Theorems of H. Steinhaus, S. Picard and J. Smital

H. Steinhaus, Sur les distances des points des ensembles de mesure positive, Fund. Math. 1 (1920), 99–104.

Theorem

If $A, B \subset \mathbb{R}$ are sets of positive Lebesgue measure, then $A + B = \{a + b : a \in A \wedge b \in B\}$ and $A - B$ contains an interval.

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S. Picard, Sur les ensembles parfaits, Paris 1942.

Theorem

If $A \subset \mathbb{R}$ has the Baire property ($A = (G \setminus P_1) \cup P_2$, where G is open, P_1, P_2 are of the first category) and is of the second category, then $A + A$ contains an interval.

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M. Kuczma, J. Smital, On measures connected with the Cauchy equation, Aequationes Math. 14 (1976), 421–428.

Theorem

If $E \subset \mathbb{R}$ has a positive outer measure and $D \subset \mathbb{R}$ is a dense set in $\mathbb{R}$, then the inner Lebesgue measure of the set $\mathbb{R} \setminus (E + D)$ is equal to zero.

Z. Kominek, On the sum and difference of two sets in topological vector spaces, Fund. Math. LXXI (1971), 165–169.

Theorem

If X is a topological vector space and $S \subset X$ is a second category set having the Baire property, then $S + S$ contains a nonempty open set.

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If X is a topological vector space and $S \subset X$ is a second category set having the Baire property, then $S + S$ contains a nonempty open set.

We say that $S \subset X$ is of the first category at a point $s \in X$ if there exists a neighbourhood G of s such that $G \cap S$ is of the first category. $D(S)$ is the set of all points of the space X at which S is not of the first category.

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K. Kuratowski (Topologie): $S \subset X$ is of the first category if and only if $D(S) = \emptyset$; $D(S)$ is closed; $D(S_1 \cup S_2) = D(S_1) \cup D(S_2)$; $D(S) \subset Cl(S)$; if $S_1 \subset S_2$ then $D(S_1) \subset D(S_2)$; $D(S) = Cl(Int(D(S)))$.

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Z. Kominek: If $S \subset X$ is of second category and has the Baire property, then $D(S) \cap (D(S'))'$ contains a non-empty open set.

Z. Kominek, Some generalization of the theorem of S. Picard, Prace Naukowe Uniwersytetu Śląskiego w Katowicach Nr 37, Prace Matematyczne IV, 1973, 31–33.

Theorem

Let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that h is a homeomorphism with respect to each variable separately. If $A, B \subset \mathbb{R}$ are second category Baire sets, then the set $h(A \times B)$ contains an interval.

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M. E. Kuczma, M. Kuczma, An elementary proof and an extension of a theorem of Steinhaus, Glasnik Mat. 6 (26) (1971), 11–18.

Theorem

If $A, B \subset \mathbb{R}$ have positive inner Lebesgue measure and $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^2$ is a region such that $A \times B \subset D$, $f \in C^1$, $f'_x \neq 0$, $f'_y \neq 0$ in D, then $f(A \times B)$ contains an interval.

Z. Kominek, Measure, category, and the sums of sets, The Amer. Math. Monthly, Vol. 90, No 8 (Oct. 1983), 561–562.

Theorem

There exist disjoint sets $A, B \subset \mathbb{R}$ *such that* $A \cup B = \mathbb{R}$, A *is a second category Baire set,* B *has infinite Lebesgue measure and* $A + B$ *does not contain any nonempty open interval.*

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Theorem

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Let C be a perfect nowhere dense set with positive Lebesgue measure. Put $B = (Q - C) \cup (Q + C)$ and $A = \mathbb{R} \setminus B$. Then $Q - B \subset B$ and all conditions are fulfilled, because $(A + B) \cap Q = \emptyset$.

Z. Kominek, On a decomposition of the space of real numbers, Glasnik Matematički, Vol. 19 (1984), 231–233.

Theorem

If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is such that* $f(x, \cdot)$ *and* $f(\cdot, y)$ *are homeomorphisms (for each* y *and* x*), then for each first category set* $A_0 \subset \mathbb{R}$ *with positive Lebesgue measure there exist* $C, D \subset \mathbb{R}$ *such that*

1. $C \cap D = \emptyset$, $C \cup D = \mathbb{R}$
2. $A_0 \subset C$
3. C *is of the first category (and positive measure)*
4. $\mathit{Int}\, f(C, D) = \emptyset$.

Lemma

Let X be a set and $g_n : T_n \to X$, $T_n \subset X$, $n = 1, 2, \ldots$ be arbitrary functions. For an arbitrary set $A_0 \subset X$ there exists a set $A \subset X$ containing A_0 such that $\bigcup_{n=1}^{\infty} g_n(A) \subset A$.

Proof.

Put $A = X$.

Original proof: If $A_k = \bigcup_{n=1}^{\infty} g_n(A_{k-1})$, $k = 1, 2, \ldots$, then $A = \bigcup_{k=0}^{\infty} A_k$ satisfies the assumptions $\left(A = \bigcup_{k=1}^{\infty} A_k\right)$ $\square$

Z. Kominek, H. J. Miller, Some remarks on a theorem of Steinhaus, Glasnik Matematički, Vol. 20 (1985), 337–344.

Theorem

If $A, B \subset \mathbb{R}^n$, with A a Lebesgue measurable set having positive Lebesgue measure and B a set having positive outer Lebesgue measure, then $A + B$ contains an n-dimensional ball.

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Remark

If both sets A, B have positive outer Lebesgue measure, then it can happen that $Int(A + B) = \emptyset$.
Let H be a Hamel basis for $\mathbb{R}$, T - the subspace spanned by $H \setminus \{h_0\}$, $h_0 \in H$. If $A = T \cap (0, 1)$, then $m_e(A) > 0$ and $Int(A + A) = \emptyset$.

W. Sander, Verallgemeinerungen eines Satzes von S. Picard, Mannuscripta Math. 16 (1975), 11–25.

Theorem

If $A, B \subset \mathbb{R}^n$, A is a second category Baire set and B is a second category set, then $A + B$ contains an n-dimensional ball.

Z. Kominek, H. I. Miller, Some remarks on a theorem of Steinhaus, Glasnik Matematički, Vol. 20 (1985), 337–344.

Theorem

Suppose $A, B \subset \mathbb{R}$ and that A is measurable and $m(A) > 0$, $m_e(B) > 0$. Suppose further that H is an open set containing $A \times B$ and that f is a real valued function defined on H.
If $a \in A$ is a density point of A, $b \in B$ is an outer density point of B and the partial derivatives f'_x and f'_y of f are continuous in some neighbourhood of (a, b) and $f'_x(a, b) \neq 0$ and $f'_y(a, b) \neq 0$, then $f(a, b)$ contains an interval.

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Theorem

let I_1, I_2 be open intervals and let $H = I_1 \times I_2$. Let f be a real valued function defined on H such that the functions $\{f^x\}_{x \in I_1}$, $\{f^y\}_{x \in I_2}$ are homeomorphisms of I_2 (respectively I_1) onto their ranges, where $f^x : I_2 \to \mathbb{R}$ with $f^x(y) = f(x, y)$ and $f^y : I_1 \to \mathbb{R}$ with $f^y(x) = f(x, y)$. If $A \subset \mathbb{R}$ is a second category Baire set and $B \subset \mathbb{R}$ is a second category set and $A \times B \subset H$ then $f(A \times B)$ contains an interval.

R. Ger, Z. Kominek, M. Sablik, Generalized Smital's Lemma and a Theorem of Steinhaus, Radovi Matematički, Vol. 1 (1985), 101–119.

Motivation: Smital's Lemma implies:

1. Steinhaus theorem,
2. Frechet-Sierpiski Theorem: each measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x_1 + x_2) = f(x_1) + f(x_2)$ is of the form $f(x) = f(1) \cdot x$,
3. each measurable and microperiodic function from $\mathbb{R}$ to $\mathbb{R}$ is constant almost everywhere
4. each nonmeasurable subspace of the linear space $\mathbb{R}$ over $\mathbb{Q}$ is saturated nonmeasurable.

If (X, ρ) is a separable metric space and $m^*\colon 2^X \to [0, \infty]$ is a metric outer measure, m^* is said to satisfy $5r$-condition provided for every bounded set $E \subset X$ there exist two positive constants $c(E)$ and $C(E)$ such that

$$m^*(\bar{K}(x, 5r)) < C(E) \cdot m^*(\bar{K}(x, r))$$

for all $x \in E$ and all $r \in (0, c(E))$.

R. Ger, Z. Kominek, M. Sablik, Generalized Smital's Lemma and a Theorem of Steinhaus, Radovi Matematički, Vol. 1(1985), 101–119 (a continuation).

Theorem

Let (X, ρ) be a separable metric space with an ordinary metric ρ and let $m^*: 2^X \to [0, \infty]$ be a metric outer measure satisfying the $5r$-condition. Suppose that we are given two sets $E, D \subset X$, a point $x_0 \in E$ and a map $f : E \times X \to X$ such that

1. *$f(\{x_0\} \times D)$ is dense in X;*
2. $\lim_{r\to\infty} \frac{m^*\left(f(E\times\{y\})\cap\bar{K}(f(x_0,y),r)\right)}{m^*\left(\bar{K}(f(x_0,y),r)\right)} = 1$

uniformly with respect to $y \in D$. Then, for any open ball $K(z, \eta) \subset X$, we have

$$m^*\left(K(z, \eta) \cap f(E \times D)\right) = m^*(K(z, \eta));$$

in other words:

$$m^*(X \setminus f(E \times D)) = 0.$$

Remark

For $X = \mathbb{R}$, m^*- an outer Lebesgue measure and $f(x, y) = x + y$ we obtain Smital's Lemma.

Theorem

Let X, Y and Z be Baire spaces. Assume that a set $E \subset X$ is of second category at each of its points and $D \subset Y$ is dense in Y. Let $h : X \times Y \to Z$ be such that the partial maps $h(x, \cdot)$ and $h(\cdot, y)$ are homeomorphisms of Y and X onto Z, respectively, for all $x \in X$ and $y \in D$. Then the set $Z \setminus h(E \times D)$ contains no second category set having the Baire property.

Z. Kominek, On an equivalent form of a Steinhaus's Theorem, Mathematica, Tome 30(53), No 1, 1988 25–27.
Consider conditions:

(1) $(X, +)$ is a separable Baire topological group (not necessarily commutative);

(2) m is a measure defined on some σ-algebra $\mathfrak{M}$ of subsets of X such that

$$\forall_{A\in\mathfrak{M}}\,[m(A) > 0 \Rightarrow \exists_{F\subset A}(m(F) > 0 \text{ and } \ F \text{ is a compact set.}]$$

(3) $\forall_{A\in\mathfrak{M}}\forall_{D\subset X}\big[m(E) > 0,\ card\ D \leq \aleph_0,\ cl\ D = X \Rightarrow m(X \setminus (E + D)) = 0\big]$

(4) $\forall_{A,B\in\mathfrak{M}}\,[m(A) > 0\ and\ m(B) > 0 \Rightarrow Int(-A + B) \neq \emptyset]$

Theorem

Let assumptions (1) and (2) be fulfilled. Condition (3) implies the condition (4)

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Let assumptions (1) and (2) be fulfilled. If every compact subset belongs to the σ-algebra $\mathfrak{M}$, then (4) implies (3).

Remark

Measurability is important.

Z. Kominek, W. Wilczyński, On sets for which the difference set is the whole space, Rocznik Naukowo-Dydaktyczny WSP w Krakowie, Zeszyt 207, Prace Matematyczne XVI, 1999, 45–51.

Theorem

Let $A \subset \mathbb{R}^p$ be a Lebesgue measurable set. If

$$\limsup_{r\to\infty} \frac{m_p(A \cap K_p(0,r))}{m_p(K_p(0,r))} = \lambda > \frac{1}{2},$$

then $A - A = \mathbb{R}^p$.

S. Solecki, Amenability, free subgroups and Haar null sets in locally compact groups, Proc. London Math. Soc. 93(3), 2006, 693–722.
N. H. Bingham, A. J. Ostaszewski, The Steinhaus theorem and regular variation: de Bruijn and after, Indagationes Mathematicae 24 (2013), 679–692.

More friendly:

W. Wilczyński, A. Kharazishvili, On the translations of measurable sets and sets with the Baire property, Bull. of the Acad. of Sciences of Georgia 145(1) (1992), 43–46 (in Russian, English and Georgian summary).

Theorem

If $A \subset \mathbb{R}$ is a measurable set, then for each sequence $\{x_n\}_{n\in\mathbb{N}}$ convergent to 0 the sequence of characteristic functions $\{\chi_{A+x_n}\}_{n\in\mathbb{N}}$ converges in measure to χ_A.

Remark

There exists a measurable set $A \subset \mathbb{R}$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ convergent to 0 such that $\{\chi_{A+x_n}\}_{n\in\mathbb{N}}$ does nor converge almost everywhere to χ_A.

Theorem

If A has the Baire property, then $\{\chi_{A+x_n}\}_{n\in\mathbb{N}}$ converges to χ_A except on a set of the first category.