Radii of Elements in Finite-Dimensional Power-Associative Algebras

or, a more intuitively:

Generalizing the Spectral Radius without Dealing with Spectra

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The Minimal Polynomial Revisited

Let $\mathcal{A}$ denote a finite-dimensional algebra over an arbitrary field $\mathbb{F}$.

Throughout the talk, we shall assume that $\mathcal{A}$ is *power-associative*. By this we mean that while $\mathcal{A}$ is not necessarily associative, the subalgebra of $\mathcal{A}$ generated by any one element is associative, or equivalently, that powers of each element in $\mathcal{A}$ are uniquely defined.

Definition. We recall that a *minimal polynomial* of an element a in a power-associative algebra over a field $\mathbb{F}$ is a monic polynomial of lowest positive degree with coefficients in $\mathbb{F}$ that annihilates a.

With this familiar definition, we can state the following non-surprising result:

Theorem. *Let $\mathcal{A}$ be a finite-dimensional power-associative algebra over an field $\mathbb{F}$. Then every element a in our algebra possesses a unique minimal polynomial.*

Example. Let $\mathbb{F}^{n\times n}$ be the algebra of $n\times n$ matrices over $\mathbb{F}$ with the usual matrix operations. Fix an idempotent matrix $M\in\mathbb{F}^{n\times n}$, $M\neq I$, and consider the set

$$\mathcal{A}=\{MAM,\ \ A\in\mathbb{F}^{n\times n}\}$$

with the same familiar matrix operations. Then $\mathcal{A}$ is a subalgebra of $\mathbb{F}^{n\times n}$ which contains the matrix M. In fact, M is the unit element in $\mathcal{A}$, so the minimal polynomial of M in $\mathcal{A}$ is

$$p(t)=t-1.$$

On the other hand, the unit element in $\mathbb{F}^{n\times n}$ is I, and it is easily seen that the minimal polynomial of M as an element in $\mathbb{F}^{n\times n}$ is

$$q(t)=t^2-t.$$

It follows that *the minimal polynomial of an element* a *may depend not only on the element, but also on the underlying algebra*.

The above example is a special case of a more general phenomenon:

Theorem[MP] [G, Trans. AMS, 2007]. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be finite-dimensional power-associative algebras over a field* $\mathbb{F}$*, such that* $\mathcal{A}$ *is a subalgebra of* $\mathcal{B}$*. Let* a *be an element of* $\mathcal{A}$*, and let* p *and* q *be the minimal polynomials of* a *in the algebras* $\mathcal{A}$ *and in* $\mathcal{B}$*, respectively. Then either* $p=q$ *or* $q(t)=tp(t)$*.*

The Radius of an Element in a FDPA Algebra

From now on, we shall restrict attention to the case where the base field $\mathbb{F}$ of our algebra is either $\mathbb{R}$ or $\mathbb{C}$.

Further, we shall abbreviate the expression finite-dimensional power-associative by FDPA.

Main Definition [G, TAMS]. Let $\mathcal{A}$ be a FDPA algebra over $\mathbb{R}$ or $\mathbb{C}$, let a be an element of $\mathcal{A}$, and let p be the minimal polynomial of a in $\mathcal{A}$. Then, the *radius* of a in $\mathcal{A}$ is defined as

$$r(a) = \max\{|\lambda| : \lambda \in \mathbb{C}, \lambda \text{ is a root of } p\}.$$

Unlike the minimal polynomial of an element a *in* $\mathcal{A}$ (which may depend on $\mathcal{A}$), the radius $r(a)$ is independent of our algebra in the following sense:

Proposition. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be FDPA algebras over* $\mathbb{R}$ or $\mathbb{C}$, *such that* $\mathcal{A}$ *is a subalgebra of* $\mathcal{B}$. *Then the radii of* a *in the algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *coincide.*

Proof. Let p and q be the minimal polynomials of a *in the algebras* $\mathcal{A}$ *and* $\mathcal{B}$, respectively. By Theorem[MP], either $p = q$ *or* $q(t) = tp(t)$. Hence, the non-zero roots of p and q are identical; so

$$\max\{|\lambda| : \lambda \in \mathbb{C}, \lambda \text{ is a root of } p\} = \max\{|\lambda| : \lambda \in \mathbb{C}, \lambda \text{ is a root of } q\},$$

and we are done. ■

The radius has been computed for elements in several well-known FDPA algebras. For example, it was shown that *if $\mathcal{A}$ is an arbitrary matrix algebra over $\mathbb{R}$ or $\mathbb{C}$ with the usual matrix operations, then the radius of a matrix $A \in \mathcal{A}$ is the classical spectral radius,*

$$\rho(A) = \max\{|\lambda| : \lambda \in \mathbb{C}, \lambda \text{ is an eigenvalue of } A\}.$$

The following theorem, which is the heart of the matter, tells us that the radius retains some of the basic properties of the spectral radius not only for matrix algebras with the usual operations, but in the general FDPA case as well:

Main Theorem [G, TAMS]. *Let $\mathcal{A}$ be a FDPA algebra over a field $\mathbb{F}$, either $\mathbb{R}$ or $\mathbb{C}$. Then*:

(a) *The radius is nonnegative.*

(b) *The radius is homogeneous, i.e., for all $a \in \mathcal{A}$ and $\alpha \in \mathbb{F}$,*

$$r(\alpha a) = |\alpha| r(a).$$

(c) *For all $a \in \mathcal{A}$ and $k = 1,2,3,\ldots$,*

$$r(a^k) = r(a)^k.$$

(d) *The radius vanishes only on nilpotent elements of $\mathcal{A}$.*

(e) *The radius is a continuous function on $\mathcal{A}$.*

A Non-Associative Example: The Cayley–Dickson Algebras

The Cayley–Dickson algebras constitute a series of algebras, $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ over the reals, where $\dim \mathcal{A}_n = 2^n$.

The first five Cayley–Dickson algebras are the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, the octonions $\mathbb{O}$, and the sedenions $\mathbb{S}$, with dimensions 1, 2, 4, 8, and 16, respectively.

While $\mathbb{R}$ and $\mathbb{C}$ are both commutative and associative, $\mathbb{H}$ is no longer commutative, and $\mathbb{O}$ and $\mathbb{S}$ are not even associative. As it is, $\mathbb{O}$ is alternative, and $\mathbb{S}$ is merely power-associative.

An algebra $\mathcal{A}$ is called *alternative* if its subalgebra generated by any two elements is associative. We recall that $\mathcal{A}$ is *power-associative* if the subalgebra generated by any one element is associative. So, *if $\mathcal{A}$ is alternative, then $\mathcal{A}$ is power-associative*.

Despite the deteriorating associativity properties of the low-dimensional Cayley–Dickson algebras, we do have:

Theorem. *All Cayley–Dickson algebras are power-associative*.

We note, in passing, that in recent years, the Cayley–Dickson algebras have gained renewed interest via several important applications such as the use of quaternions in GPS technology, and the employment of octonions and higher Cayley–Dickson algebras in modern physics (e.g., Quantum Field Theory, and the Born-Infeld modeld).

The Cayley–Dickson Doubling Process

It is well-known that the Cayley–Dickson algebras can be obtained, inductively, from each other by the following Cayley–Dickson doubling process:

We begin this process by setting $\mathcal{A}_0 = \mathbb{R}$, and by defining a^*, the *conjugate* of a real number a, to equal a. Then, assuming that $\mathcal{A}_{n-1}$, $n \geq 1$, has been determined, we define $\mathcal{A}_n$ to be the set of all ordered pairs

$$\mathcal{A}_n = \{(a, b) : \ a, b \in \mathcal{A}_{n-1}\},$$

such that addition and scalar multiplication are taken componentwise on the Cartesian product $\mathcal{A}_{n-1} \times \mathcal{A}_{n-1}$, conjugation is given by

$$(a, b)^* = (a^*, -b),$$

and multiplication is given by

$$(a, b)(c, d) = (ac - d^*b, da + bc^*).$$

With this definition, each element $a \in \mathcal{A}_n$ is of the form $a = (a_1, \ldots, a_{2^n})$, $a_j \in \mathbb{R}$; and it follows that the conjugate of a is given by

$$a^* = (a_1, -a_2, \ldots, -a_{2^n}),$$

and the unit element in $\mathcal{A}_n$ is

$$\mathbf{1}_n = (1, 0, \ldots, 0).$$

To illustrate the Cayley–Dickson doubling process, let us start again with $\mathcal{A}_0 = \mathbb{R}$, and observe that

$$\mathcal{A}_1 = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{R}\},$$

where

$$\begin{aligned}
&(\alpha, \beta)^* = (\alpha, -\beta), \\
&(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma - \delta\beta, \delta\alpha + \beta\gamma), \\
&\mathbf{1}_1 = (1, 0).
\end{aligned}$$

Therefore, we may identify $\mathcal{A}_1$ with $\mathbb{C}$ upon writing

$$z = \alpha + i\beta$$

as

$$(\alpha, \beta).$$

We point out that by construction, $\mathcal{A}_{n-1}$ can be viewed as a subalgebra of $\mathcal{A}_n$. It follows that since $\mathcal{A}_3$, the algebra of the octonions, is no longer associative, *all the Cayley–Dickson algebras for $n \geq 3$ are not associative*.

Radius of Elements in the Cayley–Dickson Algebras

Theorem [G & T. Laffey, Proc. AMS, 2015]. *The radius of an element* $a = (a_1, \ldots, a_{2^n}) \in \mathcal{A}_n$ *is the Euclidean norm of* a, *i.e.,*

$$r(a) = |a| \equiv \sqrt{a_1^2 + \cdots + a_{2^n}^2}\,.$$

With this result, we can easily obtain the following helpful observation:

Corollary. *The Cayley–Dickson algebras are void of nonzero nilpotent elements.*

Proof. By the Main Theorem, the radius vanishes only on nilpotent elements. Since the radius on the Cayley–Dickson algebras happens to be a norm, and since a norm vanishes only at $a = 0$, the proof is complete. ∎

Subnorms on FDPA Algebras

In order to discuss applications of the radius, we begin with the following definition.

Definition. Let $\mathcal{A}$ be a FDPA algebra over a field $\mathbb{F}$, either $\mathbb{R}$ or $\mathbb{C}$. Then a real-valued function

$$f : \mathcal{A} \to \mathbb{R}$$

is called a *subnorm* if for all $a \in \mathcal{A}$ and $\alpha \in \mathbb{F}$,

$$f(a) > 0, \quad a \neq 0,$$
$$f(\alpha a) = |\alpha| f(a).$$

We recall that a real-valued function N is a *norm* on $\mathcal{A}$ if for all pairs $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{F}$,

$$N(a) > 0, \quad a \neq 0,$$
$$N(\alpha a) = |\alpha| N(a),$$
$$N(a+b) \leq N(a) + N(b).$$

Hence, *a norm is a subadditive subnorm*.

We remark that while in our finite-dimensional setting, a norm is always a continuous function on $\mathcal{A}$, a subnorm may fail to be continuous when $\dim \mathcal{A} \geq 2$.

A Formula for the Radius

Having defined a subnorm, we may state a variant of a result which is well-known in the context of complex Banach algebras, and which is often referred to as the Gelfand formula.

Theorem[GF] [G, Lin. & Multilin. Algebra, 2007]. *Let f be a continuous subnorm on a FDPA algebra $\mathcal{A}$ over $\mathbb{R}$ or $\mathbb{C}$, and let r denote the radius on $\mathcal{A}$. Then for every $a \in \mathcal{A}$,*

$$\lim_{k\to\infty} f(a^k)^{1/k} = r(a).$$

Example 1. If f is a continuous subnorm on a matrix algebra $\mathcal{A}$ over $\mathbb{R}$ or $\mathbb{C}$ with the usual matrix operations, then

$$\lim_{k\to\infty} f(A^k)^{1/k} = \rho(A), \quad A \in \mathcal{A}.$$

Example 2. If f is a continuous subnorm on the Cayley–Dickson algebra $\mathcal{A}_n$, then

$$\lim_{k\to\infty} f(a^k)^{1/k} = |a|, \quad a \in \mathcal{A}_n.$$

We observe for instance, that for each fixed p, $0 < p \le \infty$, the function

$$|a|_p = (|a_1|^p + \cdots |a_{2^n}|^p)^{1/p}, \quad a = (a_1, \ldots, a_{2^n}) \in \mathcal{A}_n,$$

is a continuous subnorm on $\mathcal{A}_n$ (a norm if and only if $1 \le p \le \infty$). Hence, by Example 2,

$$\lim_{k\to\infty} \left|a^k\right|_p^{1/k} = |a|.$$

Stability of Subnorms

In order to discuss a second applications of the radius, we need the following definition:

Definition. Let $\mathcal{A}$ be a FDPA algebra over $\mathbb{R}$ or $\mathbb{C}$. Then a subnorm f on $\mathcal{A}$ is *stable* if there exists a constant $\sigma > 0$ such that for all $a \in \mathcal{A}$ and $k = 1, 2, 3, \ldots,$

$$f(a^k) \leq \sigma f(a)^k.$$

The notion of stability plays an important role in several areas of mathematics, e.g., functional analysis, and numerical analysis of time-dependent partial differential equations.

With the above definition of stability we may now quote:

TheoremST [G, TAMS]. *If f is a continuous subnorm on a FDPA algebra $\mathcal{A}$ over $\mathbb{R}$ or $\mathbb{C}$, and if $\mathcal{A}$ be void of nonzero nilpotent elements, then f is stable if and only if $f \geq r$ on $\mathcal{A}$.*

If $f \geq r$ on $\mathcal{A}$, we shall often say that f majorizes the radius on $\mathcal{A}$.

By the Main Theorem, if the algebra $\mathcal{A}$ is void of nonzero nilpotents, it is easily seen that the radius is a continuous subnorm on $\mathcal{A}$. Hence, TheoremST implies:

Corollary. *Let $\mathcal{A}$, a FDPA algebra over $\mathbb{R}$ or $\mathbb{C}$, be void of nonzero nilpotent elements. Then the radius is the smallest stable continuous subnorm on $\mathcal{A}$.*

We recall, for example, that the Cayley–Dickson algebras are void of nonzero nilpotents. Hence, TheoremST and its corollary can be rephrased to read:

Theorem. *Let f be a continuous subnorm on $\mathcal{A}_n$. Then*:

(a) *f is stable if and only if f majorizes the Euclidean norm on $\mathcal{A}_n$.*

(b) *The Euclidean norm is the smallest continuous stable subnorm on $\mathcal{A}_n$.*

For example, we recall that for each fixed p, $0 < p \leq \infty$, the function

$$|a|_p = (|a_1|^p + \cdots |a_{2^n}|^p)^{1/p}, \quad a = (a_1, \ldots, a_{2^n}) \in \mathcal{A}_n,$$

is a continuous subnorm on $\mathcal{A}_n$ (a norm for $1 \leq p \leq \infty$). So $|\cdot|_p$ is stable on $\mathcal{A}_n$ if and only if

$$|a|_p \geq |a| \text{ on } \mathcal{A}_n,$$

which holds precisely when $0 < p \leq 2$.

On the other hand, Theorem^{GF} tells us that *all continuous subnorms on a FDPA algebra over $\mathbb{R}$ or $\mathbb{C}$ satisfy the Gelfand formula.*

Confronting Theorem^{GF} with the above example for $p > 2$ (where $|\cdot|_p$ is an unstable norm), we realize that *while continuity of a subnorm f is enough to force the Gelfand formula, it is not enough to force stability, not even when f is a norm, and the underlying algebra is void of nonzero nilpotents.*

Theorem[ST] stated that *if f is a continuous subnorm on a FDPA algebra $\mathcal{A}$ over $\mathbb{R}$ or $\mathbb{C}$, and if $\mathcal{A}$ is void of nonzero nilpotent elements, then f is stable if and only if $f \geq r$ on $\mathcal{A}$.*

We claim that *the assumption that $\mathcal{A}$ is void of nonzero nilpotent elements cannot be dropped*, not even when $\mathcal{A}$ is an associative algebra and f is a norm.

Indeed (compare [A. Palacios, J. Algebra, 2000]), let $\mathbb{C}^{n\times n}$ be the algebra of $n\times n$ complex matrices with the usual operations, and let $B \neq 0$ be a fixed nilpotent matrix in $\mathbb{C}^{n\times n}$ such that $B^2 = 0$. Consider the 2-dimensional subalgebra of $\mathbb{C}^{n\times n}$ generated by I and B, i.e.,

$$\mathcal{A} = \{\alpha I + \beta B : \alpha, \beta \in \mathbb{C}\}.$$

Clearly, $\mathcal{A}$ is an associative algebra, which contains nonzero nilpotent elements, i.e. B.

Define now a norm on $\mathcal{A}$:

$$N(\alpha I + \beta B) = \max\{|\alpha|, |\beta|\}, \quad \alpha I + \beta B \in \mathcal{A}.$$

Then for every matrix $\alpha I + \beta B$,

$$N(\alpha I + \beta B) \geq |\alpha| = \rho(\alpha I + \beta B),$$

so N majorizes the radius on $\mathcal{A}$.

On the other hand, the matrix $I + B$ satisfies $(I+B)^k = I + kB$. Hence

$$\lim_{k\to\infty} N((I+B)^k) = \infty,$$

which is enough to imply that N is unstable on $\mathcal{A}$.

Our last example showed that in the presence of nonzero nilpotents, a norm on an algebra of complex matrices may fail to be stable, even when this norm majorizes the radius.

In view of this example, we shall now state two results which provide criteria for the stability of norms on complex matrices without ruling out nonzero nilpotent elements.

The first of these results, known as the Kreiss Matrix Theorem, is unusually general since it deals with arbitrary sets of matrices rather than with algebras:

Theorem [H.-O. Kreiss, BIT, 1962]. *Let $\mathcal{S}$ be an arbitrary set of matrices in $\mathbb{C}^{n\times n}$, and let N be a norm on $\mathbb{C}^{n\times n}$. Then,* there exists a constant $\sigma > 0$ such that

$$N(A^k) \le \sigma N(A)^k, \quad A \in \mathcal{S}, \quad k = 1, 2, 3, \ldots,$$

if and only If there exists a constant $\tau > 0$ such that for each $A \in \mathcal{S}$ and all $z \in \mathbb{C}$ with $|z| > 1$, the resolvent $(A - zI)^{-1}$ exists and

$$N((A - zI)^{-1}) \le \frac{\tau}{|z| - 1}.$$

The second result, whose proof heavily hinges on of Kreiss' theorem, is:

Theorem [S. Friedland & C. Zenger, Lin. Algebra & Its Appl., 1984]. *Let N be a norm on $\mathbb{C}^{n\times n}$. Then N is stable if and only If $N \ge \rho$ on $\mathbb{C}^{n\times n}$.*

Discontinuous Subnorms

It was mentioned earlier that if $\dim \mathcal{A} \geq 2$, then contrary to norms, *subnorms on $\mathcal{A}$ may fail to be continuous*.

To support this statement, let f be a continuous subnorm on a FDPA algebra $\mathcal{A}$ over a field $\mathbb{F}$, either $\mathbb{R}$ or $\mathbb{C}$, where $\dim \mathcal{A} \geq 2$. Fix an element $a_0 \neq 0$ in $\mathcal{A}$, and consider the linear subspace of $\mathcal{A}$ generated by a_0,

$$\mathbf{V} = \{\alpha a_0 : \alpha \in \mathbb{F}\}.$$

Fix $\kappa > 1$, and define

$$g_\kappa(a) = \begin{cases} \kappa f(a), & a \in \mathbf{V}, \\ f(a), & a \in \mathcal{A} \setminus \mathbf{V}. \end{cases}$$

Evidently, g_κ is a subnorm on $\mathcal{A}$. Moreover, g_κ is discontinuous at a_0, since

$$\lim_{\substack{a \to a_0 \\ a \notin \mathbf{V}}} g_\kappa(a) = f(a_0) \neq g_\kappa(a_0).$$

We note that despite its discontinuity, g_κ is well-behaved in the sense that it satisfies the Gelfand formula, it is stable, and it does majorize the radius on $\mathcal{A}$.

A Subnorm which Is Discontinuous Everywhere

We conclude this talk by displaying a subnorm which, in contrast to g_κ, is discontinuous everywhere.

To this end, we need the following short detour.

Consider the Cauchy functional equation

$$\varphi(x+y) = \varphi(x) + \varphi(y), \quad x, y \in \mathbb{R}.$$

It is well known that the only continuous solutions of this equation are of the form

$$\varphi(x) = \gamma x$$

where γ is an arbitrary real constant.

It is also known ([G. Hamel, Math. Ann., 1905]) that the Cauchy equation has discontinuous solutions, and that all such solutions are discontinuous everywhere.

Finally, given a positive constant c, it is known that one may select a discontinuous solution of the Cauchy equation which is c-periodic.

With these facts, we can now state:

Theorem [G & W.A.J. Luxemburg, Lin. & Multilin. Algebra, 2001]. *Let f be a continuous subnorm on $\mathbb{C}$, and let φ be a discontinuous π-periodic solution of the Cauchy equation*

$$\varphi(x+y)=\varphi(x)+\varphi(y), \quad x,y\in\mathbb{R}.$$

Consider the function

$$h_\varphi(z)=f(z)\,e^{\varphi(\arg z)}, \quad z\in\mathbb{C},$$

where $\arg z$ denotes the principal argument of z, $0\le \arg z<2\pi$, and $\arg 0=0$. Then:

(a) h_φ *is a subnorm on* $\mathbb{C}$.

(b) h_φ *is discontinuous everywhere in* $\mathbb{C}$.

(c) h_φ *is stable on* $\mathbb{C}$.

(d) h_φ *does not satisfies the Gelfand formula on* $\mathbb{C}$.

(e) h_φ *does not majorize the radius on* $\mathbb{C}$.

Referring to part (d) of the theorem, it can be shown that *the sets of complex numbers on which h_φ satisfies the Gelfand formula and the set on which h_φ violates this formula, are both dense in $\mathbb{C}$.*

Similarly, regarding part (e), it can be shown that *the set of complex numbers where h_φ majorizes the radius and the set where h_φ fails to do so, are again dense in $\mathbb{C}$.*