On weakly locally uniformly rotund norms which are not locally uniformly rotund

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Definitions

Let $X$ be a Banach space. Recall that if $x, y, x_n$ (for $n \in \mathbb{N}$) are arbitrary vectors from the unit sphere, then the norm $\| \cdot \|$ on $X$ is called:

- **strictly convex** if
  \[ \| x + y \| = 2 \implies x = y; \]

- **weakly locally uniformly rotund (wLUR)** if
  \[ \| x_n + x \| \xrightarrow{weakly} 2 \implies x_n \xrightarrow{weakly} x; \]

- **locally uniformly rotund (LUR)** if
  \[ \| x_n + x \| \xrightarrow{} 2 \implies \| x_n - x \| \xrightarrow{} 0. \]
Examples

- The norm

\[
\|\|x\|\| = \|x\|_{\ell_\infty} + \left( \sum_{n=1}^{\infty} \left| \frac{x(n)}{n} \right|^2 \right)^{\frac{1}{2}}
\]

on \( \ell_\infty \) is strictly convex, but not \( wLUR \).

- The norm

\[
\|\|x\|\| = \|x\|_{\ell_\infty} + \left( \sum_{n=1}^{\infty} 2^{-n} |x(n)|^2 \right)^{\frac{1}{2}}
\]

on \( c_0 \) is \( wLUR \), but not \( LUR \).
Markushevich bases and Pełczyński theorem

Assume $X$ is a Banach space. Recall that an $M$-basis

$$\{(e_n, e_n^*): n \in \mathbb{N}\} \subset X \times X^*$$

of $X$ is called:

- **bounded** if
  $$\sup\{\|e_n\| \cdot \|e_n^*\|: n \in \mathbb{N}\} < \infty;$$

- **shrinking** if $\{e_n^*: n \in \mathbb{N}\}$ is linearly dense in $X^*$.

**Theorem (Pełczyński, 1976)**

*If $X^*$ is separable, then $X$ has a bounded and shrinking $M$-basis.*
Main theorem

Theorem

Assume $X$ is an infinite-dimensional Banach space. If $X^*$ is separable, then $X$ admits an equivalent wLUR norm which is not LUR.

Proof (sketch).

Let $\{ (e_n, e_n^*) : n \in \mathbb{N} \}$ be a bounded and shrinking $M$-basis of $X$. We may assume $\| e_n \| = 1$ for each $n \in \mathbb{N}$. Define an equivalent norm on $X$ by

$$
\| x \|_0 = \max \left\{ \frac{1}{2} \| x \| , \sup_{n \in \mathbb{N}} | e_n^* x | \right\}
$$

and another one by

$$
\| x \|_0 = \max \left\{ \frac{1}{2} \| x \| , \sup_{n \in \mathbb{N}} | e_n^* x | \right\}
$$

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Proof ctd.

\[ \|x\|^2 = \|x\|_0^2 + \sum_{n=1}^{\infty} 4^{-n} |e_n^* x|^2. \]

To see that \(\|\cdot\|\) is wLUR take \(x_n, x \in X\) with \(\|x_n\| = 1 = \|x\|\) and \(\|x_n + x\| \to 2\). Note that

\[ e_k^* x_n \to e_k^* x \quad \text{for each } k \in \mathbb{N} \]

and by density of \(\text{span}\{e_n^*: n \in \mathbb{N}\}\) we have \(x_n \xrightarrow{\text{weakly}} x\). To see that \(\|\cdot\|\) is not LUR note that

\[ \|e_1 + e_n\| \to \|e_1\| = \frac{\sqrt{5}}{2} \]

\[ \|2e_1 + e_n\| \to \sqrt{5}, \]

but \(\|e_n\| \to 1\).
Questions

◇ Does a separable space fail to have the Schur property if and only if it admits an equivalent \( wLUR \) norm which is not LUR?

◇ Does a space admitting an equivalent \( wLUR \) norm which is not LUR have an infinite-dimensional subspace with separable dual?

Remark
The answer to both questions cannot be positive simultaneously. Indeed, Popov constructed a hereditarily \( \ell_1 \) space without the Schur property.

A. Pełczyński: *All separable Banach spaces admit for every $\epsilon > 0$ fundamental total and bounded by $1 + \epsilon$ biorthogonal sequences*. Studia Math. **55** (1976), 295–304.
