THE ALIENATION PHENOMENON AND ASSOCIATIVE RATIONAL OPERATIONS

Roman Ger

Abstract. The alienation phenomenon of ring homomorphisms may briefly be described as follows: under some reasonable assumptions, a map $f$ between two rings satisfies the functional equation

$$(*) \quad f(x + y) + f(xy) = f(x) + f(y) + f(x)f(y)$$

if and only if $f$ is both additive and multiplicative. Although this fact is surprising for itself it turns out that that kind of alienation has also deeper roots. Namely, observe that the right hand side of equation $(*)$ is of the form $Q(f(x), f(y))$ with the map $Q(u, v) = u + v + uv$ being a special rational associative operation. This gives rise to the following question: given an abstract rational associative operation $Q$ does the equation

$$f(x + y) + f(xy) = Q(f(x), f(y))$$

force $f$ to be a ring homomorphism (with the target ring being a field)?

Plainly, in general, that is not the case. Nevertheless, the 2-homogeneity of $f$ happens to be a necessary and sufficient condition for that effect provided that the range of $f$ is large enough.

Institute of Mathematics
Silesian University
Bankowa 14
40-007 Katowice
Poland
e-mail: roman.ger@us.edu.pl

(2010) Mathematics Subject Classification: 39B52, 39B82.
Key words and phrases: alienation phenomenon, additive and multiplicative mappings.