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A REALLY SIMPLE ELEMENTARY PROOF OF THE UNIFORM BOUNDEDNESS PRINCIPLE IN *F*-SPACES

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Abstract. We give a proof of the uniform boundedness principle for linear continuous maps from F-spaces into topological vector spaces which is elementary and also quite simple.

Within the general framework of topological vector spaces, the uniform boundedness principle states the following.

Theorem 1 (see [3, Theorem 2.6]). *Let $\mathbb{X}$ be an F-space and $\mathbb{Y}$ be a topological vector space. Let Γ be a family of linear continuous operators from $\mathbb{X}$ into $\mathbb{Y}$. Suppose that $\Gamma(x) := \{T(x) \colon T \in \Gamma\}$ is bounded for every $x \in \mathbb{X}$. Then Γ is equicontinuous.*

We refer the reader to [5] for an overview of the history of this important theorem. According to this paper, the proof of the uniform boundedness principle nowadays considered standard, which relies on the Baire category theorem, goes back to [1], and Banach, Steinhaus and Saks must get credit for it; the two first-named mathematicians were the authors of the article, and the role of the last-named one as the referee of it was crucial.

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As the Baire category theorem can be considered a deep result, the challenge of finding an elementary proof of the uniform boundedness principle as simple as possible has attracted some attention from specialists. Sokal [4] succeeded in this task by finding an elementary proof that can fairly be considered simpler than the previously released ones.

As Sokal, as well as Banach and Steinhaus, limited himself to dealing with Banach spaces, it is natural to look for a simple elementary argument that allows us to prove the uniform boundedness principle in the generality established in Theorem 1. In this regard, we emphasize that the proof from [4] heavily depends on computations that involve norms of vectors and operators, so it can not be verbatim transferred to the case when Γ is a family of operators from an F-space into another one.

In this note, we give a candidate to be such a proof of Theorem 1.

The proof

Recall that an F-norm is a subadditive map $\|\cdot\|$ from a vector space $\mathbb{X}$ over the real or complex field into $[0,\infty)$ such that $\|x\| > 0$ if $x \neq 0$, and $\sup_{|t|\le 1} \|tx\| \le \|x\|$ and $\lim_{t\to 0} \|tx\| = 0$ for all $x \in \mathbb{X}$. An F-space is a vector space $\mathbb{X}$ with an F-norm $\|\cdot\|$ such that $\mathbb{X}$ is complete with respect to the translation invariant metric $(x,y) \mapsto \|x-y\|$. It is known (see [2]) that a (Hausdorff) topological vector space is first-countable and complete if and only if there is an F-norm $\|\cdot\|$ over $\mathbb{X}$ so that

$$\{\{x \in \mathbb{X} \colon \|x\| < r\} : r \in (0,\infty)\}$$

is a basis of neighbourhoods of the origin, and $\mathbb{X}$ endowed with $\|\cdot\|$ is an F-space. Since our proof of Theorem 1 will rely on handling neighbourhoods of the origin, we will look at F-spaces as first-countable complete topological vector spaces. For convenience, we will denote by $\mathcal{U}_{\mathbb{X}}$ the set consisting of all neighbourhoods of the origin in an arbitrary topological vector space $\mathbb{X}$.

For further reference, we write down a ready consequence of completeness which is closely related to the fact that the convergence of the norms of a series in a Banach space implies norm convergence of the series.

LEMMA 1. *Let $\mathbb{X}$ be an F-space. Let $(V_n)_{n=1}^\infty$ be a sequence such that $\{V_n \colon n \in \mathbb{N}\}$ is a basis of neighbourhoods of the origin, and $V_{n+1} + V_{n+1} \subseteq V_n$ for all $n \in \mathbb{N}$. Suppose that a sequence $(x_n)_{n=1}^\infty$ in $\mathbb{X}$ satisfies $x_n \in V_n$ for all $n \in \mathbb{N}$. Then $\sum_{n=1}^\infty x_n$ converges to a vector in $V_1 + \overline{V_1}$.*

Proof. A straighforward induction on k yields $\sum_{n=j+1}^{j+k} x_n \in V_j$ for all j, $k \in \mathbb{N}$. Hence, $\sum_{n=1}^{\infty} x_n$ is a Cauchy series, so it converges; besides, if x denotes its sum, $x - \sum_{n=1}^{j} x_n \in \overline{V_j}$ for all $j \in \mathbb{N}$. Applying this with $j = 1$ we obtain $x = x_1 + x - x_1 \in V_1 + \overline{V_1}$. □

The following lemma states that we can weaken the equicontinuity condition by allowing translations and dilations of the neighbourhoods involved.

Lemma 2. *Let Γ be a family of linear mappings from a topological vector space $\mathbb{X}$ into another one $\mathbb{Y}$. Suppose that for every $U \in \mathcal{U}_{\mathbb{Y}}$ there exist $V \in \mathcal{U}_{\mathbb{X}}$, $z \in \mathbb{X}$ and $r \in (0, \infty)$ such that $T(z+V) \subseteq rU$ for all $T \in \Gamma$. Then, the family Γ is equicontinuous.*

Proof. Let $W \in \mathcal{U}_{\mathbb{Y}}$. Pick $U \in \mathcal{U}_{\mathbb{Y}}$ with $(U - U)/2 \subseteq W$. Let V, z and r be as in the statement. Choose $B \in \mathcal{U}_{\mathbb{X}}$ balanced with $B \subseteq \varepsilon V$, where $\varepsilon = 1/r$. For all $T \in \Gamma$ we have

$$\begin{aligned} T(B) &\subseteq T\left(\frac{(\varepsilon z + B) - (\varepsilon z - B)}{2}\right) = \frac{T(\varepsilon z + B) - T(\varepsilon z - B)}{2} \\ &\subseteq \frac{T(\varepsilon(z+V)) - T(\varepsilon(z+V))}{2} \subseteq \frac{\varepsilon r U - \varepsilon r U}{2} \subseteq W. \end{aligned}$$

□

Proof of Theorem 1. Fix a countable basis $\{B_n : n \in \mathbb{N}\}$ of neighbourhoods of the origin. Suppose that Γ is not equicontinuous, and pick $U \in \mathcal{U}_{\mathbb{Y}}$ balanced such that the condition of Lemma 2 breaks down. Choose $W \in \mathcal{U}_{\mathbb{Y}}$ balanced with $W + W \subseteq U$. Then, with the conventions that $T_0 = 0$ and $V_0 = \mathbb{X}$, we recursively construct $(V_n)_{n=1}^{\infty}$ in $\mathcal{U}_{\mathbb{X}}$, $(x_n)_{n=1}^{\infty}$ in $\mathbb{X}$, and $(T_n)_{n=1}^{\infty}$ in Γ such that, for all $n \in \mathbb{N}$,

- V_n is closed and small enough so that $V_n \subseteq B_n$, $V_n + V_n \subseteq V_{n-1}$ and $T_{n-1}(V_n + V_n) \subseteq (n-1)W$; and
- T_n and x_n witness that the condition of Lemma 2 does not hold for $V = V_n$, $z = \sum_{k=1}^{n-1} x_k$ and $r = n$, that is, $x_n \in V_n$ and $y_n := T_n(\sum_{k=1}^{n} x_k) \notin nU$.

Since $\{V_n : n \in \mathbb{N}\}$ is a basis of neighbourhoods of the origin, Lemma 1 allows us safely define $x = \sum_{n=1}^{\infty} x_n \in \mathbb{X}$, and infer that $z_n := x - \sum_{k=1}^{n} x_k \in V_{n+1} + V_{n+1}$, whence $T_n(z_n) \in nW$, for all $n \in \mathbb{N}$. Consequently, $T_n(x) = y_n + T_n(z_n) \notin nW$ for all $n \in \mathbb{N}$. This fact witnesses that $\Gamma(x)$ is not bounded. □

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