Recent results and problems related to convexity

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Dedicated to the 70th birthday of
Professors ROMAN GER and ZYGFRYD KOMINEK

Let $I \subseteq \mathbb{R}$ be a nonvoid interval throughout this talk.
For a given two-variable mean $M : I^2_\leq := \{(x,y) \in I^2 \mid x \leq y\} \to \mathbb{R}$, we consider the class of functions $f : I \to \mathbb{R}$ satisfying the inequality

$$f(M(x,y)) \leq \frac{y - M(x,y)}{y - x} f(x) + \frac{M(x,y) - x}{y - x} f(y) \qquad x, y \in I,\ x < y.$$

Such functions will be called *M*-convex.

In the particular case when M is of the form

$$M(x,y) = A_t(x,y) := tx + (1-t)y \qquad x, y \in \mathbb{R}$$

for some $t \in]0,1[$, then the notion of M-convexity reduces to

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \qquad x, y \in I,\ x < y,$$

which could be called asymmetric *t*-convexity. We speak about (symmetric) *t*-convexity if

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which could be called asymmetric t-convexity. We speak about (symmetric) t-convexity if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \qquad x, y \in I.$$

Among the many implications related to (symmetric) t-convexity properties we mention the following ones:

1. If f is Jensen convex then it is $\mathbb{Q}$-convex, i.e., t-convex for all $t \in [0, 1] \cap \mathbb{Q}$ (Kuczma [6]);
2. If f is t-convex for some $t \in]0, 1[$, then it is Jensen convex (Daróczy–Páles [1]);
3. If f is t-convex for some $t \in]0, 1[$, then, by a result of Kuhn [7], there exists a subfield K of $\mathbb{R}$ such that

$$\{s \in]0, 1[\mid f \text{ is } s\text{-convex}\} =]0, 1[\cap K.$$

4. Conversely, if K is a subfield of $\mathbb{R}$, then there exists a function $f : I \to \mathbb{R}$ such that the above equality holds (Ger [2]).

For generalizations in terms of (a, b)-convex functions, see the paper by Kominek [5]. For more general results in terms of higher-order convexity notions refer to the paper by Gilányi and Páles [3].

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In 2014, for every transcendental number $t \in]0,1[$, Lewicki and Olbryś [8] constructed a function $f : I \to \mathbb{R}$ such that

$$\begin{cases} f(tx+(1-t)y) < tf(x)+(1-t)f(y) \\ f((1-t)x+ty) > (1-t)f(x)+tf(y) \end{cases} \qquad (x, y \in I,\ x < y).$$

Hint: Let $f = d$, where $d : \mathbb{R} \to \mathbb{R}$ is an algebraic derivation which is positive at t. Then, for all $x, y \in \mathbb{R}$,

$$\begin{aligned} &tf(x)+(1-t)f(y)-f(tx+(1-t)y) \\ &\qquad = td(x)+(1-t)d(y)-d(tx+(1-t)y) = d(t)(y-x). \end{aligned}$$

Therefore, for a transcendental $t \in]0,1[$, the notions of asymmetric and symmetric t-convexity properties are different from each other.

It is unknown if these two properties are equivalent to each other for rational, or more generally, for algebraic $t \in]0,1[$.

The particular case $t = \frac{1}{3}$ also has not been clarified yet.

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An old method

If you cannot solve a problem then generalize it!

Descendants of n-tuples of means (Kiss–Páles [4])

Given an $n \geq 2$ member sequence of two-variable means $M_1, \dots, M_n : I^2_\leq \to \mathbb{R}$, the i^{th} element of an increasing sequence $N_1, \dots, N_n : I^2_\leq \to \mathbb{R}$ of two-variable means such that, for all $(x, y) \in I^2_\leq$,

$$\begin{aligned} N_1(x,y) &= M_1(x, N_2(x,y)), \\ N_i(x,y) &= M_i(N_{i-1}(x,y), N_{i+1}(x,y)) \qquad (i \in \{2, \dots, n-1\}), \\ N_n(x,y) &= M_n(N_{n-1}(x,y), y) \end{aligned}$$

hold, is called the i^{th} descendant of the n-tuple $(M_1, \dots, M_n)$.

Observe that the above system of equations states that, for $(x, y) \in I^2_\leq$, the vector $(N_1(x,y), \dots, N_n(x,y)) \in [x,y]^n_\leq$ is a fixed point of the mapping $\varphi_{(x,y)} : [x,y]^n_\leq \to \mathbb{R}^n$ defined by

$$\varphi_{(x,y)}(t_1, \dots, t_n) := \big(M_1(x, t_2), \dots, M_i(t_{i-1}, t_{i+1}), \dots, M_n(t_{n-1}, y)\big).$$

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The existence of the descendants is proved by using the following fixed point theorem which is a useful consequence of the so-called Halper–Bergman Fixed Point Theorem:

Theorem ([4])

Let $c_1, \dots, c_m \in \mathbb{R}^n$ and $\gamma_1, \dots, \gamma_m \in \mathbb{R}$ and assume that the polyhedron $K \subseteq \mathbb{R}^n$ defined by

$$K := \big\{x \in \mathbb{R}^n \mid \langle c_k, x\rangle \le \gamma_k,\ k \in \{1, \dots, m\}\big\}$$

is bounded. Let $f : K \to \mathbb{R}^n$ be a continuous function with the following property

$$\langle c_k, f(x)\rangle \le \gamma_k \qquad \text{for all } x \in K \text{ and for all } k \in \{1, \dots, m\} \text{ such that } \langle c_k, x\rangle = \gamma_k.$$

Then the set of fixed points of f is a nonempty compact subset of K.

Corollary ([4])

Given an $n \geq 2$, for any sequence of continuous two-variable means $M_1, \dots, M_n : I^2_{\leq} \to \mathbb{R}$, there exits an increasing sequence $N_1, \dots, N_n : I^2_{\leq} \to \mathbb{R}$ of two-variable means such that, for all $(x, y) \in I^2_{\leq}$,

$$\begin{aligned} N_1(x,y) &= M_1(x, N_2(x,y)), \\ N_i(x,y) &= M_i(N_{i-1}(x,y), N_{i+1}(x,y)) \qquad (i \in \{2, \dots, n-1\}), \\ N_n(x,y) &= M_n(N_{n-1}(x,y), y). \end{aligned}$$

Proof

For $x < y$ in I let $K := [x, y]^n_{\leq}$ and apply the fixed point theorem to the mapping $\varphi_{(x,y)} : [x, y]^n_{\leq} \to \mathbb{R}^n$ defined by

$$\varphi_{(x,y)}(t_1, \dots, t_n) := (M_1(x, t_2), \dots, M_i(t_{i-1}, t_{i+1}), \dots, M_n(t_{n-1}, y)).$$

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Theorem ([4])

Let $n \geq 2$, $M_1, \ldots, M_n : I^2_{\leq} \to \mathbb{R}$ be a sequence of continuous two-variable strict means, and let $N_1, \ldots, N_n : I^2_{\leq} \to \mathbb{R}$ be a strictly increasing sequence of the descendants of $(M_1, \ldots, M_n)$.
Assume that $f : I \to \mathbb{R}$ is convex with respect to each of the means $M_1, \ldots, M_n$. Then f is also convex with respect to each of the means $N_1, \ldots, N_n$

The proof of this theorem is based on the following inequality related to second-order divided differences.

Lemma (Chain inequality, Nikodem–Páles [10])

Let $f : H \to \mathbb{R}$, $n \geq 2$ and $x_0 < x_1 < \cdots < x_n < x_{n+1}$ be elements of H. Then, for all $k \in \{1, \ldots, n\}$, we have

$$\min_{1 \leq i \leq n} [x_{i-1}, x_i, x_{i+1}; f] \leq [x_0, x_k, x_{n+1}; f] \leq \max_{1 \leq i \leq n} [x_{i-1}, x_i, x_{i+1}; f].$$

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Proof of the Theorem

Let $(x, y) \in I^2_<$ and define

$$x_0 := x, \qquad x_1 := N_1(x, y), \quad \ldots, \quad x_n := N_n(x, y), \qquad x_{n+1} := y.$$

Then, $x_0 < x_1 < \cdots < x_n < x_{n+1}$ and, for $i \in \{1, \ldots, n\}$, we have that

$$x_i = M_i(x_{i-1}, x_{i+1}).$$

Thus, by the M_i-convexity of f,

$$[x_{i-1}, x_i, x_{i+1}; f] = [x_{i-1}, M_i(x_{i-1}, x_{i+1}), x_{i+1}; f] \geq 0.$$

Using the Chain Inequality, for all $k \in \{1, \ldots, n\}$, it follows that

$$[x_0, x_k, x_n; f] \geq 0,$$

i.e.,

$$[x, N_k(x, y), y; f] \geq 0.$$

Remark

In general, the descendants of some given means are not uniquely determined. For instance, let $n \geq 2$, and let $M_1 := \max$, $M_n := \min$ and $M_i := \mathcal{A}_{\frac{1}{2}}$ for $i \in \{2, \ldots, n-1\}$ over the interval $\mathbb{R}$. Then, for $(x, y) \in \mathbb{R}^2_<$, the fixed point equation $(t_1, \ldots, t_n) = \varphi_{(x,y)}(t_1, \ldots, t_n)$ is equivalent to

$$(t_1, \ldots, t_n) = \left(t_2, \frac{t_1 + t_3}{2}, \ldots, \frac{t_{n-2} + t_n}{2}, t_{n-1}\right).$$

An easy computation shows that this equality is satisfied if and only if $t_1 = \cdots = t_n$. Therefore, the fixed point set of this map is given by $\{(t_1, \ldots, t_n) \mid t_1 = \cdots = t_n \in [x, y]\} =$diag$[x, y]^n$.

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The class of Matkowski means (Matkowski [9])

We say that a function $M : I \times I \to \mathbb{R}$ is a generalized weighted quasi-arithmetic mean or a Matkowski mean if there exist continuous, strictly increasing functions $f, g : I \to \mathbb{R}$, such that, for all $x, y \in I$, we have

$$M(x,y) = (f+g)^{-1}(f(x)+g(y)) =: \mathcal{M}_{f,g}(x,y).$$

If $h : I \to \mathbb{R}$ is a continuous, strictly increasing function and $t \in]0,1[$, then $\mathcal{M}_{th,(1-t)h}$ is a weighted quasi-arithmetic mean.

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Descendants of a chain of Matkowski means ([4])

Let $n \geq 2$ and $f_1, \ldots, f_n, g_1, \ldots, g_n : I \to \mathbb{R}$ be continuous, strictly increasing functions. For $(x, y) \in I^2_<$, define $\varphi_{(x,y)} : [x, y]^n_\leq \to \mathbb{R}^n$ by

$$\varphi_{(x,y)}(t_1, \ldots, t_n) := \big(\mathcal{M}_{f_1, g_1}(x, t_2), \ldots, \mathcal{M}_{f_i, g_i}(t_{i-1}, t_{i+1}), \ldots, \mathcal{M}_{f_n, g_n}(t_{n-1}, y)\big).$$

Then, for $(x, y) \in I^2_<$, the fixed point set $\Phi_{(x,y)}$ of the mapping $\varphi_{(x,y)}$ is nonempty and compact. Furthermore, $\Phi_{(x,y)}$ is a singleton if

$$a_i := \mathrm{Lip}\left[f_i \circ (f_{i-1} + g_{i-1})^{-1}\right] < +\infty \qquad (i \in \{2, \ldots, n\}),$$

$$b_i := \mathrm{Lip}\left[g_i \circ (f_{i+1} + g_{i+1})^{-1}\right] < +\infty \qquad (i \in \{1, \ldots, n-1\}),$$

and if the constants $w_1, \ldots, w_{n-1}$ defined by $w_{-1} := w_0 := 1$ and

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Let $n \geq 2$, $p, q, h_1, \dots, h_{n-1} : I \to \mathbb{R}$ be continuous, strictly increasing functions. Set further $h_0 := h_n := 0$, and assume that, for some $j \in \{1, \dots, n\}$, the sequence of means $M_1, \dots, M_n : I^2_{\leq} \to I$ is defined by

$$M_i(x,y) = \begin{cases} \mathcal{M}_{p+h_{i-1},\, h_i}(x,y)) & \text{if } i \in \{1,\dots,j-1\}, \\ \mathcal{M}_{p+h_{j-1},\, q+h_j}(x,y) & \text{if } i = j, \\ \mathcal{M}_{h_{i-1},\, q+h_i}(x,y), y) & \text{if } i \in \{j+1,\dots,n\}. \end{cases}$$

Then, for all $i \in \{1, \dots, n\}$, the i^{th} descendant N_i of $(M_1, \dots, M_n)$ is given by

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Descendants of a chain of weighted quasi-arithmetic means ([4])

Let $n \geq 2$, $t_1, \dots, t_n \in]0,1[$, and let $h : I \to \mathbb{R}$ be a continuous, strictly increasing function. Let $(M_1, \dots, M_n) : I^2_{\leq} \to \mathbb{R}$ be defined by

$$M_i(x,y) = \mathcal{M}_{t_i h,\,(1-t_i)h}(x,y), \qquad i \in \{1,\dots,n\},\ (x,y) \in I^2_{\leq}$$

Then, for all $i \in \{1,\dots,n\}$, the i^{th} descendant N_i of $(M_1,\dots,M_n)$ is given by $N_i(x,y) = \mathcal{M}_{s_i h,\,(1-s_i)h}(x,y)$, where

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For example, if $n = 2$, then (after some simplifications),

$$s_1 = \frac{t_1}{1 - t_1 + t_1 t_2}, \qquad s_2 = \frac{t_1 t_2}{1 - t_1 + t_1 t_2}.$$

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Theorem (about assymmetric t-convexity) ([4])

Given a function $f : I \to \mathbb{R}$, the following statements hold:

1. if $t, s_1, s_2 \in \mathcal{C}_f$ with $s_1 < s_2$, then $ts_2 + (1-t)s_1 \in \mathcal{C}_f$;
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Corollary

For a function $f : I \to \mathbb{R}$ the following statements hold:

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Open problem

Find a complete characterization of the set $\mathcal{C}_f$.

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