UNIVERSALLY KURATOWSKI–ULAM SPACES
AND OPEN-OPEN GAMES

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Abstract. We examine the class of spaces in which the second player has a winning strategy in the open-open game. We show that this spaces are not universally Kuratowski–Ulam. We also show that the games $G$ and $G_7$ introduced by P. Daniels, K. Kunen, H. Zhou [Fund. Math. 145 (1994), no. 3, 205–220] are not equivalent.

1. Introduction

First we shall recall some game introduced in [2] called $G_2$. Let $X$ be a topological space equipped with a topology $\mathcal{T}$ and let $\mathcal{B} \subseteq \mathcal{T}$ be its base. The length of the game is $\omega$. Two players I and II take turns playing. At the $n$-th move II chooses a family $\mathcal{P}_n$ consisting of open non-empty subset of $X$ such that $\text{cl} \bigcup \mathcal{P}_n = X$, then I picks a $V_n \in \mathcal{P}_n$. I wins iff $\text{cl} \bigcup_{n \in \omega} V_n = X$. Otherwise player II wins. Denote by $D_{\text{cov}}$ a collection of families $\mathcal{F}$ consisting of open sets with $\text{cl} \bigcup \mathcal{F} = X$. We say that $\sigma_{\text{cov}} : (\bigcup D_{\text{cov}})^{<\omega} \to D_{\text{cov}}$ is a winning strategy for player II in the game $G_2$ whenever, for any sequence $U_0, U_1, \ldots$ consisting of non-empty open subsets with $U_0 \in \sigma_{\text{cov}}(\emptyset) = \mathcal{P}_0 \in D_{\text{cov}}$ and $U_n \in \sigma_{\text{cov}}(U_0, U_1, \ldots, U_{n-1}) = \mathcal{P}_n \in D_{\text{cov}}$, for all $n \in \omega$, there holds $\text{cl} \bigcup_{n \in \omega} U_n \neq X$. 

Received: 10.03.2015. Revised: 6.06.2015.
Key words and phrases: II-favorable space, uK-U space, tiny sequence.
Andrzej Kucharski thanks European Science Foundation for their support through the grant 3007 within the INFTY program.
In the paper [2] the authors introduced an open-open game. We say that
\( G \) is an open-open game of length \( \omega \) if two players take turns playing; a round
consists of player I choosing a non-empty open set \( U \subseteq X \) and player II
choosing a non-empty open \( V \subseteq U \); I wins if the union of II’s open sets is
dense in \( X \), otherwise II wins. Suppose that there exists a function
\[ s_{op} : \bigcup \{ T^n : n \geq 0 \} \to T \]
such that for each sequence \( V_0, V_1, \ldots \) consisting of non-empty elements of
\( T \) with \( s_{op}(V_0) \subseteq V_0 \) and \( s_{op}(V_0, V_1, \ldots, V_n) \subseteq V_n \), for all \( n \in \omega \), there holds
\( \text{cl} \bigcup_{n \in \omega} V_n \neq X \). Then the function \( s_{op} \) is called a winning strategy for II
player in the open-open game and we say that the space \( X \) is II-favorable.

It is known [2] that the open-open game \( G \) is equivalent to \( G_2 \). We consider
only games with the length equal to \( \omega \). In [2] the authors introduced a game
\( G_7 \) which is played as follows: In the \( n \)-th inning II chooses \( O_n \), a family of
open sets with \( \bigcup O_n \) dense in \( X \). I responds with \( T_n \), a finite subfamily of \( O_n \);
I wins if \( \bigcup_{n \in \omega} T_n \) is dense subset of \( X \); otherwise, II wins.

According to A. Szymański [13] a sequence \( \{ P_n : n \in \omega \} \) of open families
in \( X \) is a tiny sequence if
1. \( \bigcup P_n \) is dense in \( X \) for all \( n \in \omega \)
2. if \( F_n \) is a finite subfamily of \( P_n \) for each \( n \in \omega \) then \( \bigcup \{ \bigcup F_n : n \in \omega \} \) is
not dense in \( X \).

We call a sequence \( \{ P_n : n \in \omega \} \) of open families in \( X \) a 1-tiny sequence if
1. \( \bigcup P_n \) is dense in \( X \) for all \( n \in \omega \)
2. if \( F_n \) is a member of \( P_n \) for each \( n \in \omega \) then \( \bigcup \{ F_n : n \in \omega \} \) is not dense
in \( X \).

M. Scheepers used in the paper [12] negation of the existence of tiny se-
quence, and 1-tiny sequence - called these properties \( S_{fin}(D, D) \) and \( S_1(D, D) \)
respectively. In this paper we refer to notions tiny sequence and 1-tiny se-
quence, because in some situations (Theorem 1.1 and 1.2) we can define them.

Recall another game \( G_4 \) introduced in [2]. In the \( n \)-th inning player I
chooses finite open family \( A_n \). Player II responds with a finite, open family
\( B_n \) with \( |B_n| = |A_n| \) and for each \( V \in A_n \) there exists \( W \in B_n \) such that
\( W \subseteq V \). I wins if \( \bigcup_{n \in \omega} \bigcup B_n \) is dense subset of \( X \); otherwise, II wins. One
can prove that the game \( G_7 \) is equivalent to the game \( G_4 \) in a way similar to
the proof of the equivalence between games \( G \) and \( G_2 \).

From now on we consider only c.c.c. spaces.

**Theorem 1.1 (M. Scheepers [12], Theorem 2).** II has a winning strategy
in the game \( G_7 \) if and only if there exists a tiny sequence.
Theorem 1.2 (M. Scheepers [12, Theorem 14]). Player II has a winning strategy in the game $G_2$ if and only if there exists a 1-tiny sequence.

2. The main results

Recall that $X$ is called a II-favorable space if player II a has winning strategy in the game $G$. If player I has a winning strategy in the game $G$ then we say that the space is I-favorable.

The following theorem was proven by K. Kuratowski and S. Ulam, see [9]. In order to formulate it, let us recall that: a $\pi$-base is a family of open, nonempty sets such that any open set contains a set from this family, and the $\pi$-weight of a space is the smallest cardinality of a $\pi$-base in this space.

Let $X$ and $Y$ be topological spaces such that $Y$ has countable $\pi$-weight. If $E \subseteq X \times Y$ is a nowhere dense set, then there is a meager set $P \subseteq X$ such that the section $E_x = \{y : (x,y) \in E\}$ is nowhere dense in $Y$ for each point $x \in X \setminus P$.

A space $Y$ is universally Kuratowski–Ulam (for short, uK-U space), whenever for a topological space $X$ and a nowhere dense set $E \subseteq X \times Y$ the set

$$\{x \in X : \{y : (x,y) \in E\} \text{ is not nowhere dense in } Y\}$$

is meager in $X$, see D. Fremlin [6] (compare [3]). In the paper [7] authors have shown that a compact I-favorable space is universally Kuratowski–Ulam and posed a question: Does there exist a compact universally Kuratowski–Ulam space which is not I-favorable? We will partially answer to this question, namely we will prove that a II-favorable space is not universally Kuratowski–Ulam space.

Theorem 2.1. Let $X$ be a dense in itself space with a $\pi$-base $B = \bigcup_{n \in \omega} B_n$, where $B_n$ is a maximal family of pairwise disjoint open sets for $n \in \omega$ and let $Y$ be II-favorable c.c.c. space. Then the Kuratowski–Ulam theorem does not hold in $X \times Y$.

Proof. By Theorem 1.2 there is a 1-tiny sequence $\{P_n : n \in \omega\}$. Since the space $Y$ satisfies c.c.c., we can assume that each $P_{n+1}$ is a countable, open, pairwise disjoint family. We can also assume that every $P_{n+1}$ is a refinement of $P_n$, i.e. each member of $P_{n+1}$ is a subset of a member of $P_n$. Let $\{V^n_\sigma : \sigma \in {}^n\mathbb{N}\}$ be an enumeration of the family $P_n$ such that for each $\tau \in {}^{n-1}\mathbb{N}$, $\{V^n_{\tau \bowtie k} : k \in \mathbb{N}\} = P_n$. 


We can assume that $B_{n+1}$ is a refinement of $B_n$ and $|\{V \in B_{n+1} : V \subseteq U\}| \geq \omega$ for each $U \in B_n$. For each $n \in \mathbb{N}$ fix a function $f_n : B_n \rightarrow {}^*\mathbb{N}$ such that for a fixed $U \in B_n$ we have

\[(2.1) \quad \{f_{n+1}(V) : V \in B_{n+1} \text{ and } V \subseteq U\} = f_n(U)^\infty.\]

Therefore, there holds the condition:

\[(2.2) \quad \text{if } V \subset U \text{ then } f_{n+1}(V) \supset f_n(U) \text{ for every } V \in B_{n+1} \text{ and } U \in B_n.\]

Consider an open set

\[F = \bigcup \left\{ \bigcup \{U \times V_n^{f_n(U)} : U \in B_n \} : n \in \mathbb{N} \right\}.\]

We shall show that $F$ is dense and $F_x = \{y \in Y : (x, y) \in F\}$ is not dense for each $x \in X$. If $x \in X \setminus \bigcap \{\bigcup B_n : n \in \mathbb{N}\}$ then it is easy to see that $E_x$ is not dense. If $x \in \bigcap \{\bigcup B_n : n \in \mathbb{N}\}$ then by condition (2.2) there is $\sigma \in {}^*\mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $U_n \in B_n$ with $f_n(U_n) = \sigma|n$ and $x \in \bigcap \{U_n : n \in \mathbb{N}\}$, hence $F_x = \bigcup \{V_n^{f_n(U)} : n \in \mathbb{N}\}$. Since $V_n^{f_n(U)} \in P_n$ for each $n \in \mathbb{N}$ and $\{P_n : n \in \omega\}$ is a 1-tiny sequence the set $\bigcup \{V_n^{f_n(U)} : n \in \mathbb{N}\}$ is not dense.

Now we show that $F$ is a dense set. Let $U \times W$ be any open set. Since $B$ is a $\pi$-base there are $n \in \mathbb{N}$ and $U_0 \in B_n$ such that $U_0 \subseteq U$. Let $\sigma = f_n(U_0)$, since \{\[V_{\sigma|k}^{n+1} : k \in \mathbb{N}\]\} is a dense family, we get that $W \cap V_{\sigma|k}^{n+1} \neq \emptyset$ for some $k \in \mathbb{N}$. By (1), we may take $U_1 \subseteq U_0$ such that $U_1 \in B_{n+1}$ and $f_{n+1}(U_1) = \sigma|k$. Thus $U_1 \times V_{f_{n+1}(U_1)}^{n+1} \cap U \times W \neq \emptyset$. \hfill $\square$

Since $\mathbb{R}$ with natural topology satisfies assumption of the above theorem and every universally Kuratowski–Ulam space is c.c.c. space, we get the following theorem.

**Theorem 2.2.** A II-favorable space is not universally Kuratowski–Ulam space.

Following [10] pp. 86–91 recall category measure space. If $X$ is a topological space with finite measure $\mu$ defined on the $\sigma$-algebra $S$ of sets having the Baire property, and if $\mu(E) = 0$ if and only if $E$ is of a meager set, then $(X, S, \mu)$ is called a category measure space. An example of a regular Baire space which is a category measure space, is an open interval $(0, 1)$ with Lebesgue measure $\mu_1$ and density topology $T_d$, see [10]. For density topology and measurable set $A \subseteq (0, 1)$ the following conditions are equivalent:
(1) \(\mu_1(A) = 0\),
(2) \(A\) is closed and nowhere dense.

In the space \(((0,1), T_d)\) there is a 1-tiny sequence but there is no tiny sequence. Indeed, define a 1-tiny sequence in the following way: let \(P_n = \{U : U \in T_d \text{ and } \mu_1(U) \leq \frac{1}{3^n}\}\). If \(\{U_n : n \in \mathbb{N}\}\) is a family chosen by player I then \(\mu_1(\bigcup\{U_n : n \in \mathbb{N}\}) \leq \frac{1}{2}\). Therefore \(\{U_n : n \in \mathbb{N}\}\) is not a dense family. Now assume that there exists a tiny sequence \(\{P_n : n \in \mathbb{N}\}\). In each stage we choose a finite subfamily \(R_n \subset P_n\) such that \(\mu_1(\bigcup\bigcup R_i : i \leq n) \geq 1 - \frac{1}{n}\), hence we get a dense family \(\bigcup\{R_n : n \in \mathbb{N}\}\).

The authors of the paper [2] posed a question (Question 4.3): Does a player have a winning strategy in the game \(G\) if and only if the same player has a winning strategy in the game \(G_7\). The author of paper [12] showed that if \(\text{cov}(\mathcal{M}) < \mathfrak{d}\) the answer is NO. We show that games \(G\) and \(G_7\) are not equivalent.

**Corollary 2.3.** The game \(G\) is not equivalent to the game \(G_7\).

**Proof.** By Theorem 1.2 a winning strategy of II player in the game \(G\) is equivalent to the existence of a 1-tiny sequence and by Theorem 1.1 the existence of a winning strategy of player II in the game \(G_7\) is equivalent to the existence of a tiny sequence. Since in the space \(((0,1), T_d)\) there is a 1-tiny sequence but there is no tiny sequences we get that games \(G\) and \(G_7\) are not equivalent. \(\square\)

Since the game \(G_7\) is equivalent to the game \(G_4\), we get the following:

**Corollary 2.4.** The game \(G\) is not equivalent to the game \(G_4\).

### 3. Some remarks

It is known that on the \(\omega_1\) with discrete topology II player has a winning strategy in the game \(G_7\), but one can pose a question:

*Is it possible to construct a tiny sequence \(\{P_n : n \in \omega\}\) on a discrete space of the size \(\omega_1\) with \(|P_n| = \omega\) for all \(n \in \omega\) ?*

The following Remark 3.1 gives us the answer - it is possible if and only if the dominating number is equal \(\omega_1\). This is reformulation of well know results about critical cardinal number, see W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki [5]; D. Fremlin, A.W. Miller [4] and B. Tsaban [14].

Recall that \(f \leq^* g\) denotes that for almost all \(n \in \omega\) holds \(f(n) \leq g(n)\), where \(f, g\) are functions defined on natural numbers. A family \(\mathcal{R} \subseteq \omega\omega\) is
a dominating family if for each $f \in \omega^\omega$ there is $g \in \mathcal{R}$ such that $f \leq^* g$. The dominating number $\mathfrak{d}$ is the smallest cardinality of a dominating family:

$$\mathfrak{d} = \min\{|\mathcal{R}| : \mathcal{R} \text{ is dominating}\}.$$ 

**Remark 3.1.** The smallest cardinality $\kappa$ such that there exists a tiny sequence $\{P_n : n \in \omega\}$ on the discrete space of the size $\kappa$ with $|P_n| = \omega$ for all $n \in \omega$ is equal to $\mathfrak{d}$.

**Proof.** Let $X$ be any discrete space for which there exists a tiny sequence $\{P_n : n \in \omega\}$. We can assume that every $P_n$ is a partition of $X$ into countably many blocks $\{X^n_0, X^n_1, \ldots\}$, so we may define for each $x \in X$ a function $f_x : \omega \to \omega$ in the following way: $f_x(n) = k$ whenever $x \in X^n_k$. Take an arbitrary function $f : \omega \to \omega$, and any $x \in X \setminus \bigcup \{\bigcup \{X^n_k : k \leq f(n)\} : n < \omega\}$, then $f$ is dominated by the function $f_x$. It shows that $\{f_x : x \in X\}$ is a dominating family, hence $|X| \geq \mathfrak{d}$.

Now, let $\mathcal{F} \subseteq \omega^\omega$ be a dominating family of the cardinality $\mathfrak{d}$. Without loss of generality assume that for each function $f : \omega \to \omega$ there is $g \in \mathcal{F}$ such that $f(n) < g(n)$ for all $n < \omega$. We treat $\mathcal{F}$ as a discrete topological space. For $n, k \in \omega$ put $A^n_k = \{f \in \mathcal{F} : f(n) \leq k\}$ and set $P_n = \{A^n_k : k < \omega\}$. Of course, each family $P_n$ is increasing and has the union equal to $\mathcal{F}$. From each $P_n$ take some single $A^n_{f(n)}$ where $f : \omega \to \omega$. If $\bigcup \{A^n_{f(n)} : n < \omega\}$ was equal to $\mathcal{F}$, then it would contain such a function $g$ that $g(n) > f(n)$ for all $n \in \omega$, but it is not the case. Therefore $\{P_n : n \in \omega\}$ is a tiny sequence. \qed

Recall a definition of a Baire number $\text{cov}(\mathcal{M})$ for the ideal $\mathcal{M}$ of meager subsets of real line $\mathbb{R}$:

$$\text{cov}(\mathcal{M}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{M} \text{ and } \bigcup \mathcal{A} = \mathbb{R}\}.$$ 

T. Bartoszyński [1] proved that $\text{cov}(\mathcal{M})$ is the cardinality of the smallest family $\mathcal{F} \subseteq \omega^\omega$ such that

$$\forall (g \in \omega^\omega) \exists (f \in \mathcal{F}) \forall (n \in \omega) f(n) \neq g(n).$$ 

We get another well known characterization of such families by a 1-tiny sequence.

**Remark 3.2.** The smallest cardinality $\kappa$ such that there exists a 1-tiny sequence $\{P_n : n \in \omega\}$ on the discrete space of the size $\kappa$ with $|P_n| = \omega$ for all $n \in \omega$ is equal to $\text{cov}(\mathcal{M})$. 

We give the proof for the sake of completeness. We shall prove that the smallest cardinality of a family $\mathcal{F} \subseteq \omega^\omega$ such that

\[(*) \quad \forall (g \in \omega^\omega) \exists (f \in \mathcal{F}) \forall (n \in \omega) f(n) \neq g(n)\]

is equal to the smallest cardinality $\kappa$ such that there exists a 1-tiny sequence \(\{P_n : n \in \omega\}\) on the discrete space $\kappa$ with $|P_n| = \omega$ for all $n \in \omega$.

**Proof.** Let $\mathcal{F} = \{f_\alpha : \alpha < \kappa\} \subseteq \omega^\omega$ be a family with the property $(\ast)$. Define $A_n^i = \{f \in \mathcal{F} : f(i) = n\}$ for every $i, n \in \omega$. Let $\mathcal{P}_i = \{A_n^i : n \in \omega\}$ for $i \in \omega$. We will show that $\{\mathcal{P}_i : i \in \omega\}$ is a 1-tiny sequence. Assume that we have chosen $A_n^{i_k} \in \mathcal{P}_i$ for each $i \in \omega$. Define a function $g(i) = n_i$ for $i \in \omega$. Since $\mathcal{F}$ satisfies $(\ast)$ there is $f \in \mathcal{F}$ such that $f(i) \neq g(i)$ for each $i \in \omega$. Therefore we get $f \in \mathcal{F} \setminus \bigcup\{A_n^{i_k} : i \in \omega\}$.

Let $\{\mathcal{P}_n : n \in \omega\}$ be a 1-tiny sequence with $|\mathcal{P}_n| = \omega$ and $\bigcup \mathcal{P}_n = \kappa$ for each $n \in \omega$. We can assume that each $\mathcal{P}_n$ consists of pairwise disjoint subsets of $\kappa$. Let $\{A_n^k : k \in \omega\}$ be an enumeration of $\mathcal{P}_n$. We define a function $f_x \in \omega^\omega$ for each $x \in \kappa$ in the following way: $f_x(i) = n$, where $x \in A_n^i$ for each $i \in \omega$. The family $\{f_x : x \in \kappa\}$ satisfies $(\ast)$. Indeed, let $g \in \omega^\omega$ be any function. Since $\{\mathcal{P}_n : n \in \omega\}$ is a 1-tiny sequence, choose $x \in \kappa \setminus \bigcup\{A_n^{i(g)} : i \in \omega\}$. Finally, observe that $f_x(i) \neq g(i)$ for every $i \in \omega$. \(\square\)

We shall recall definition of the bounding number

$$b = \min \{ |\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ and } \forall (g \in \omega^\omega) \exists (f \in \mathcal{F}) \neg((f \leq^* g)) \}.$$  

We say that a sequence $\{\mathcal{P}_n : n \in \omega\}$ of open families in $X$ is a $b$-tiny sequence if

1. $\bigcup \mathcal{P}_n$ is dense in $X$ for all $n \in \omega$;
2. if $\mathcal{F}_n$ is a finite subfamily of $\mathcal{P}_n$ for each $n \in \omega$, then there exists strictly increasing sequence $\{n_i : i \in \omega\}$ such that $\bigcup \{\bigcup \mathcal{F}_{n_i} : i \in \omega\}$ is not dense in $X$.

We get the next reformulation of the bounding number.

**Remark 3.3.** The smallest cardinality $\kappa$ such that there exists a $b$-tiny sequence $\{\mathcal{P}_n : n \in \omega\}$ on the discrete space of the size $\kappa$ with $|\mathcal{P}_n| = \omega$ for all $n \in \omega$ is equal to $b$. 

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Acknowledgement. The authors are indebted to the Referee for very careful reading of the paper and valuable comments.

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