Roman Ger, Ludwig Reich

A generalized ring homomorphism equation

Monatshefte für Mathematik 159, 225–233 (2010)

X, Y rings, $f: X \to Y$.

Under which hypotheses can ring homomorphisms $f: X \to Y$ can be characterized by a single functional equation in two variables?

$$\begin{cases} f(x+y) = f(x) + f(y) \\ f(xy) = f(x) \cdot f(y) \end{cases}, \quad x, y \in X \quad (RH)$$

A) J. Dhombres 1988

X, Y unitary rings, X 2-divisible, $f(0)=0$

$f: X \to Y$,

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y) \quad (D)$$

$\Rightarrow$ f is a ring homomorphism

(Add the equations of (RH) side by side!)

G – R

②

→ J. Dhombres

$$a f(xy) + b f(x) f(y) + c f(x+y) + d f(x) + d f(y) = 0,$$

$x, y \in X$ (E0)

$f: X \to Y$, X a unitary ring in which division by 2 can be performed; Y a skew field, a, b, c, d in the center of Y

B) H. Alzer (2004, Problem)

$$f(x+y) - f(xy) \overset{(\leq)}{=} f(x) + f(y) - f(x) \cdot f(y) \quad (A)$$

(Subtract equations of (RH) side by side)

$f: \mathbb{K} \to \mathbb{R}$

R. Ger Publ. Math. Debrecen $\underline{52}$ (1998)

Rocznik Nauk.-Dydakt. Prace. Mat $\underline{17}$ (2000)

<u>Problem</u>:

X, Y unitary rings; Y commutative, without divisors of zero. What are the solutions of

$$a f(xy) + b f(x) f(y) + c f(x+y) + d f(x) + k f(y) = 0 \quad (E):$$

$x, y \in X$

(? homomorphisms, additive functions, exponential functions; dependence on the parameters $a, b, \ldots, k \in Y$)

We do not assume that division by 2 is possible in X.

$\mathcal{S}(E)$ the set of all solutions $f: X \to Y$ of (E) with $f \neq 0$, ~~$f(0)=0$~~

The paper gives an "almost" complete description of $\mathcal{S}(E)$

(1) Assume that not all coefficients a, b, c, d, k in (E) are zero. Then each $f \in \mathcal{S}(E)$ is a ring homomorphism if and only if $k = d = -c \neq 0$, and $b = -a \neq 0$

(a) In the set $\mathcal{S}(E)$ in certain special cases <u>additive</u> or <u>multiplicative</u> or <u>gen. exponential</u> functions occur. <u>Logarithmic</u> functions do not appear (since f is defined at 0!)

(m) There are cases where solutions exist which are quite different from the expected ones (from (i) and (a))

Assume $c \neq 0, a \neq 0, b = 0$

Then

α) f arbitrary nonconstant, if $a = c$

β) f even ; $f(2x) = 0$ for $x \in X$

If $a = 2c$, then the formula

$$F(x + 2X) = f(x), \quad x \in X$$

defines an even function $F: X/_{2X} \to Y$ which solves (E) on $X/_{2X}$.

Similar questions motivated by derivations

$f: \mathbb{R} \to \mathbb{R}$ (or $f: \mathbb{K} \to \mathbb{K}$, $\mathbb{K}$ field of char. 0)

$$\begin{cases} f(x+y) = f(x) + f(y) \\ f(xy) = x f(y) + y f(x) \end{cases} \qquad x, y \in \mathbb{R}$$

$\Downarrow$

$$f(x+y) \pm f(xy) = f(x) + f(y) \pm (x f(y) + y f(x))$$

$x, y \in \mathbb{R}$

$\Downarrow$?

$$a f(xy) + b x f(y) + c y f(x) + d f(x+y) + e f(x) + k f(y) = 0, \quad x, y \in X$$

X, Y rings; $f: X \to Y$;

$a, \dots, k \in Y$ (given)

Zygfryd Kominek – Ludwig Reich – – Jens Schwaiger ①

On additive functions fulfilling some additional conditions

ÖAW, Sitzungsberichte Math.-nat. Kl. Abt II, **207**, 35–42 (1998)

Let $D \subset \mathbb{R}^2$; $f : \mathbb{R} \to \mathbb{R}$ additive

Under which conditions on D is it true

that $f(x) \cdot f(y) = 0$ for all $(x,y) \in D$

implies $f = 0$

[Original question by G. Szabó for

$D = \{(x,y) | x^2 + y^2 = 1\}$

A) $f : \mathbb{R} \to \mathbb{R}$ additive, $D = \{(x,y) | x^2 + y^2 = 1\}$

Then

$f(x) \cdot f(y) = 0$, $\forall (x,y) \in D \Rightarrow f = 0$

[$u = \frac{3x+4y}{5}$, $v = \frac{4x-3y}{5}$; $x^2+y^2 = 1 \Leftrightarrow u^2+v^2 = 1$

$0 = f(u) \cdot f(v) = \frac{12}{25}(f(x)^2 - f(y)^2) \Rightarrow f = 0$ on $[0,1] \Rightarrow f = 0$]

K - S - R

B) X a real normed space, Y a linear space.

$f: X \to Y$ additive.

then

$[\|x\|^2 + \|y\|^2 = 1 \Rightarrow f(x) = 0$ or $f(y) = 0] \Longrightarrow f = 0$

C) G an abelian group

$\mathbb{K}$ a field of characteristic 0

$v, w: G \to \mathbb{K}$ generalized polynomials

such that $\lim_{\mathbb{Q}} v(G) = \lim_{\mathbb{Q}} w(G) = \mathbb{K}$

$D := \{(v(x), w(x)) \in \mathbb{K} \times \mathbb{K} \mid x \in G\}$

Assume that $f: G \to \mathbb{K}$ is additive

and $f(v(x)) \cdot f(w(x)) = 0$ for all $x \in G$

then $f = 0$.

(Uses a result of F. Halter-Koch, Reich and Schwaiger on products of additive functions)

(1') Let $v, w : \mathbb{R} \to \mathbb{R}$ be ordinary polynomials of degree at least 1. The C) is true for additive functions $f : \mathbb{R} \to \mathbb{R}$ and sets $D = \{ (v(x), w(x)) \mid x \in \mathbb{R} \}$.

The product " $f(x) \cdot f(y) = 0$ " in the problem may be generalized as follows:

" $f(x) \cdot f(y) = 0,\ (x,y) \in D$ " $\rightsquigarrow$ " $Q(f(x), f(y)) = 0,\ (x,y) \in D$ " where Q is an ordinary polynomial, $Q \neq 0$.

D) $f : \mathbb{R} \to \mathbb{R}$ additive

$p, q : \mathbb{R} \to \mathbb{R}$ generalized polynomials of degree 1, and that $p(\mathbb{R}), q(\mathbb{R})$ contain Hamel bases

$D := \{ (p(u), q(u)) \mid u \in \mathbb{R} \} (\subset \mathbb{R}^2)$

$Q \in \mathbb{R}[X, Y]$ which does not leave a factor $AX + BY + C$ with $AB \neq 0$

If $[(x,y)\in D \Rightarrow Q(f(x), f(y))=0]$, then $f=0$.

E) The results described above hold true for additive functions $f:\mathbb{R}^n\to\mathbb{R}$ and sets $D\subset\mathbb{R}^{2n}$ which are measurable with positive measure. Similarly for sets $D\subset\mathbb{R}^{2n}$ of second category with the Baire property.

F) $D\subset\mathbb{R}^2$ a hyperbola.

Sets $D=\{(u(t), v(t)) \mid t\in T\subset\mathbb{R}\}(\subset\mathbb{R}^2)$ having the following invariance property:

There exists a rational (2,2)-matrix M such that $M\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}\in D$, $\forall t\in T$,

$\mathrm{lin}_{\mathbb{Q}}\, u(T)=\mathrm{lin}_{\mathbb{Q}}\, v(T)=\mathbb{R}$

Detlef Gronau, Maciej Sablik

A functional equation arising from an asymptotic formula for iterates

Annales Mathematicae Silesianae 8, 43-187 (1994)

Asymptotic behaviour of iterates in the neighbourhood of a fixed point of the iterated mapping: D. Gronau, L. Berg

D open real interval, $0 \in D$

$f : D \to \mathbb{R}$, $f(0) = 0$.

Assume that $\lim\limits_{n\to\infty} n f^n\left(\frac{x}{n}\right)$ exists uniformly on an interval $\mathcal{U} \subset D$, $0 \in \mathcal{U}$.

$$\varphi(x) := \lim_{n\to\infty} n f^n\left(\frac{x}{n}\right), \quad x \in \mathcal{U}$$

$(\Rightarrow \varphi(0) = 0)$

D. Gronau: If $f \in \mathcal{C}^2$, $f'(0) = 1$, then

$$f^{kn}\left(\frac{x}{n}\right) = \frac{1}{k}\varphi(kx) + o\left(\frac{1}{n}\right)$$

for $n \to \infty$, $kx \in \mathcal{U}$, $k \in \mathbb{N}$

$\Rightarrow \varphi(0)=0$

$$\varphi^{m}(x) = \frac{1}{m}\varphi(mx), \quad x\in U, m\in N \quad (I)$$

1) Let $\varphi : D \to \mathbb{R}$ be a solution of (I) for a fixed $m \geq 2$ on a set U, containing a neighbourhood of 0. Let φ be differentiable at 0. Then:

(i) If m is even, then $\varphi'(0) \in \{0, 1\}$.

(ii) If m is odd, then $\varphi'(0) \in \{0, 1, -1\}$.

(iii) If $\varphi'(0)=0$, then $\varphi(x)=0$ for all x with $\frac{1}{m}x \in U$.

2) (Main result)

Let $\varphi : D \to \mathbb{R}$ be a solution of (I) for a fixed $m > 1$, U an open interval, $0 \in U \subset D$. Suppose that φ is continuous on U and twice differentiable at 0. If $\underline{\varphi'(0)=1}$, then

$$\varphi(x) = \frac{x}{1-bx}, \text{ with } b = \frac{1}{2}\frac{d^2\varphi}{dx^2}(0),$$

for $x \in W$ where

$$W = \begin{cases} m\mathcal{U} \cap [-\infty, b^{-1}] , & \text{if } b>0 \\ m\mathcal{U} , & \text{if } b=0 \\ m\mathcal{U} \cap (b^{-1}, \infty) , & \text{if } b<0. \end{cases}$$

the proof is delicate.

3) Let $\varphi : D \to \mathbb{R}$ be a real solution of (I) for a fixed natural $m > 1$ on a open interval containing 0. Suppose that φ is continuous on $\mathcal{U}$ and twice differentiable at 0. If $\varphi'(0) = -1$, then

$$\varphi(x) = -x , \text{ for all } x \in m\mathcal{U} .$$

(~~If~~

G-S

Equation (I) has, under weaker regularity conditions on φ (e.g. φ nondecreasing) many more solutions. The paper discusses construction of such solutions by means of certain sequences of functions, defined by using iteration.

Wojciech Jabłoński – Ludwig Reich (1)

$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

$\mathbb{K}[[X]]$ the ring of formal power series in X over $\mathbb{K}$, $(\Gamma, \circ)$ the group of invertible formal series; $(G,+)$ an abelian group

Study of homomorphisms

$\Theta : (G,+) \longrightarrow (\Gamma, \circ)$

$t \in G \qquad \Theta(t)(X) = \Theta_t(X) = c_1(t)X + c_2(t)X^2 + \cdots,$

$c_1(t) \neq 0 \qquad (\Theta_t)_{t\in G}$ "one-parameter group"

Problems 1) Construction of these homomorphisms

2) Explicit formulas for the "coefficient functions" c_ν

3) Algebraic structure of the groups $\Theta(G)$

L^1_∞, L^1_s differentiable groups, ($1 \leq s$) arising in the theory of geometric objects of J. Aczél and St. Gołąb)

J - R

Describe the homomorphisms of abelian groups $(G, +)$

$\theta : (G, +) \longrightarrow L^1_\infty$

this problem is essentially the same as the problem for homomorphisms $\theta : (G,+) \to (\Gamma, \circ)$

Homomorphisms θ

$\theta : (G, +) \longrightarrow L^1_s \qquad (1 \leq s < \infty)$

This is essentially the same problem as finding the homomorphisms

$\theta : G, +) \longrightarrow (\Gamma^s, \circ)$

where $(\Gamma^s, \circ)$ is the group of invertable s-truncated series

$\Gamma(\mathbb{C}[[X]])^s \cong \{ c_0 + c_1 X + \dots + c_s X^s \mid c_0, c_1 \dots, c_s \in \mathbb{C} \}$

with an appropriate definition of $+, \cdot$ and $\circ$]

J. – R

Further problems

a) Extensibility of homomorphisms

$\Theta : (G, +) \to L'_{s}$ to homomorphisms

$\tilde{\Theta} : (G, +) \to L_{s+r}$ $(r \geq 1)$

b) Stability of the translation equation in power series rings

c) Connections with Aczél–Jabotinsky differential equations

Method: Combination of results about analytic iteration groups with the investigation of the infinite system of functional equations for the coefficient functions (equivalent to the translation equation)

W. Jabłoński – L. Reich
A new approach to the description of one-parameter groups of formal power series in one indeterminate.
Aequationes Math. 87 (2014), 247–284

Detlef Gronau & Janusz Matkowski

I) Geometrical convexity and generalizations of the Bohr-Mollerup theorem on the Gamma-function

Mathematica Pannonica 4/1, 153-168 (1992)

II) Geometrically convex solutions of certain difference equations and generalized Bohr-Mollerup theorems

Results in Mathematics 26, 290-297 (1994)

I) E. Artin, in his development of the theory of Gamma-function in the real case, used the Bohr-Mollerup theorem as important tool:

Γ is the unique logarithmically convex solution of the difference equation

$$g(x+1) = x \cdot g(x) \qquad (1)$$

$$x \in (0, \infty), \quad g(1) = 1 .$$

G - M

[g logarithmically convex on $(0,\infty)$, if $\log\circ g$ convex on $(0,\infty)$]

[Artin's proof is very elegant. In the theory of Γ as meromorphic function, Wielandt's theorem plays a similar role as the Bohr-Mollerup theorem in the real case.]

Gronau & Matkowski proved a far reaching generalization of the Bohr-Mollerup result

<u>Definition</u>: $g : (0,\infty) \to (0,\infty)$ is <u>geometrically convex</u>, if $\log\circ g\circ \exp$ is convex on $(0,\infty)$

$$\Leftrightarrow \quad g(x^\lambda \cdot y^{1-\lambda}) \le g(x)^\lambda g(y)^{1-\lambda}, \quad x, y \in (0,\infty), \quad \lambda \in (0,1)$$

($\log\circ g\circ \exp$ is a conjugate of g)

G - M.

1) Let $g:(0,\infty)\to(0,\infty)$ be a solution of (1), geometrically convex on an interval (a,∞) for a certain $a>0$. Then $g=\Gamma$.

2) If $g:(0,\infty)\to(0,\infty)$ is a solution of (1) and g logarithmically convex on an interval (a,∞) for an $a\geq 0$, then $g=\Gamma$.

Jensen geometrical convexity in the theory of Γ

$g: I\to(0,\infty)$ geometrically Jensen convex on the interval I, if

$$g(\sqrt{xy})\leq\sqrt{g(x)\,g(y)} \quad \text{for all } x,y\in I$$

3) Let $g:(0,\infty)\to(0,\infty)$ be bounded above on a neighbourhood of a point, and geometrically Jensen convex on an interval (a,∞) with $a\geq 0$, and assume that g satisfies (1). Then $g=\Gamma$.

P)

Relations to Krull's Normallösung

1) Let the function $G:(0,\infty)\to(0,\infty)$ be logarithmically concave on a neighbourhood of ∞, and suppose that for a $\delta>0$

$$\lim_{x\to\infty}\frac{G(x+\delta)}{G(x)}=1 \qquad (L).$$

Then for every real $c>0$ there exists at most one solution $g:(0,\infty)\to(0,\infty)$ of the difference equation

$$g(x+1)=G(x)\cdot g(x),\ x\in(0,\infty) \qquad (D)$$

with $g(1)=c$ and which is geometrically convex on a neighbourhood of ∞.

2) (Main result) (Generalization of Krull's theorem)

Let $G:(0,\infty)\to(0,\infty)$ satisfy, for a $\delta>0$,

$$\lim_{x\to\infty}\frac{G(x+\delta)}{G(x)}=1.$$

G–A

Suppose that there exists $a \geq 0$ such that G is logarithmically concave on (a, ∞) and that there exists a $y \in (a, \infty)$ such that $G(y) = 1$

Then we have:

For each $c > 0$ there exists one solution g of

$$g(x+1) = G(x) \cdot g(x), \; x \in (0, \infty) \qquad (D)$$

which is <u>geometrically convex</u> on a neighbourhood of ∞ and such that $g(1) = c$

[g is geometrically convex at least on $(y+1, \infty)$]

I) and II) contain a lot of examples illustrating the interplay of the various convexity concepts.

Janusz Morawiec & Ludwig Reich (1)

I) On probality distribution solutions of a functional equation.

Bull. Polish Academy of Sciences, Mathematics 53, 389-399 (2005)

II) The set of probability distribution solutions of a linear functional equation.

Annales Polonici Mathematici 93, 253-261 (2008)

I) M. Corsolini

$$\varphi(x) = p\varphi(f_1(x)) + q\varphi(f_2(x)), \quad x \in \mathbb{R} \qquad (CE)$$

$p, q \geq 0, \ p+q=1$

$\varphi : \mathbb{R} \to \mathbb{R}, \ \varphi|_{(-\infty,0]} = 0, \ \varphi|_{[1,\infty)} = 1$

φ unknown

M – R

$$f_1(x) = \frac{x-\beta}{1-\beta}, \quad f_2(x) = \begin{cases} \frac{x}{\alpha} & , \text{ if } x \le \beta \\ \frac{x(\alpha-\beta)+\beta(1-\alpha)}{\alpha(1-\beta)} & , \text{ if } x \ge \beta \end{cases}$$

$\alpha, \beta \in (0,1)$

Origin of this equation: game theory?

J. Morawiec: $\alpha \le \beta$

$\boxed{\alpha > \beta}$

$\mathcal{I} := \{\varphi : \mathbb{R} \to \mathbb{R} \mid \varphi$ increasing, $\varphi|_{(-\infty,0]} = 0$, $\varphi|_{[1,\infty)} = 1\}$

$\mathcal{C} := \{\varphi : \mathbb{R} \to \mathbb{R} \mid \varphi$ continuous, $\varphi|_{(-\infty,0]} = 0$, $\varphi|_{[1,\infty)} = 1\}$

We are interested in solutions in the sets $\mathcal{I}$ and $\mathcal{C}$, and the relations between these sets of solutions.

(CE) has always solutions in $\mathcal{I}$.

1) (CE) has at most one solution in $\mathcal{C}$.

2) Existence of solutions of (CE) in $\mathcal{C}$:

$$\varphi_1(x) = \chi\big|_{[0,\infty)}(x) \; ; \; \varphi_{n+1}(x) = p\,\varphi_n(f_1(x)) + q\,\varphi_n(f_2(x)) \quad (n \geq 1)$$

$x \in \mathbb{R}$

$\Rightarrow \phi(x) = \lim_{n\to\infty} \varphi_n(x)$ exists ($x \in \mathbb{R}$),

ϕ is a solution of (CE)!

(CE) has a solution in $\mathcal{C}$ if and only if ϕ is continuous. If ϕ is continuous, then it is strictly increasing on $[0, 1]$, either absolutely continuous or singular.

3) $q \leq \alpha - p\beta \Rightarrow$ (CE) has no solution in the class $\mathcal{C}$.

4) (CE) has a solution in $\mathcal{C}$ if and only if it has a solution φ in $\mathcal{I}$ with $\varphi \neq \chi\big|_{[1,\infty)}$

M–R

$\varphi \in \mathcal{C}$ solution of (CE) $\Rightarrow$

[$\psi \in I$ is a solution of (CE) if and only if $\exists$ $\lambda \in [0,1)$ such that $\psi(x) = \begin{cases} \lambda\varphi(x), & x \in (-\infty, 1] \\ 1 & x \in [1,\infty) \end{cases}$]

5) $q \geq \alpha$

Let φ be the unique solution of (CE) in $\mathcal{C}$. Then every solution of (CE) in the class I has the form (from 4)

6) If $q \leq \alpha - p\beta$, then $\chi|_{[1,\infty]}$ is the only solution of (CE) in the class $\tilde{I}$

(CE) has ~~no~~ solution in $\mathcal{C}$.

7) Assume $0 < p \log \frac{1}{1-\beta} + q \log \frac{\alpha-\beta}{\alpha(1-\beta)}$.

Then (CE) has no solution in the class $\mathcal{C}$.

$\chi|_{[1,\infty)}$ is the unique solution of (CE).

M-R

Application of a theorem of Karol Baron concerning the equation

$$\varphi(x) = \int_\Omega \varphi(\tau(x,\omega))\, dP(\omega)$$

((CE) is a particular case of this equation).

II)

$$F(x) = \int_\Omega F(\tau(x,\omega))\, dP(\omega) \qquad (1)$$

$(\Omega, \mathcal{A}, P)$ probability space

$\tau : \mathbb{R} \times \Omega \to \mathbb{R}$ given

$\tau(x, \cdot)$ measurable for all x

$\tau(\cdot, \omega)$ strictly increasing and continuous for almost all ω

$F : \mathbb{R} \to [0,1]$ unknown

$\mathcal{C}$ set of all continuous probability distribution solutions F of (1)

$\mathcal{I} = \{ F : \mathbb{R} \to [0,1] \mid F$ is a nondecreasing solution of (1), $F^+(-\infty) = 0$, $F_-(+\infty) = 1 \}$

M — R

II Relations between the sets $\mathcal{C}$ and $\underset{\sim}{\mathcal{I}}$.
Existence and uniqueness of solutions
Connections with refinement equations
and with integral Cauchy equation
from probability theory.

$\underline{\mathbb{F}} := \{x \in \mathbb{R} \mid \tau(x,\omega) = x$ for almost all $\omega \in \Omega\}$

$E_0 = \underline{\mathbb{F}} \cup \{-\infty, \infty\}$

1) $\underline{\mathbb{F}} = \emptyset \Rightarrow \mathcal{C} = \underset{\sim}{\mathcal{I}}$

2) $\underline{\mathcal{C} = \emptyset}$ then:

$\underline{\mathbb{F}} = \emptyset \Rightarrow \underset{\sim}{\mathcal{I}} = \emptyset$

$\underline{\mathbb{F}} \neq \emptyset \Rightarrow \underset{\sim}{\mathcal{I}} \neq \emptyset$ A detailed description of the F's in $\underset{\sim}{\mathcal{I}}$ is possible

3) $\underline{\mathcal{C} \neq \emptyset}$ then

$|\mathcal{C}| = 1$ or $|\mathcal{C}| = \mathfrak{c}$

$\underline{\mathbb{F}} = \emptyset \Rightarrow \underset{\sim}{\mathcal{I}} = \mathcal{C}$

$\underline{\mathbb{F}} \neq \emptyset \Rightarrow \underset{\sim}{\mathcal{I}} \supsetneq \mathcal{C}$ A detailed description of the F's in $\underset{\sim}{\mathcal{I}}$ is possible

4) $|\mathcal{C}| = 1$

then:

$$\mathbb{E} = \emptyset \Rightarrow \underset{\sim}{I} = \{F\}$$

$$\mathbb{E} \neq \emptyset \Rightarrow \underset{\sim}{I} \supsetneq \{F\}$$

Connections with known results of K. Baron, R. Kapica, Kapica & Morawiec are discussed

K. Baron, F. Halter-Koch, P. Volkmann
On orthogonally exponential functions
Archiv Math. 72, 185-191 (1995)

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M. Sablik
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ÖAW, Anzeiger Math.-nat.-Kl. Abt II, 137
11-15 (2000)

—

R. Ger
A Pexider type equation in normed linear
spaces. ÖAW Sitzungsber. Abt. II 206, 291-~~297~~
303 (1997)

P. Schöpf
Solutions of $\|f(\xi+\eta)\| = \|f(\xi)+f(\eta)\|$.
Math. Pannonica 8, 117-127 (1997)

P. Schöpf
Orthogonality and proportional norms.
ÖAW, Anzeiger Math-nat. Kl, Abt II, 133,
11-16 (1996

Conferences

(1)

1) D. Gronau and L. Reich (Eds.)
Selected Topics in Functional Equations. Proceedings of a Seminar Held in Graz, Austria, May 22-23, 1986. Berichte der Mathematisch-Statistischen Sektion in der Forschungsgesellschaft Joanneum, 285-296; IV+234 (1988)

2) D. Gronau and L. Reich (Eds.)
Selected Topics in Functional Equations and Iteration Theory. Proceedings of the Austrian-Polish Seminar, Universität Graz, Austria, October 24-26, 1991. Grazer Mathematische Berichte 316, VI+238 (1992)

- 2) contains an article about M. Kuczma by K. Baron

1) Contributions by J. Aczél and Zb. Gajda.

Conferences

3) R. Ger (Ed.) Polish-Austrian Seminar on Functional Equations and Iteration Theory. Cieszyn, October 26-31, 1993. Annales Mathematicae Silesianae 8, 1994

4) D. Gronau and L. Reich (Eds.): Contributions to the Theory of Functional Equations III. Proceedings of the Colloquium Held in Graz, Austria, October 20-21, 2000. Grazer Mathematische Berichte 344, IV+68. (2001)