S$-metric space, abbreviated as PIV-$S$-metric space, as a generalization of both the PIV-metric space and $S$-metric space. The study utilizes this new framework to establish a fixed point theorem and a best proximity point theorem. Additionally, the paper proves the existence and uniqueness of the best proximity point within this context. Several illustrative examples are provided to demonstrate the practical applications of the main findings.

## 1. Introduction

The theory of fixed points continues to be a widely used tool across various branches of mathematics, with the classical Banach contraction principle being a well-known example [7]. This principle is particularly significant as it allows for the resolution of integral equations, differential equations, and fractional differential equations by reducing them to the problem of identifying a self-mapping's fixed points. The assurance of a unique fixed point on a complete metric space provided by the Banach contraction principle has sparked numerous extensions by researchers, expanding its applicability in

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various contexts (see [2–6, 8, 10, 12, 15–19, 21]). In 1994, the concept of partial metric spaces was introduced by Matthews during the study of denotational semantics of data-flow networks. Expanding on this notion, Shukla [21] made significant progress two decades later by generalizing both partial metric spaces and $b-$metric spaces, leading to the establishment of the class of partial $b-$metric spaces. This new framework was utilized to establish a fixed point theorem as an analog of the Banach contraction principle.

In 2007, Huang and Zhang [12] introduced the concept of cone metric spaces by substituting the set of real numbers (range set of the metric) with an ordered Banach space. Later, in 2014, Ma et al. [17] expanded on this concept by inventing $C^*$-algebra valued metric spaces, which give a more comprehensive framework than ordinary metric spaces by substituting a unital $C^*$-algebra for the range set. Within this context, they successfully proved fixed point results. The subsequent year, Ma et al. [16] took it a step further and introduced $C^*$-algebra valued $b$-metric spaces, thereby broadening their work and establishing additional fixed point results, including an application involving integral type operators. In 2019, Chandok [9] generalized $C^*$-algebra valued metric spaces to $C^*$-algebra valued partial metric spaces and demonstrated some fixed point theorems. By developing the novel class of $C^*$-algebra valued partial $b$-metric spaces in 2020, Mlaiki et al. [19] expanded both $C^*$-algebra valued partial metric spaces and $C^*$-algebra valued $b$-metric spaces. They utilized this innovative concept to prove fixed point theorems and presented an application for solving integral type equations.

In 2012, Sedghi et al. [20] introduced the concept of $S$-metric spaces and established fixed point theorems for implicit relations. Concurrently, in the same year, Iranmanesh et al. [13] pioneered the idea of PIV-metric spaces and proved fixed point results within this framework. Subsequently, in 2019, Iranmanesh et al. [14] further expanded on this concept and presented additional fixed point and the results on best proximity point in PIV-metric spaces.

Building on the previous observations, we propose the notion of PIV-metric space and $S$-metric space, culminating in the concept of PIV-$S$-metric space. By doing so, we not only unify these two notions but also employ this new framework to establish fixed point results. Additionally, we provide an illustrative example to showcase the practical applications and utility of our main findings.

## 2. Preliminaries

Now, we gather a few pertinent definitions and facts that will be relevant for the subsequent discussion:

In 1994, Matthews put forth the definition of a partial metric space, which is as follows:

Definition 2.1 ([18]). Let $V \neq \emptyset$. A mapping $p\colon V \times V \to \mathbb{R}^+$ is called partial metric on $U$ if (for all $\chi, \wp, \theta \in V$):
(1) $\chi = \wp \Leftrightarrow p(\chi,\chi) = p(\chi,\wp) = p(\wp,\wp)$,
(2) $p(\chi,\chi) \leq p(\chi,\wp)$,
(3) $p(\chi,\wp) = p(\wp,\chi)$,
(4) $p(\chi,\theta) \leq p(\chi,\wp) + p(\wp,\theta) - p(\wp,\wp)$.
The pair $(V,p)$ is called a partial metric space.

Since then this fundamental concept has played a significant role in various mathematical investigations and continues to be relevant in subsequent research.

Remark 2.1. Obviously, if $p(\chi,\chi) = 0$ for all $\chi \in V$, then $(V,p)$ is a metric space.

Throughout the paper, we denote by $(\mathbb{B},\oplus)$ an idempotent space and $\mathbb{B}^+ := \{\chi \in \mathbb{B} : \chi \geq_\oplus 0_\mathbb{B}\}$, where $0_\mathbb{B}$ is a zero element in $\mathbb{B}$.

Definition 2.2 ([13]). We define an order relations on idempotent space $(\mathbb{B},\oplus)$ by

$$\chi \leq_\oplus \wp \quad \Leftrightarrow \quad \chi \oplus \wp = \wp.$$

Also, we write $\wp \geq_\oplus \chi$ instead of $\chi \leq_\oplus \wp$. Similarly,

$$\chi <_\oplus \wp \quad \Leftrightarrow \quad \chi \oplus \wp = \wp \quad \text{and} \quad \chi \neq \wp.$$

Also, we write $\wp >_\oplus \chi$ instead of $\chi <_\oplus \wp$.

Example 2.1 ([13]). Let $\mathbb{B} = \mathbb{R}$ endowed with $\chi \oplus \wp = \max\{\chi,\wp\}$ or $\chi \oplus \wp = \min\{\chi,\wp\}$ for all $\chi,\wp \in \mathbb{R}$. Then $(\mathbb{B},\oplus)$ is an idempotent space.

Example 2.2 ([13]). Let $\mathbb{B}$ be a set of real matrices. The conforming matrices $M_1 = (\chi_{ij})$, $M_2 = (\wp_{ij})$ satisfy the conventional rule of matrix addition together with multiplication by a scalar $\alpha \in \mathbb{R}$ as follows

$$\{M_1 + M_2\} = \chi_{ij} \oplus \wp_{ij}, \quad \{\alpha M_1\} = \alpha\chi_{ij}.$$

Then $(\mathbb{B},\oplus)$ is an idempotent space.

Definition 2.3 ([13]). A vector space $\mathbb{B}$ over field $\mathbb{R}$ is said to be an idempotent space if it satisfies the following (for all $\chi, \wp, \theta \in \mathbb{B}$):

(i) $\chi \oplus (\wp \oplus \theta) = (\chi \oplus \wp) \oplus \theta$;

(ii) $\chi \oplus \chi = \chi$.

An idempotent space $(\mathbb{B}, \oplus)$ is commutative if $\chi \oplus \wp = \wp \oplus \chi$.

Definition 2.4 ([13]). Assume that $(\mathbb{B}, \leq)$ is a partially ordered set. We define

$$\overset{\oplus}{\max}\{l, m\} = \begin{cases} l, & \text{if } m \leq l, \\ m, & \text{if } l < m, \\ 0_{\mathbb{B}}, & \text{otherwise,} \end{cases}$$

for $l, m \in \mathbb{B}$, and

$$\overset{\oplus}{\max}\{\chi_1, \chi_2, \cdots, \chi_n\} = \overset{\oplus}{\max}\left\{ \overset{\oplus}{\max}\{\chi_1, \chi_2, \cdots, \chi_{n-1}\}, \chi_n \right\},$$

for $\chi_1, \chi_2, \cdots, \chi_n \in \mathbb{B}$.

Definition 2.5 ([13]). Let $(\mathbb{B}, \leq)$ be a partially ordered vector space. Let $\{\chi_n\}$ be a sequence in $\mathbb{B}$ and $\chi \in \mathbb{B}$. If $\forall\ 0_{\mathbb{B}} < a \ \ \exists\ N \in \mathbb{N} \ \ \forall\ n \geq N$ $(\chi_n - \chi < a)$, then the sequence $\{\chi_n\}$ is called convergent and converges to $\chi$, where $\chi$ is called limit of $\{\chi_n\}$. We write then $\lim_{n\to\infty} \chi_n = \chi$ or $\chi_n \to \chi$ as $n \to \infty$.

Definition 2.6 ([13]). Consider a partially ordered vector space $(\mathbb{B}, \leq)$, and let us focus on its order relation. We say that the order relation on $\mathbb{B}$ has a positive cone ordering property when the following conditions hold: For any vector $0_{\mathbb{B}} \leq r \leq s$ and scalar inequalities $0 \leq l \leq \wp$, the resulting inequalities are as follows:

$$0_{\mathbb{B}} \leq lr \leq ls, \quad l\chi \leq \wp\chi$$

for all $0_{\mathbb{B}} \leq \chi \in \mathbb{B}$. This property essentially ensures a well-behaved ordering system within the vector space, where the positive cone contributes to maintaining the ordering of vectors and scalars consistently.

Definition 2.7 ([13]). Let $(\mathbb{B}, \leq)$ be a partially ordered vector space. If the order relation on $\mathbb{B}$ has the positive cone ordering property, then $\mathbb{B}$ is a normal space.

Now, we recall the definition of PIV-metric space as follows:

Definition 2.8 ([13]). Let $V \neq \emptyset$. An idempotent valued $S$-metric on $V$ is a function $d_I\colon V \times V \to \mathbb{B}^+$ that satisfies the following (for all $\chi, \varsigma, \theta \in V$):
(i) $d_I(\chi, \varsigma) = d_I(\chi, \chi) = d_I(\varsigma, \varsigma) \Leftrightarrow \chi = \varsigma$;
(ii) $d_I(\chi, \chi) \leq_{\oplus} d_I(\chi, \varsigma)$;
(iii) $d_I(\chi, \varsigma) = d_I(\varsigma, \chi)$;
(iv) $d_I(\chi, \varsigma) \leq_{\oplus} d_I(\chi, \theta) \oplus d_I(\theta, \varsigma)$.
The triplet $(V, \mathbb{B}, d_I)$ is called PIV-metric space.

Example 2.3 ([13]). Let $V = [0, \infty)$, $\mathbb{B} = \mathbb{R}$ endowed with $\chi \oplus \varsigma = \max\{\chi, \varsigma\}$. Define a mapping $d\colon V \times V \to \mathbb{B}^+$ by

$$d(\chi, \varsigma) = \chi \oplus \varsigma, \quad \forall\ \chi, \varsigma \in V.$$

Then the triplet $(V, \mathbb{B}, d)$ is a PIV-metric space.

The concept of $S$-metric space was initiated by Sedghi et al. [20] in 2012 and runs as follows.

Definition 2.9 ([20]). Let $V \neq \emptyset$. An $S$-metric space is a function $d\colon V \times V \times V \to \mathbb{R}^+$ that satisfies the following (for all $\chi, \varsigma, \theta, \sigma \in V$):
(i) $d(\chi, \varsigma, \theta) \geq 0$;
(ii) $d(\chi, \varsigma, \theta) = 0 \ \Leftrightarrow \chi = \varsigma = \theta$;
(iii) $d(\chi, \varsigma, \theta) \leq d(\chi, \chi, \sigma) + d(\varsigma, \varsigma, \sigma) + d(\theta, \theta, \sigma)$.
The pair $(V, d)$ is called $S$-metric space.

The following notion of partial $S$-metric space is proposed by Asil et al. [1].

Definition 2.10 ([1]). Let $V \neq \emptyset$. A partial $S$-metric space is a function $d\colon V \times V \times V \to \mathbb{R}^+$ that satisfies the following (for all $\chi, \varsigma, \theta, \sigma \in V$):
(i) $d(\chi, \chi, \chi) = d(\varsigma, \varsigma, \varsigma) = d(\chi, \chi, \varsigma) \Leftrightarrow \chi = \varsigma$;
(ii) $d(\chi, \chi, \chi) \leq d(\chi, \varsigma, \theta)$;
(iii) $d(\chi, \varsigma, \theta) \leq d(\chi, \chi, \sigma) + d(\varsigma, \varsigma, \sigma) + d(\theta, \theta, \sigma) - 2d(\sigma, \sigma, \sigma)$.
The pair $(V, d)$ is called partial $S$-metric space.

Remark 2.2. Clearly, every $S$-metric space can be considered as a partial $S$-metric space. However, it is important to note that the converse is not always true in general. In other words, not every partial $S$-metric space can be regarded as an $S$-metric space. The distinction between the two lies in the specific conditions and properties that they satisfy, making them distinct concepts within the realm of mathematical spaces.

## 3. Idempotent-valued $S$-metric space

Now, we introduce the following definition of PIV-$S$-metric space.

Definition 3.1. Let $V \neq \emptyset$. A PIV-$S$-metric on $V$ is a function $d\colon V \times V \times V \to \mathbb{B}^+$ that satisfies the following (for all $\chi, \wp, \theta, \sigma \in V$):
(i) $d(\chi,\chi,\chi) = d(\wp,\wp,\wp) = d(\chi,\chi,\wp) \Leftrightarrow \chi = \wp$;
(ii) $d(\chi,\chi,\chi) \leq_{\oplus} d(\chi,\wp,\theta)$;
(iii) $d(\chi,\chi,\wp) = d(\wp,\wp,\chi)$;
(iv) $d(\chi,\wp,\theta) \leq_{\oplus} d(\chi,\chi,\sigma) \oplus d(\wp,\wp,\sigma) \oplus d(\theta,\theta,\sigma)$.
The triplet $(V, \mathbb{B}, d)$ is called PIV-$S$-metric space.

Example 3.1. Let $V = [0,\infty)$, $\mathbb{B} = \mathbb{R}$ endowed with the operation $\chi \oplus \wp = \max\{\chi, \wp\}$. Define a mapping $d\colon V \times V \times V \to \mathbb{B}^+$ by

$$d(\chi,\wp,\theta) = \chi \oplus \wp \oplus \theta, \quad \forall\, \chi,\wp,\theta \in V.$$

Then the triplet $(V, \mathbb{B}, d)$ is a PIV-$S$-metric space. When we connect points $\chi, \wp, \theta$ with lines, forming a triangle, and choose a point $\sigma$ inside this triangle, the inequality

$$d(\chi,\wp,\theta) = d(\chi,\chi,\sigma) \oplus d(\wp,\wp,\sigma) \oplus d(\theta,\theta,\sigma)$$

holds true. This expression signifies the fulfillment of a specific property within the PIV-$S$-metric space, demonstrating its unique characteristics and structure.

Example 3.2. Let $Y \neq \emptyset$ and $V = B(Y, [0,\infty))$ be the set of bounded mappings with order-bounded range. Suppose $\mathbb{B} = B(V, (\mathbb{R}, \oplus))$ with $(T_1 \oplus T_2)(\chi) = T_1(\chi) \oplus T_2(\chi)$ and $\chi \oplus \wp = \max\{\chi, \wp\}$. Define a mapping $d\colon V \times V \times V \to \mathbb{B}^+$ by

$$d(T_1, T_2, T_3)(\chi) = \max\{T_1(\chi), T_2(\chi), T_3(\chi)\}, \quad \forall\, T_1, T_2, T_3 \in V.$$

Then the triplet $(V, \mathbb{B}, d)$ is a PIV-$S$-metric space.

Example 3.3. Let $V \neq \emptyset$, $d_I$ endowed with the operation $\oplus$ PIV-metric on $V$. Then

$$d(\chi,\wp,\theta) = d_I(\chi,\theta) \oplus d_I(\wp,\theta)$$

is a PIV-$S$-metric on $V$.

Let $(V, \mathbb{B}, d)$ be a PIV-$S$-metric space. Then the open ball with radius $\epsilon >_{\oplus} 0_{\mathbb{B}}$ and center $\chi$ is defined by

$$B_d(\chi, \epsilon) = \{\varsigma \in V : d(\varsigma, \varsigma, \chi) <_{\oplus} d(\chi, \chi, \chi) \oplus \epsilon\},$$

and the closed ball with radius $\epsilon >_{\oplus} 0_{\mathbb{B}}$ and center $\chi$ is defined by

$$B_d(\chi, \epsilon) = \{\varsigma \in V : d(\varsigma, \varsigma, \chi) \leq_{\oplus} d(\chi, \chi, \chi) \oplus \epsilon\}.$$

The family of open balls

$$\Lambda = \{B_d(\chi, \epsilon) : \chi \in V,\ \epsilon >_{\oplus} 0_{\mathbb{B}}\},$$

where $B_d(\chi, \epsilon)$ represents the open ball centered at $\chi$ with radius $\epsilon$, constitutes a basis for a certain topology $\tau_{\oplus}$ on $V$. This means that $\tau_{\oplus}$ is a collection of open sets in $V$, and any open set in $\tau_{\oplus}$ can be expressed as a union of these open balls from $\Lambda$. The topology $\tau_{\oplus}$ is built upon the notion of open balls defined by the $S$-metric space, which plays a crucial role in describing the open sets and the structure of the space $V$.

Definition 3.2. Let $(V, \mathbb{B}, d)$ be a PIV-$S$-metric space. We say that

(i) A sequence $\{\chi_n\} \subset V$ is called convergent to $\chi$ if and only if

$$\lim_{n \to \infty} d(\chi_n, \chi_n, \chi) = d(\chi, \chi, \chi).$$

(ii) A sequence $\{\chi_n\} \subset V$ is called Cauchy if and only if $\lim_{n,m \to \infty} d(\chi_n, \chi_n, \chi_m)$ exists and is finite.

(iii) The PIV-$S$-metric space $(V, \mathbb{B}, d)$ is said to be complete if every Cauchy sequence $\{\chi_n\}$ in $V$ converges to a point $\chi \in V$ such that

$$d(\chi, \chi, \chi) = \lim_{n \to \infty} d(\chi_n, \chi_n, \chi) = \lim_{n,m \to \infty} d(\chi_n, \chi_n, \chi_m).$$

Definition 3.3. Let $(V, \mathbb{B}, d)$ be a PIV-$S$-metric space. A mapping $f\colon V \to V$ is called a continuous at point $\chi \in V$ if for any $\chi_n \to \chi$ implies that $f\chi_n \to f\chi$.

Our main result runs as follows:

THEOREM 3.1. *Let $(V, \mathbb{B}, d)$ be a complete PIV-$S$-metric space and $f\colon V \to V$ satisfy the following:*

$$d(f\chi, f\wp, f\theta) \preceq_{\oplus} \Theta\left(\overset{\oplus}{\max}\{d(\chi, \wp, \theta), d(\chi, \chi, f\chi), d(\wp, \wp, f\wp), d(\theta, \theta, f\theta)\}\right) \tag{3.1}$$

*for all $\chi, \wp, \theta \in V$, where $\Theta\colon \mathbb{B}^+ \to \mathbb{B}^+$ is a continuous, non-decreasing function such that $\lim_{n\to\infty} \Theta^n(b) = 0_{\mathbb{B}}$ and $\Theta(b) <_{\oplus} b$ for $b \in \mathbb{B}^+$. Then $f$ has a unique fixed point $\xi \in V$ and $d(\xi, \xi, \xi) = 0_{\mathbb{B}}$.*

PROOF. Choose $\chi_0 \in V$ and define $\chi_{n+1} = f\chi_n$ for all $n \in \mathbb{N}_0$. For any $n \in \mathbb{N}_0$, we have

$$\begin{aligned} d(\chi_{n+1}, \chi_{n+1}, \chi_n) &= d(f\chi_n, f\chi_n, f\chi_{n-1}) \\ &\preceq_{\oplus} \Theta\Big(\overset{\oplus}{\max}\{d(\chi_n, \chi_n, \chi_{n-1}), d(\chi_n, \chi_n, f\chi_n), d(\chi_n, \chi_n, f\chi_n), \\ &\qquad d(\chi_{n-1}, \chi_{n-1}, f\chi_{n-1})\}\Big) \\ &= \Theta\Big(\overset{\oplus}{\max}\{d(\chi_n, \chi_n, \chi_{n-1}), d(\chi_n, \chi_n, f\chi_n), d(\chi_{n-1}, \chi_{n-1}, f\chi_{n-1})\}\Big) \\ &= \Theta\Big(\overset{\oplus}{\max}\{d(\chi_n, \chi_n, \chi_{n-1}), d(\chi_n, \chi_n, \chi_{n+1}), d(\chi_{n-1}, \chi_{n-1}, \chi_n)\}\Big) \\ &= \Theta\Big(\overset{\oplus}{\max}\{d(\chi_n, \chi_n, \chi_{n-1}), d(\chi_n, \chi_n, \chi_{n+1}), d(\chi_n, \chi_n, \chi_{n-1})\}\Big) \\ &= \Theta\Big(\overset{\oplus}{\max}\{d(\chi_n, \chi_n, \chi_{n-1}), d(\chi_n, \chi_n, \chi_{n+1})\}\Big). \end{aligned}$$

Assume that

$$\overset{\oplus}{\max}\{d(\chi_n, \chi_n, \chi_{n-1}), d(\chi_n, \chi_n, \chi_{n+1})\} = d(\chi_n, \chi_n, \chi_{n+1}),$$

then we have

$$\begin{aligned} d(\chi_{n+1}, \chi_{n+1}, \chi_n) &\preceq_{\oplus} \Theta\big(d(\chi_n, \chi_n, f\chi_n)\big) \\ &= \Theta\big(d(\chi_n, \chi_n, \chi_{n+1})\big) = \Theta\big(d(\chi_{n+1}, \chi_{n+1}, \chi_n)\big), \end{aligned}$$

a contradiction. Thus

$$\overset{\oplus}{\max}\{d(\chi_n, \chi_n, \chi_{n-1}), d(\chi_n, \chi_n, \chi_{n+1})\} = d(\chi_n, \chi_n, \chi_{n-1}).$$

Therefore, (3.1) gives rise

$$d(f\chi_n, f\chi_n, f\chi_{n-1}) = d(\chi_{n+1}, \chi_{n+1}, \chi_n) \preceq_{\oplus} \Theta\big(d(\chi_n, \chi_n, \chi_{n+1})\big). \tag{3.2}$$

By continuing this process, we get

$$d(\chi_{n+1}, \chi_{n+1}, \chi_n) \preceq_{\oplus} \Theta^n\big(d(\chi_0, \chi_0, \chi_1)\big). \tag{3.3}$$

This demonstrates that $\lim_{n\to\infty} d(\chi_{n+1}, \chi_{n+1}, \chi_n) = 0_{\mathbb{B}}$. Now, we assert that the sequence $\{\chi_n\}$ is Cauchy in $V$. For $n, m \in \mathbb{N}_0$ $(n < m)$, we have

$$\begin{aligned}
d(\chi_n,\chi_n, \chi_m) &\preceq_{\oplus} d(\chi_n, \chi_n, \chi_{n+1}) \oplus d(\chi_n, \chi_n, \chi_{n+1}) \oplus d(\chi_m, \chi_m, \chi_{n+1})\\
&= d(\chi_n, \chi_n, \chi_{n+1}) \oplus d(\chi_m, \chi_m, \chi_{n+1}) \text{ (By (ii) of Definition 2.3)}\\
&\preceq_{\oplus} d(\chi_n, \chi_n, \chi_{n+1}) \oplus \{d(\chi_m, \chi_m, \chi_{n+2}) \oplus d(\chi_m, \chi_m, \chi_{n+2})\\
&\quad \oplus d(\chi_{n+1}, \chi_{n+1}, \chi_{n+2})\}\\
&= d(\chi_n, \chi_n, \chi_{n+1}) \oplus d(\chi_{n+1}, \chi_{n+1}, \chi_{n+2}) \oplus d(\chi_m, \chi_m, \chi_{n+2})\\
&\preceq_{\oplus} d(\chi_n, \chi_n, \chi_{n+1}) \oplus d(\chi_{n+1}, \chi_{n+1}, \chi_{n+2})\\
&\quad \oplus \{d(\chi_m, \chi_m, \chi_{n+3}) \oplus d(\chi_{n+2}, \chi_{n+2}, \chi_{n+3})\}\\
&\preceq_{\oplus} d(\chi_n, \chi_n, \chi_{n+1}) \oplus d(\chi_{n+1}, \chi_{n+1}, \chi_{n+2})\\
&\quad \oplus d(\chi_{n+2}, \chi_{n+2}, \chi_{n+3}) \oplus d(\chi_m, \chi_m, \chi_{n+3})\\
&\preceq_{\oplus} d(\chi_n, \chi_n, \chi_{n+1}) \oplus d(\chi_{n+1}, \chi_{n+1}, \chi_{n+2}) \oplus \cdots \oplus d(\chi_m, \chi_m, \chi_{m-1})\\
&= d(\chi_{n+1}, \chi_{n+1}, \chi_n) \oplus d(\chi_{n+2}, \chi_{n+2}, \chi_{n+1}) \oplus \cdots \oplus d(\chi_m, \chi_m, \chi_{m-1}).
\end{aligned}$$

Now, by employing (3.3), we get

$$\begin{aligned}
d(\chi_n, \chi_n, \chi_m) &\preceq_{\oplus} \Theta^n\big(d(\chi_1, \chi_1, \chi_0)\big) \oplus \Theta^{n+1}\big(d(\chi_1, \chi_1, \chi_0)\big)\\
&\quad \oplus \Theta^{n+2}\big(d(\chi_1, \chi_1, \chi_0)\big) \oplus \cdots \oplus \Theta^{m-1}\big(d(\chi_1, \chi_1, \chi_0)\big)\\
&\preceq_{\oplus} \Theta^n\big(d(\chi_1, \chi_1, \chi_0)\big) \oplus \Theta^n\big(d(\chi_1, \chi_1, \chi_0)\big)\\
&\quad \oplus \Theta^n\big(d(\chi_1, \chi_1, \chi_0)\big) \oplus \cdots \oplus \Theta^n\big(d(\chi_1, \chi_1, \chi_0)\big)\\
&= \Theta^n\big(d(\chi_1, \chi_1, \chi_0)\big) \text{ (By (ii) of Definition (2.3))}.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} d(\chi_n, \chi_n, \chi_m) = 0_{\mathbb{B}}.$$

Hence, the sequence $\{\chi_n\}$ is Cauchy in $V$. Since $V$ is a complete PIV-$S$-metric space then there exists $\xi \in V$ such that

$$d(\xi,\xi,\xi) = \lim_{m\to\infty} d(\chi_n,\chi_n,\xi) = \lim_{n,m\to\infty} d(\chi_n,\chi_n,\chi_m) = 0_{\mathbb{B}}. \tag{3.4}$$

Now, to show that $\xi \in V$ is a fixed point of $f$, we note that

$$\begin{aligned} d(\xi,\xi,f\xi) &\leq_\oplus d(\xi,\xi,\chi_{n+1}) \oplus d(T\xi,T\xi,\chi_{n+1}) \\ &= d(\xi,\xi,\chi_{n+1}) \oplus d(f\xi,f\xi,f\chi_n). \end{aligned}$$

From (3.2), we have $d(f\xi,f\xi,f\chi_n) \leq_\oplus \Theta\big(d(\xi,\xi,\chi_n)\big)$, then

$$\begin{aligned} d(\xi,\xi,f\xi) &\leq_\oplus d(\xi,\xi,\chi_{n+1}) \oplus \Theta\big(d(\xi,\xi,\chi_n)\big) \\ &= d(\chi_{n+1},\chi_{n+1},\xi) \oplus \Theta\big(d(\chi_n,\chi_n,\xi)\big). \end{aligned}$$

Using (3.4) in the above inequality, we get $d(\xi,\xi,f\xi) = 0_{\mathbb{B}}$ which implies $f\xi = \xi$. Hence, $\xi$ is a fixed point of $f$.

For the uniqueness part, let us assume that there exist two points $\xi$ and $\xi^*$ in the space $V$ such that $f\xi = \xi$ and $f\xi^* = \xi^*$. In other words, $\xi$ and $\xi^*$ are fixed points of the mapping $f$. Then by employing (3.1), we have

$$\begin{aligned} d(\xi,\xi,f\xi^*) &= d(f\xi,f\xi,f\xi^*) \\ &\leq_\oplus \alpha \overset{\oplus}{\max}\{d(\xi,\xi,\xi^*), d(\xi,\xi,f\xi), d(\xi,\xi,f\xi), d(\xi^*,\xi^*,f\xi^*)\} \\ &= \Theta\big(\overset{\oplus}{\max}\{d(\xi,\xi,\xi^*), d(\xi,\xi,\xi), d(\xi^*,\xi^*,\xi^*)\}\big) \\ &= \Theta\big(d(\xi,\xi,\xi^*)\big) <_\oplus d(\xi,\xi,\xi^*), \end{aligned}$$

a contradiction. Hence $\xi = \xi^*$. Therefore, $\xi$ is a unique fixed point of $f$.

Finally, we show that $d(\xi,\xi,\xi) = 0_{\mathbb{B}}$. Let us suppose, $d(\xi,\xi,\xi) >_\oplus 0_{\mathbb{B}}$. Then (3.1) implies that

$$\begin{aligned} d(\xi,\xi,\xi) &= d(f\xi,f\xi,f\xi^*) \\ &\leq_\oplus \Theta\big(\overset{\oplus}{\max}\{d(\xi,\xi,\xi), d(\xi,\xi,f\xi), d(\xi,\xi,f\xi), d(\xi,\xi,f\xi)\}\big) \\ &= \Theta\big(\overset{\oplus}{\max}\{d(\xi,\xi,\xi), d(\xi,\xi,\xi), d(\xi,\xi,\xi)\}\big) \\ &= \Theta\big(d(\xi,\xi,\xi)\big) <_\oplus d(\xi,\xi,\xi), \end{aligned}$$

which leads to a contradiction. Consequently, $d(\xi,\xi,\xi) = 0_{\mathbb{B}}$. □

Example 3.4. Let $V=[1,\infty)$, $\mathbb{B}=\mathbb{R}$ endowed with the operation $\chi \oplus \varsigma = \max\{\chi, \varsigma\}$. Define a mapping $d\colon V\times V\times V \to \mathbb{B}^+$ by

$$d(\chi,\varsigma,\theta)=\chi\oplus\varsigma\oplus\theta, \quad \forall\ \chi,\varsigma,\theta\in V.$$

Then the triplet $(V,\mathbb{B},d)$ is a complete PIV-$S$-metric space. Define a mapping $f\colon V\to V$ by

$$f\chi=\frac{\chi+2}{3} \quad \text{for all} \quad \chi\in V$$

and define $\Theta\colon \mathbb{B}^+\to\mathbb{B}^+$ by $\Theta(b)=\frac{b}{2}$. Here, one can easily seen that $\Theta$ is continuous and non-decreasing function. Now, we have

$$\begin{aligned}
d(f\chi,f\varsigma,f\theta) &= \frac{\chi+2}{3}\oplus\frac{\varsigma+2}{3}\oplus\frac{\theta+2}{3}\\
&= \max\left\{\frac{\chi+2}{3},\frac{\varsigma+2}{3},\frac{\theta+2}{3}\right\}=\frac{1}{3}\max\{\chi,\varsigma,\theta\}\\
&\le \frac{1}{3}\max\Bigg\{\left(\frac{\chi+2}{3},\frac{\varsigma+2}{3},\frac{\theta+2}{3}\right),\left(\chi,\chi,\frac{\chi+2}{3}\right),\\
&\quad \left(\varsigma,\varsigma,\frac{\varsigma+2}{3}\right),\left(\theta,\theta,\frac{\theta+2}{3}\right)\Bigg\}\\
&= \frac{1}{3}\overset{\oplus}{\max}\Bigg\{\left(\frac{\chi+2}{3}\oplus\frac{\varsigma+2}{3}\oplus\frac{\theta+2}{3}\right),\left(\chi\oplus\chi\oplus\frac{\chi+2}{3}\right),\\
&\quad \left(\varsigma\oplus\varsigma\oplus\frac{\varsigma+2}{3}\right),\left(\theta\oplus\theta\oplus\frac{\theta+2}{3}\right)\Bigg\}\\
&\le_{\oplus} \Theta\big(\overset{\oplus}{\max}\{d(\chi,\varsigma,\theta),d(\chi,\chi,f\chi),d(\varsigma,\varsigma,f\varsigma),d(\theta,\theta,f\theta)\}\big).
\end{aligned}$$

Therefore, all the prerequisites of Theorem 3.1 are met, confirming that $\xi=1$ stands as the unique fixed point of the given mapping $f$.

Corollary 3.1. *Consider a complete PIV-$S$-metric space $(V,\mathbb{B},d)$ and a mapping $f\colon V\to V$ that fulfills the following:*

$$d(f\chi,f\varsigma,f\theta)\le_{\oplus}\alpha d(\chi,\varsigma,\theta)$$

*for all $\chi,\varsigma,\theta\in V$, where $\alpha\in[0,1)$. Then $f$ has a unique fixed point $\xi\in V$ and $d(\xi,\xi,\xi)=0_{\mathbb{B}}$.*

## 4. Best proximity point results in PIV-$S$-metric space

In this section, we delve into the study of best proximity points within the framework of PIV-$S$-metric spaces. The significance of best proximity point results lies in their relevance to best approximation outcomes from this perspective. The inception of best proximity point results can be traced back to Fan [11], who demonstrated that for a continuous mapping $f\colon K \to X$ defined on a nonempty compact convex subset $K$ of a Hausdorff locally convex topological vector space $V$, equipped with the metric $d$, there exists a point $\chi \in K$ such that $d(\chi, f\chi) = \inf\{d(\wp, f\chi) : \wp \in K\}$. Notably, when $f$ is a self-mapping, the best proximity point transforms into a fixed point. The best approximation theorem ensures the existence of an approximate solution, while the best proximity point theorem proves instrumental in resolving the problem and offering an optimal approximate solution.

Consider two nonempty subsets, denoted as $A$ and $B$, of a metric space $(V, d)$. In the context of a mapping $f\colon A \to B$, an element $\chi \in A$ is referred to as a fixed point if $f\chi = \chi$. It is important to highlight that the requirement $f(A) \cap A \neq \emptyset$ is a vital condition for the presence of a fixed point in the mapping $f$. However, it's worth noting that this condition is necessary but not sufficient. If the intersection of $A$ and $f(A)$ is empty, it implies that there is no fixed point for $f$. In such cases, it is customary to seek an element $\chi$ that is in some way closest to $f\chi$, with the hope of finding an approximate solution even though a fixed point may not exist.

Proposition 4.1. *Let $(V, \mathbb{B}, d)$ be a PIV-$S$-metric space and $S_d\colon V \times V \to \mathbb{B}$ defined as follows*

$$S_d(\chi, \wp) = d(\chi, \chi, \wp) \oplus d(\wp, \wp, \chi). \tag{4.1}$$

*Then $(V, \mathbb{B}, S_d)$ is a PIV-metric space.*

Proof. Observe that the conditions (i)–(iii) are satisfied. Now to for (iv), we have

$$\begin{aligned} S_d(\chi, \wp) = d(\chi, \chi, \wp) \oplus d(\wp, \wp, \chi) &\leq_{\oplus} (d(\chi, \chi, \theta) \oplus d(\chi, \chi, \theta) \oplus d(\wp, \wp, \theta)) \\ &\quad \oplus (d(\wp, \wp, \theta) \oplus d(\wp, \wp, \theta) \oplus d(\chi, \chi, \theta)) \\ &= (d(\chi, \chi, \theta) \oplus d(\wp, \wp, \theta)) \oplus (d(\wp, \wp, \theta) \oplus d(\theta, \theta, \theta)) \\ &= d(\theta, \theta, \theta) \oplus d(\theta, \theta, \chi) \oplus d(\theta, \theta, \wp) \oplus d(\wp, \wp, \theta) = S_d(\chi, \theta) \oplus S_d(\theta, \wp). \end{aligned}$$

Hence, $(V, \mathbb{B}, S_d)$ is a PIV-metric space. □

Definition 4.1. Let $A$ and $B$ be two nonempty subsets of $(V, \mathbb{B}, d)$. Denote the distance between $A$ and $B$ by

$$S_d(A, B) = \overset{\oplus}{\inf}\{S_d(\chi, \wp) : \chi \in A,\ \wp \in B\}.$$

Define the following

$$A_0 = \{\chi \in A : S_d(\chi, \wp) = S_d(A, B) \text{ for some } \wp \in B\} \quad \text{and}$$

$$B_0 = \{\chi \in B : S_d(\chi, \wp) = S_d(A, B) \text{ for some } \wp \in A\}.$$

In the same way, we have developed the best proximity point analysis in the framework of PIV-$S$-metric space $(V, \mathbb{B}, d)$.

An element $\chi \in A$ is said to be best proximity point of the map $f \colon A \to B$ if

$$S_d(\chi, fA) = S_d(A, B).$$

The global minimum of the mapping $\chi \to d(\chi, f\chi)$ is identified as the best proximity point, as it satisfies the condition $S_d(\chi, f\chi) \geq_{\oplus} S_d(A, B)$ for every $\chi \in A$. In other words, the best proximity point minimizes the distance between a point and its image under the mapping, making it an optimal approximate solution within the space $A$ and $B$. If the mapping is a self-mapping, it is evident that the best proximity point coincides with a fixed point.

The fundamental objective of the best proximity point theory is to establish sufficient conditions that guarantee the existence of such points. By providing these conditions, the theory offers a valuable tool to determine when best proximity points are attainable, allowing for the identification of optimal approximate solutions in various scenarios.

In what follows, we introduce two notions.

Definition 4.2. Let $(V, \mathbb{B}, d)$ be a PIV-$S$-metric space and $A$, $B$ two nonempty subsets of $V$. The set $B$ is said to be approximatively compact with respect to $A$ if every sequence $\{\wp_n\} \subseteq B$ satisfying the condition $S_d(\chi, \wp_n) \to S_d(\chi, B)$ has a convergent subsequence for some $\chi$ in $A$.

Definition 4.3. Let $A$ and $B$ be two nonempty subsets of a PIV-$S$-metric space $(V, \mathbb{B}, d)$. The mapping $f \colon A \to B$ is said to be $\oplus-$proximal contraction mapping if

$$\left.\begin{aligned} S_d(x, f\chi) &= S_d(A, B) \\ S_d(y, f\wp) &= S_d(A, B) \end{aligned}\right\} \text{ implies } \quad d(x, x, y) \leq_{\oplus} \alpha d(\chi, \chi, \wp), \tag{4.2}$$

where $\alpha \in [0, 1)$, for all $x, y, \chi, \wp \in A$.

THEOREM 4.1. *Consider two nonempty subsets, $A$ and $B$ of a complete PIV-$S$-metric space $(V, \mathbb{B}, d)$, $A_0$ is nonempty, and $B$ is approximately compact with respect to $A$. Let us assume that $f\colon A \to B$ is a $\oplus-$proximal contraction mapping satisfying $f(A_0) \subseteq B_0$. Then $f$ has a unique best proximity point $\xi \in A$ such that $S_d(\xi, f\xi) = S_d(A, B)$.*

PROOF. Fix an arbitrary point $\chi_0 \in A_0$ and take $f\chi_0 \in f(A_0) \subseteq B_0$ into account. We can choose $\chi_1 \in A_0$ such that $S_d(\chi_1, f\chi_0) = S_d(A, B)$. Also, since $f\chi_1 \in f(A_0) \subseteq B_0$, there exists $\chi_1 \in A_0$ such that $S_d(\chi_2, f\chi_1) = S_d(A, B)$. Continuing this process, we can construct a sequence $\{\chi_n\}$ in $A_0$ such that

$$S_d(\chi_{n+1}, f\chi_n) = S_d(A, B), \quad \forall\, n \in \mathbb{N}_0,$$

which shows that

$$\begin{aligned} S_d(x, f\chi) &= S_d(A, B), \\ S_d(y, f\varsigma) &= S_d(A, B), \end{aligned}$$

where $x = \chi_n,\ \chi = \chi_{n-1},\ y = \chi_{n+1}$ and $\varsigma = \chi_n$. Then from (4.2), we obtain

$$\begin{aligned} d(\chi_n, \chi_n, \chi_{n+1}) &\leq_\oplus \alpha d(\chi_{n-1}, \chi_{n-1}, \chi_n) \\ &\leq_\oplus \alpha^2 d(\chi_{n-2}, \chi_{n-2}, \chi_{n-1}) \\ &\vdots \\ &\leq_\oplus \alpha^n d(\chi_0, \chi_0, \chi_1), \end{aligned}$$

which on making $n \to \infty$, gives

$$\lim_{n\to\infty} d(\chi_n, \chi_n, \chi_{n+1}) = 0_{\mathbb{B}}.$$

Now, we assert to show that $\{\chi_n\}$ is a Cauchy sequence in $(V, \mathbb{B}, d)$. Let us suppose, on the contrary, that there exists $\epsilon >_\oplus 0_{\mathbb{B}}$ and a subsequence $\{\chi_{n_k}\}$ of $\{\chi_n\}$ such that

$$d(\chi_{m_k}, \chi_{m_k}, \chi_{n_k}) \geq_\oplus \epsilon \quad \text{for } n_k \geq m_k > k. \tag{4.3}$$

Also, for any $m_k$, we can choose $n_k$ with $n_k > m_k$ which satisfies (4.3). Hence

$$d(\chi_{m_k}, \chi_{m_k}, \chi_{n_k - 1}) <_\oplus \epsilon.$$

Setting $\Gamma_n = d(\chi_n, \chi_n, \chi_{n-1})$, we have

$$\begin{aligned}\epsilon &\leq_{\oplus} d(\chi_{m_k}, \chi_{m_k}, \chi_{n_k}) = d(\chi_{n_k}, \chi_{n_k}, \chi_{m_k})\\ &\leq_{\oplus} d(\chi_{n_k}, \chi_{n_k}, \chi_{n_k-1}) \oplus d(\chi_{n_k}, \chi_{n_k}, \chi_{n_k-1}) \oplus d(\chi_{m_k}, \chi_{m_k}, \chi_{n_k-1})\\ &= d(\chi_{n_k}, \chi_{n_k}, \chi_{n_k-1}) \oplus d(\chi_{m_k}, \chi_{m_k}, \chi_{n_k-1}) <_{\oplus} \Gamma_{n_k} \oplus \epsilon.\end{aligned}$$

By taking limit as $k \to \infty$, we get

$$\lim_{k\to\infty} d(\chi_{m_k}, \chi_{m_k}, \chi_{n_k}) = \epsilon.$$

Further, we have

$$\begin{aligned}d(\chi_{m_k},\chi_{m_k}, \chi_{n_k}) &\leq_{\oplus} d(\chi_{m_k}, \chi_{m_k}, \chi_{n_k-1}) \oplus d(\chi_{n_k}, \chi_{n_k}, \chi_{m_k-1})\\ &\leq_{\oplus} d(\chi_{m_k}, \chi_{m_k}, \chi_{n_k-1}) \oplus d(\chi_{n_k}, \chi_{n_k}, \chi_{n_k-1})\\ &\quad \oplus d(\chi_{m_k-1}, \chi_{m_k-1}, \chi_{n_k-1})\\ &<_{\oplus} (\Gamma_{m_k} \oplus \Gamma_{n_k}) \oplus d(\chi_{m_k-1}, \chi_{m_k-1}, \chi_{n_k-1})\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}d(\chi_{m_k-1}, \chi_{m_k-1}, \chi_{n_k-1}) &\leq_{\oplus} d(\chi_{m_k-1}, \chi_{m_k-1}, \chi_{m_k}) \oplus d(\chi_{n_k-1}, \chi_{n_k-1}, \chi_{m_k})\\ &\leq_{\oplus} d(\chi_{m_k-1}, \chi_{m_k-1}, \chi_{m_k}) \oplus d(\chi_{n_k-1}, \chi_{n_k-1}, \chi_{n_k})\\ &\quad \oplus d(\chi_{m_k}, \chi_{m_k}, \chi_{n_k})\\ &<_{\oplus} \left(\Gamma_{m_{k-1}} \oplus \Gamma_{n_{k-1}}\right) \oplus d(\chi_{m_k}, \chi_{m_k}, \chi_{n_k}).\end{aligned} \tag{4.5}$$

By taking limit as $k \to \infty$ in (4.5) and using (4.4), we get

$$\lim_{k\to\infty} d(\chi_{m_k-1}, \chi_{m_k-1}, \chi_{n_k-1}) = \epsilon.$$

From (4.2) with $x = \chi_{m_k},\ \chi = \chi_{m_k-1},\ y = \chi_{n_k}$ and $\varsigma = \chi_{n_k-1}$, we obtain

$$\begin{aligned}d(\chi_{m_k}, \chi_{m_k}, \chi_{n_k}) &\leq_{\oplus} \alpha d(\chi_{m_k-1}, \chi_{m_k-1}, \chi_{n_k-1})\\ \Longrightarrow \lim_{k\to\infty} d(\chi_{m_k}, \chi_{m_k}, \chi_{n_k}) &\leq_{\oplus} \alpha \lim_{k\to\infty} d(\chi_{m_k-1}, \chi_{m_k-1}, \chi_{n_k-1})\\ \epsilon &\leq_{\oplus} \alpha\epsilon,\end{aligned}$$

which implies that $\epsilon = 0_{\mathbb{B}}$. Thus

$$\lim_{m,n\to\infty} d(\chi_m, \chi_{m_k}, \chi_{n_k}) = 0_{\mathbb{B}}.$$

Therefore, $\{\chi_n\}$ is a Cauchy sequence in $A$. Since $(A, \mathbb{B}, d)$ is a complete PIV-$S$-metric space then there exists $\xi \in A$ such that $\lim_{n\to\infty} \chi_n = \xi$. Furthermore, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} S_d(\xi, B) &\leq_\oplus S_d(\xi, f\chi_n) \leq_\oplus S_d(\xi, f\chi_{n+1}) \oplus S_d(\chi_{n+1}, f\chi_n) \\ &= S_d(\xi, f\chi_{n+1}) \oplus S_d(A, B). \end{aligned}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} S_d(\xi, f\chi_n) = S_d(\xi, B) = S_d(A, B).$$

The sequence $\{f\chi_n\}$ has a subsequence $\{f\chi_{n_k}\}$ that converges to some $\eta \in B$ since $B$ is approximately compact with respect to $A$. Hence,

$$S_d(\xi, \eta) = \lim_{n\to\infty} S_d(\chi_{n_k}, f\chi_{n_k}) = S_d(A, B),$$

that is, $\xi \in A_0$. Since $f\xi \in f(A_0) \subseteq B_0$ then there exists $\theta \in A_0$ such that $S_d(\theta, f\xi) = S_d(A, B)$. From (4.2) with $x = \chi_{n+1},\ \chi = \chi_n,\ y = \theta$ and $\wp = \xi$, we obtain

$$d(\chi_{n+1}, \chi_{n+1}, \theta) \leq_\oplus \alpha d(\chi_n, \chi_n, \xi) \implies \lim_{n\to\infty} d(\xi, \xi, \theta) = 0_{\mathbb{B}},$$

which implies that $d(\xi, \xi, \theta) = 0_{\mathbb{B}}$ and so $\xi = \theta$. Therefore, $S_d(\xi, f\xi) = S_d(A, B)$. Hence, $f$ has the best proximity point.

For the uniqueness part, suppose that $\xi \neq \xi^*$, $S_d(\xi, f\xi) = S_d(A, B)$ and $S_d(\xi^*, f\xi^*) = S_d(A, B)$. Employing (4.2) with $x = \chi = \xi$ and $y = \wp = \xi^*$, we have

$$d(\xi, \xi, \xi^*) \leq_\oplus \alpha d(\xi, \xi, \xi^*),$$

which implies that $d(\xi, \xi, \xi^*) = 0_{\mathbb{B}}$ and, consequently, $\xi = \xi^*$. □

Example 4.1. In Example 3.1, define a mapping $f\colon V \to V$ by

$$f\chi = \frac{\chi}{2} \quad \text{for all } \chi \in V.$$

Here, $(V, \mathbb{B}, d)$ is a complete PIV-$S$-metric space. By employing (4.1), we have

$$S_d(\chi, \wp) = \chi \oplus \wp.$$

Let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 5, 7\}$. Observe that $S_d(A, B) = 2,\ A_0 = \{4\},\ B_0 = \{2\}$ and $f(A_0) \subseteq B_0$. Suppose $S_d(x, f\chi) = S_d(A, B)$ and $S_d(y, f\varsigma) = S_d(A, B) = 2$ then $(x, \chi), (y, \varsigma) \in \{(2, 4), (2, 2)\}$. Since $S_d(x, f\chi) = S_d(A, B)$ and $S_d(y, f\varsigma) = S_d(A, B)$, then from (4.2) for $\alpha \leq 0$ and $x = y = 2$, we have

$$d(x, x, y) \leq_{\oplus} \alpha d(\chi, \chi, \varsigma).$$

Hence, all the hypotheses of Theorem 4.1 are fulfilled and $\xi = 2$ is a unique best proximity point of $f$.

## 5. Conclusion

In this paper, we presented a unification of the concepts of PIV-metric space and $S$-metric space, by introducing the notion of PIV-$S$-metric space and utilizing it to establish fixed point results. Additionally, we extended our study to prove best proximity point results within the framework of PIV-$S$-metric space. To illustrate the practical applications of our main results, we provided several examples that showcase their relevance and utility. This unified approach not only enhances our understanding of these mathematical spaces but also opens up new avenues for exploring fixed point and best proximity point properties within a broader context.

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