Convexity in the output of Professor Roman Ger

Kazimierz Nikodem

University of Bielsko–Biala, Poland

Uniwersytet Śląski, Katowice, November 20, 2015

Outline

Degrees

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- **1968** - MSc (magister), Jagiellonian University, Brench in Katowice;

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- **1976** - post-doctoral degree (doktor habilitowany), Silesian University;

Degrees

Akademia Techniczno-Humanistyczna w Bielsku-Białej

- **1968** - MSc (magister), Jagiellonian University, Brench in Katowice;
- **1971** - PhD (doktor), Silesian University;
- **1976** - post-doctoral degree (doktor habilitowany), Silesian University;
- **1990** - Professor (profesor nauk matematycznych), President of Poland, Warsaw.

Functions at Silesian University

- Since 1987 - Head of the Department of Functional Equations;

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- 1990–1992 - Vice-Rector for Teaching Affairs;
- 2005–2008 - Vice-Director for Science of the Institute of Mathematics;
- 2008–2012 - Director of the Institute of Mathematics.

PhD students

1. 1981 - Kazimierz Nikodem
2. 1985 - Irena Fidytek
3. 1985 - Zbigniew Gajda
4. 1987 - Piotr Cholewa
5. 1988 - Paweł Urban
6. 1994 - Roman Badora
7. 1998 - Justyna Sikorska
8. 2000 - Barbara Kulpa
9. 2001 - Katarzyna Domańska
10. 2002 - Tomasz Szostok
11. 2003 - Eleonora Czyba
12. 2004 - Dorota Budzik
13. 2006 - Iwona Tyrala
14. 2007 - Włodzimierz Fechner
15. 2010 - Tomasz Kochanek

Conferences, Colloquium Talks

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- over 130 talks at international symposia and conferences in Europe, America and Asia;

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- over 130 talks at international symposia and conferences in Europe, America and Asia;
- over 50 colloquium talks at many universities, among others in:
  - Chicago, Amherst, Orlando (USA)
  - Waterloo, Toronto, Ottawa (Canada)
  - Karlsruhe, Clausthal, Erlangen (Germany)
  - Graz, Innsbruck (Austria)
  - Milan, Rome (Italy)
  - Bern (Switzerland)
  - Debrecen (Hungary)
  - Hajfa (Israel)
  - Aarhus (Denmark)
  - Barcelona (Spain)
  - Chengdu (China)

Publications

$\sim$ **120 publications** in prestigious mathematical journals;

Co-author of the monograph:

Bogdan Choczewski, Roman Ger, Marek Kuczma,
Iterative Functional Equations, Cambridge University Press,
Cambridge - New-York - Sydney, 1990,
in the series "Encyclopedia of Mathematics and Applications"

second edition - 2008.

Quotations, HI

According to Google Scholar:

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According to Google Scholar:

- Number of quotations - **1852**

Quotations, HI

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- Hirsch Index - **19**

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- Number of quotations - **1852**
- Hirsch Index - **19**

16 of the papers got an award in the Marek Kuczma Contest for the best Polish Paper on Functional Equations

Research fields

- Functional equations (mainly in several variables):

  - Cauchy, Pexider, Jensen, Fischer-Muszély, Mikusiński equations

  - equations with restricted domains
(among others the habilitation dissertation:
On some functional equations with restricted domain,
Prace Naukowe Uniwersytetu Śląskiego, 1976)

  - conditional and alternative equations
  - equations almost everywhere
  - orthogonal additivity

Research fields

- Hyers-Ulam stability:
  - superstability
  - stability of additive, polynomial, exponential functions
- Set-valued maps:
  - subadditive, multiadditive maps
  - selections
- Means
- Difference operators
- Iteration groups

Research fields -convexity

- Over **25** papers (i.e. about 20%) devoted to (or related with) convexity. Main topics:

  - convex and Jensen-convex functions
  - higher-order convexity
  - approximately convex functions
  - delta-convex functions

- First publication:

Some remarks on ***convex*** *functions*, Fundamenta Mathematica 66 (1970), 255-262.

Research fields -convexity

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Some properties of polynomial functions and ***convex*** *functions of higher order*, Silesian University, 1971.

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- Prize-Winner in the first Marek Kuczma Contest for the best Polish Paper on Functional Equations:
*n-**convex** functions in linear spaces*, Aequat. Math. 10 (1974), 172-176.

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*n-**convex** functions in linear spaces*, Aequat. Math. 10 (1974), 172-176.

- PhD dissertation of the first PhD student:
K. N., ***Convex** and quadratic stochastic processes*, Silesian University, 1981.

Stationary sets

Stationary sets

Definition

X - a nonempty set; $\mathcal{K}$ - a class of functions $f : X \to \mathbb{R}$.
A set $T \subset X$ is called *stationary* for the class $\mathcal{K}$ iff for every $f \in \mathcal{K}$ the condition $f\big|_T = 0$ implies that $f = 0$.

Examples

1. Let $\mathcal{K}_1$ be the family of all continuous functions $f : \mathbb{R} \to \mathbb{R}$.
Then $T \subset \mathbb{R}$ is stationary for $\mathcal{K}_1$ iff it is dense in $\mathbb{R}$.

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2. Let $\mathcal{K}_2$ be the family of all polynomials $f : \mathbb{R} \to \mathbb{R}$ of degree at most n. Then $T \subset \mathbb{R}$ is stationary for $\mathcal{K}_2$ iff it has at least $n+1$ elements.

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3. Let $\mathcal{K}_3$ be the family of all additive functions $f : \mathbb{R} \to \mathbb{R}$. Then $T \subset \mathbb{R}$ is stationary for $\mathcal{K}_3$ iff it contains a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$.

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Jensen convex functions

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Characterize the stationary sets for the class of Jensen convex functions.

Jensen convex functions

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Characterize the stationary sets for the class of Jensen convex functions.

Recall that a function $f : D \to \mathbb{R}$ is *Jensen convex* if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \qquad \forall x, y \in D.$$

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$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \qquad \forall x, y \in D.$$

Denote by $J(D)$ the family of all Jensen convex functions on D.

A partial answer

Theorem (M. Babilonová, 1999):

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Theorem (M. Babilonová, 1999):

A set $T \subset \mathbb{R}^n$ is stationary for the class $J(\mathbb{R}^n)$ if and only if $cl\, conv_{\mathbb{Q}}\, T = \mathbb{R}^n$ and $T \in \mathcal{A}(\mathbb{R}^n)$.

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$\mathcal{A}(\mathbb{R}^n)$ is the set–class introduced by R. Ger and M. Kuczma (*On the boundedness and continuity of convex functions and additive functions*, Aequat. Math. 4, 1970):

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$\mathcal{A}(\mathbb{R}^n)$ is the set–class introduced by R. Ger and M. Kuczma (*On the boundedness and continuity of convex functions and additive functions*, Aequat. Math. 4, 1970):

$$\mathcal{A}(\mathbb{R}^n) = \{ T \subset \mathbb{R}^n : \textit{every Jensen convex function } f : D \to \mathbb{R}, \textit{ where } D \supset T \textit{ is an open convex set}, \textit{ bounded from above on } T \textit{ is continuous} \}.$$

A characterization

R. Ger and K. Nikodem, *A characterization of stationary sets for the class of Jensen convex functions*, Functional Equations-Results and Advances, Kluwer Academic Publishers, 2002, 25–28.

A characterization

Theorem 1

Let D be an open convex subset of a Hausdorff locally convex space X. A set $T \subset X$ is stationary for the class $J(D)$ if and only if $D \subset \mathit{cl}\, \mathit{conv}_{\mathbb{Q}}\, T$ and $T \in \mathcal{A}(X)$.

A characterization

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Techniczno-Humanistyczna
w Bielsku-Białej

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Theorem 2

Let $D \subset \mathbb{R}^n$ be an open convex set and T be a subset of D symmetric with respect to a point. Then T is stationary for the class $J(D)$ if and only if $conv_{\mathbb{Q}} T = D$.

Strongly convex functions

Strongly convex functions

Definitions

- $f : I \to \mathbb{R}$ is called *strongly convex* with modulus $c > 0$ if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2,$$

for all $x, y \in I$ and $t \in [0,1]$;

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- f is *strongly Jensen convex* with modulus c if

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B.T. Polyak 1966 ;
A.W. Roberts and D.E. Varberg,
N.Merentes, J.L.Sanchez, Zs.Páles, A.Gilányi, R.Ger, K.N., ...

Strongly n-convex functions

R. Ger and K. Nikodem, *Strongly convex functions of higher order* , Nonlinear Anal. 74 (2011), 661–665.

Divided differences

$I \subset \mathbb{R}$ - an interval, $n \in \mathbb{N}$, $x_0, \ldots, x_n$ - distinct points in I and $f : I \to \mathbb{R}$. Denote by $[x_0, \ldots, x_n; f]$ the divided difference of f at $x_0, \ldots, x_n$ defined by the recurrence

$$[x_0; f] = f(x_0),$$

$$[x_0, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}, \ n \in \mathbb{N}.$$

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Following Hopf (1926) and Popoviciu (1934) $f : I \to \mathbb{R}$ is called *convex of order n* (or *n-convex*) if

$$[x_0, \ldots, x_{n+1}; f] \geq 0$$

for all $x_0 < \ldots < x_{n+1}$ in I.

Strongly n-convex functions

Definition

$f : I \to \mathbb{R}$ is *strongly convex of order n with modulus c* (or *strongly n-convex with modulus c*) if

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for all $x_0 < \ldots < x_{n+1}$ in I.

For $n = 1$ condition is equivalent to

$$f(tx_0 + (1-t)x_2) \leq tf(x_0) + (1-t)f(x_2) - ct(1-t)(x_2 - x_0)^2$$

for all $x_0, x_2 \in I$ and $t \in (0, 1)$, which means that f is strongly convex with modulus c.

Characterizations by derivatives

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Theorem 3

$f : I \to \mathbb{R}$ is strongly n-convex with modulus c if and only if it is of the class C^{n-1} in I and its $(n-1)$-th derivative $f^{(n-1)}$ is strongly convex with modulus $\frac{c}{2}(n+1)!$.

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Theorem 4

let $f : I \to \mathbb{R}$ be of the class C^n. Then f is strongly n-convex with modulus c iff $f^{(n)}$ satisfies the condition

$$(f^{(n)}(x) - f^{(n)}(y))(x-y) \geq c(n+1)!(x-y)^2, \; x, y \in I.$$

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Theorem 5

Let $f : I \to \mathbb{R}$ be of the class C^{n+1} . Then f is strongly n-convex with modulus c iff $f^{(n+1)} \geq c(n+1)!$, $x \in I$.

Difference operator

$I \subset \mathbb{R}$ - an interval, $f : I \to \mathbb{R}$ and $h > 0$.
Let $\triangle_h^n$ be the difference operator of n-th order with increment h defined by the recurrence:

$$\triangle_h^0 f(x) = f(x), \quad \triangle_h^n f(x) = \triangle_h^{n-1} f(x+h) - \triangle_h^{n-1} f(x), \; n \in \mathbb{N}.$$

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A function $f : I \to \mathbb{R}$ is said to be *n-convex in the sense of Jensen* (or *Jensen n-convex*) if

$$\triangle_h^{n+1} f(x) \geq 0$$

for all $x \in I$ and $h > 0$ such that $x + (n+1)h \in I$.

Strongly Jensen n-convex functions

Definition

$f : I \to \mathbb{R}$ is *strongly n-convex with modulus $c > 0$ in the sense of Jensen* (or *strongly Jensen n-convex with modulus c*) if

$$\triangle_h^{n+1} f(x) \geq c(n+1)!\ h^{n+1}$$

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for all $x \in I$ and $h > 0$ such that $x + (n+1)h \in I$.

For for $n = 1$ this condition reduces to

$$f\left(\frac{u+v}{2}\right) \leq \frac{f(u)+f(v)}{2} - \frac{c}{4}(u-v)^2, \quad u, v \in I,$$

which means that f is strongly Jensen convex with modulus c.

Ciesielski-type theorem

Theorem 6

If a function $f : I \to \mathbb{R}$ is strongly Jensen n-convex with modulus $c > 0$ and bounded on a set $A \subset I$ having positive Lebesgue measure (or of the second cathegory and with the Baire property), then f is continuous on I and strongly n-convex with modulus c.

n-parameter families

Following Tornheim (1950) a family $\mathcal{F}$ of continuous real functions defined on $I \subset \mathbb{R}$ is called an *n-parameter family* if for any n points $(x_1, y_1), \ldots, (x_n, y_n) \in I \times \mathbb{R}$ with $x_1 < \ldots < x_n$ there exists exactly one $\varphi \in \mathcal{F}$ such that $\varphi(x_i) = y_i$ for $i = 1, \ldots, n$.

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Examples

$$\mathcal{F}_n = \{a_n x^n + \cdots + a_1 x + a_0 \ : a_0, \ldots a_n \in \mathbb{R}\},$$

$$\mathcal{F}_{n,c} = \{c x^{n+1} + a_n x^n + \cdots + a_1 x + a_0 \ : a_0, \ldots a_n \in \mathbb{R}\}.$$

A function $f : I \to \mathbb{R}$ is *convex with respect to the n-parameter family $\mathcal{F}$* (shortly, *$\mathcal{F}$-convex*) if for any $x_1 < \ldots < x_n$ in I

$$f(x) \leq \varphi_{(x_1,f(x_1)),\ldots,(x_n,f(x_n))}(x), \; x \in [x_{n-1}, x_n].$$

$\varphi_{(x_1,y_1),\ldots,(x_n,y_n)}$ is the unique function in $\mathcal{F}$ determined by $(x_1, y_1), \ldots, (x_n, y_n)$. .

Connections with strong convexity

It is well known that

$$f : I \to \mathbb{R} \text{ is } \mathcal{F}_n\text{-convex iff } f \text{ is } n\text{-convex.}$$

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We have also

Theorem 7

A function $f : I \to \mathbb{R}$ is strongly n-convex with modulus c iff f is $\mathcal{F}_{n,c}$-convex.

Dear Roman -

Happy Birthday
and congratulation to you!