Graphs of additive functions

Szymon Draga

University of Silesia, Institute of Mathematics

Some definitions

- A function \( f : \mathbb{R}^m \to \mathbb{R}^n \) is called \textit{additive} if
  \[
  f(x + y) = f(x) + f(y)
  \]
  for every \( x, y \in \mathbb{R}^m \).
- A set \( A \subset \mathbb{R}^m \) is called \textit{saturated non-measurable} if none of the sets
  \[
  A, \quad \mathbb{R}^m \setminus A
  \]
  contains a measurable set of positive measure.
- A set \( A \subset \mathbb{R}^m \) is called \textit{saturated non-Baire} if none of the sets
  \[
  A, \quad \mathbb{R}^m \setminus A
  \]
  contains a second category set with the Baire property.
Functions with big graphs

We say that a function $f : \mathbb{R}^m \to \mathbb{R}^n$ has a big graph if

$$B \cap \text{gr } f \neq \emptyset$$

for every Borel set $B \subset \mathbb{R}^{m+n}$ whose projection onto $\mathbb{R}^m$ has the cardinality of $c$.

**Theorem**

*There exists an additive function $\mathbb{R}^m$ into $\mathbb{R}^n$ with big graph.*

**Proof.**

For a proof by transfinite induction see [2].

If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a function with big graph, then

- $\text{gr } f$ is a saturated non-measurable subset of $\mathbb{R}^{m+n}$
- $\text{gr } f$ is a saturated non-Baire subset of $\mathbb{R}^{m+n}$. 
Functions with small graphs

We say that a function \( f : \mathbb{R}^m \to \mathbb{R}^n \) has a small graph if

\[
\text{card } f(\mathbb{R}^m) = \aleph_0.
\]

Theorem

There exists an additive function \( \mathbb{R}^m \) into \( \mathbb{R}^n \) with small graph.

Proof (sketch).

Define \( f \) as the extension of an arbitrary non-zero mapping from a Hamel basis of \( \mathbb{R}^m \) into \( \mathbb{Q}^n \) to the additive function.

\[\square\]

If \( f : \mathbb{R}^m \to \mathbb{R}^n \) is a function with small graph, then

- \( \text{gr } f \) is a measure-zero subset of \( \mathbb{R}^{m+n} \)
- \( \text{gr } f \) is a first category subset of \( \mathbb{R}^{m+n} \).
Injective additive functions with irregular graphs

Theorem

There exists an injective additive function $\mathbb{R}^m$ into $\mathbb{R}^n$ whose graph is both a saturated non-measurable and saturated non-Baire subset of $\mathbb{R}^{m+n}$.

Proof (sketch).

Let $(B_\alpha)_{\alpha < \mathfrak{c}}$ be a well ordering of the family of all positive measure or second category Borel subsets of $\mathbb{R}^{m+n}$. Using transfinite induction choose

$$x_\alpha \in \{ x \in \mathbb{R}^m : \text{card} (B_\alpha)_x = \mathfrak{c} \} \setminus \text{lin}_\mathbb{Q} \{ x_\beta : \beta < \alpha \}$$

$$y_\alpha \in (B_\alpha)_{x_\alpha} \setminus \text{lin}_\mathbb{Q} \{ y_\beta : \beta < \alpha \}$$

for every $\alpha < \mathfrak{c}$. Extend the mapping $x_\alpha \mapsto y_\alpha$ to the additive function. Details may be found in [1].
Injective additive functions with regular graphs

Theorem
There exists a discontinuous and injective additive function $\mathbb{R}^m$ into $\mathbb{R}^n$ whose graph is both a measure-zero and first category subset of $\mathbb{R}^{m+n}$.

Proof (sketch).
Distinguish two cases:
(I) $n > 1$
Define $f$ as a discontinuous and injective additive function ranging in $\mathbb{R}^{n-1} \times \{0\}$. Then the set $\text{gr } f$ has the desired properties.
Injective additive functions with regular graphs ctd.

Proof ctd.
(II) $n = 1$

The set

$$V = \text{lin}_\mathbb{Q} \left\{ \sum_{n=1}^{\infty} \frac{c_n}{2^n!} : c_n \in \{0, 1\} \text{ for } n \in \mathbb{N} \right\}$$

is both a measure-zero and first category linear subspace of $\mathbb{R}$. Moreover $\text{card } V = c$. Define $f$ as the extension of an arbitrary bijective mapping between Hamel bases of $\mathbb{R}^m$ and $V$ to the additive function.
Szymon Draga, *Regularity of the graphs of injective additive functions*, manuscript.