The role of Δ_2-condition in Orlicz spaces

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Henryk Hudzik
Faculty of Mathematics and Computer Science
of the Adam Mickiewicz University in Poznań
Poznań, Poland
email: hudzik@amu.edu.pl

Let Φ be an Orlicz function, that is, $\Phi : \mathbb{R} \to \mathbb{R}_+ := [0, \infty)$, Φ is convex, even, $\Phi(0) = 0$ and not identically equal to 0. The last property is equivalent to the fact that $\Phi(u) \to \infty$ as $u \to \infty$.

Let (Ω, Σ, μ) be a complete and non-trivial measure space and $L^o := L^o(\Omega, \Sigma, \mu)$ be the space of (equivalence classes of) $\Sigma-$measurable functions $x : \Omega \to \mathbb{R}$.

We define the convex modular $I_\Phi : L^o \to \mathbb{R}_+^e = [0, \infty]$ by the formula

$$I_\Phi(x) = \int_\Omega \Phi(x(t))d\mu(t).$$

Then the set

$$B_\Phi = \{x \in L^o : I_\Phi(x) \leqslant 1\}$$

is absolutely convex, that is, for any $x, y \in B_\Phi$ and any $\alpha, \beta \in \mathbb{R}$ with $|\alpha| + |\beta| \leqslant 1$, we have $\alpha x + \beta y \in B_\Phi$.

Next the Minkovsky functional

$$m_\Phi(x) = \inf\left\{\lambda > 0 : \frac{x}{\lambda} \in B_\Phi\right\}$$

is a function semi-norm on L^o, that is, $m_\Phi : L^o \to \mathbb{R}_+^e := [0, \infty]$, $m_\Phi(0) = 0$, $m_\Phi(\lambda x) = |\lambda| m_\Phi(x)$ for any $x \in L^o$ and $\lambda \in \mathbb{R}$, and $m_\Phi(x + y) \leqslant m_\Phi(x) + m_\Phi(y)$ for all $x, y \in L^o$.

It was important to distinguish the biggest possible subset of L^o on which the functional m_Φ is finite. It is easy to see that this subset is the following one

$$L^\Phi = L^\Phi(\Omega, \Sigma, \mu) = \{x \in L^o\colon I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

and it is called the Orlicz space. It is easy to check that $(m_\Phi(x) = 0 \wedge x \in L^\Phi) \Rightarrow x = 0$. Therefore, the Minkovsky functional m_Φ is a norm on the Orlicz space L^Φ. It is called the Luxemburg norm and it is denoted by $\|.\|_\Phi$.

Let us denote

$$B_{m,\Phi} = \left\{ x \in L^{\Phi} : I_{\Phi}(x) < 1 \right\}, \;\; \overline{B_{m,\Phi}} = \left\{ x \in L^{\Phi} : I_{\Phi}(x) \leqslant 1 \right\},$$

$$B_{\|\cdot\|_{\Phi}} = \left\{ x \in L^{\Phi} : \|x\|_{\Phi} < 1 \right\}, \;\; \overline{B_{\|\cdot\|_{\Phi}}} = \left\{ x \in L^{\Phi} : \|x\|_{\Phi} \leqslant 1 \right\}.$$

It is well known that $\overline{B_{m,\Phi}} = \overline{B_{\|\cdot\|_{\Phi}}}$ and that $B_{\|\cdot\|_{\Phi}} \subset B_{m,\Phi}$ for any Orlicz function Φ because $I_{\Phi}(x) \leqslant \|x\|_{\Phi}$ whenever $\|x\|_{\Phi} \leqslant 1$.

We say that $x \in L^{\Phi}$ has order continuous norm (or that x is order continuous) if for any sequence $(x_n)_{n=1}^{\infty}$ in L^{Φ} such that $0 \leqslant x_n \leqslant |x|$ $\mu-$a.e. in Ω, we have the implication: $(x_n \to 0 \ \mu -$ a.e. in $\Omega) \Rightarrow (\|x_n\|_{\Phi} \to 0)$. The set of all order continuous elements in L^{Φ} is denoted by $(L^{\Phi})_a$. If the measure space (Ω, Σ, μ) is non-atomic, then it is known that

$$(L^{\Phi})_a = E^{\Phi},$$

where $E^{\Phi} = \{x \in L^o \colon I_{\Phi}(\lambda x) < \infty \text{ for any } \lambda > 0\}$ and it is also known that $(L^{\Phi})_a = L^{\Phi}$ if and only if Φ satisfies condition $\Delta_2(\infty)$ whenever $\mu(\Omega) < \infty$ and condition $\Delta_2(\mathbb{R}_+)$ whenever $\mu(\Omega) = \infty$. Let us recall that $\Phi \in \Delta_2(\infty)$ (resp. $\Phi \in \Delta_2(\mathbb{R}_+)$) whenever there exists $K > 0$ such that $\Phi(u) \leqslant K\Phi(u)$ for $u \geqslant u_0$, where $u_0 > 0$ (resp. for all $u \geqslant 0$).

Theorem

If the measure space (Ω, Σ, μ) is non-atomic, then the following assertions are equivalent:

(a) $(L^{\Phi})_a = L^{\Phi}$,

(b) $\ell^{\infty} \not\hookrightarrow L^{\Phi}$ *order isometrically,*

(c) $B_{m,\Phi} = B_{\|.\|_{\Phi}}$,

(d) *the set* $B_{m,\Phi}$ *is open in the norm topology given by the norm* $\|.\|_{\Phi}$,

(e) *the generated Orlicz function* Φ *satisfies suitable* Δ_2-*condition, that is, condition* $\Delta_2(\infty)$ *if* $\mu(\Omega) < \infty$ *and condition* $\Delta_2(\mathbb{R}_+)$ *if* $\mu(\Omega) = \infty$.

Proof. In order to prove these equivalences let us start with proving the implication $(b) \Rightarrow (e)$. We will restrict ourselves to the case of the non-atomic finite measure (because the case of the non-atomic infinite measure is easier).

We need to prove that if μ is non-atomic and finite, and $\Phi \notin \Delta_2(\infty)$ then $(L^\Phi, \|.\|_\Phi)$ contains an order isometrically subspace of ℓ^∞. First, we will show that if $\Phi \notin \Delta_2(\infty)$, then for any $\varepsilon \in (0,1)$ there exists $x \in L^\Phi$ such that $I_\Phi(x) < \varepsilon$ and $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$. Let us note first that $\Phi \notin \Delta_{1+\frac{1}{n}}(\infty)$ for any $n \in \mathbb{N}$, whenever $\Phi \notin \Delta_2(\infty)$. This means that there exists a sequence $(u_n)_{n=1}^\infty$ such that

$$u_n < u_{n+1} \ \forall\, n \in \mathbb{N}, \qquad u_n \to \infty \text{ as } n \to \infty, \qquad \Phi(u_1)\mu(\Omega) \geqslant \varepsilon$$

and

$$\Phi\left(\left(1 + \frac{1}{n}\right) u_n\right) > 2^n \Phi(u_n).$$

Let us choose a sequence $(A_n)_{n=1}^{\infty}$ in Σ of pairwise disjoint sets such that

$$\Phi(u_n)\mu(A_n) = \frac{\varepsilon}{2^{n+1}}; \quad n = 1, 2, 3, \ldots .$$

Then, defining

$$x = \sum_{n=1}^{\infty} u_n \chi_{A_n},$$

we have

$$\begin{aligned} I_{\Phi}(x) &= \int_{\Omega} \Phi\left(x(t)\right) d\mu(t) = \sum_{n=1}^{\infty} \Phi(u_n)\mu(A_n) \\ &= \varepsilon \sum_{n=1}^{\infty} 2^{-n-1} = \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Moreover, given any $\lambda > 1$, by $1 + \frac{1}{n} \to 1$ as $n \to \infty$, one can find $m \in \mathbb{N}$ such that $1 + \frac{1}{n} \leqslant \lambda$ for any $n \geqslant m$, whence

$$\Phi(\lambda u_n) \geqslant \Phi\left(\left(1 + \frac{1}{n}\right) u_n\right) > 2^n \Phi(u_n), \ \forall\, n \geqslant m.$$

Therefore

$$
\begin{aligned}
I_\Phi(\lambda x) &= \sum_{n=1}^{\infty} \Phi(\lambda u_n)\mu(A_n) \geqslant \sum_{n=m}^{\infty} \Phi\left(\left(1+\frac{1}{n}\right)u_n\right)\mu(A_n) \\
&\geqslant \sum_{n=m}^{\infty} 2^n \Phi(u_n)\mu(A_n) \\
&= \frac{\varepsilon}{2}\sum_{n=m}^{\infty} 1 = \frac{\varepsilon}{2}\cdot\infty = \infty.
\end{aligned}
$$

Now, we start to build an order isometric copy of ℓ^∞ in L^Φ. Let, for any $n \in \mathbb{N}$, $B_n \in \Sigma$ be such that $\mu(B_n) = 2^{-n}\mu(\Omega)$ and the sets from this sequence are pairwise disjoint. Next, we can build the sequence of finite non-atomic measure spaces $(B_n, \Sigma \cap B_n, \mu|_{\Sigma \cap B_n})_{n=1}^{\infty}$. From the above construction we know that there exist a sequence $(x_n)_{n=1}^{\infty}$ in $L(\Omega, \Sigma, \mu)$ such that $x_n(t) = 0$ for any $t \in \Omega \backslash B_n$, $I_\Phi(x_n) < 2^{-n}$ and $I_\Phi(\lambda x_n) = \infty$ for any $\lambda > 1$.

Let us define the operator $\mathcal{I} : \ell^\infty \to L^\Phi$ by

$$\ell^\infty \ni c = (c_n) \to \mathcal{I}c = \sum_{n=1}^{\infty} c_n x_n \in L^\Phi, \ \forall c \in \ell^\infty.$$

Then we have

$$\begin{aligned} I_\Phi\left(\frac{\mathcal{I}c}{\|c\|_\infty}\right) &= \sum_{n=1}^{\infty} I_\Phi\left(\frac{|c_n|}{\|c\|_\infty} x_n\right) \leqslant \sum_{n=1}^{\infty} I_\Phi(x_n) \\ &\leqslant \sum_{n=1}^{\infty} 2^{-n} = 1, \end{aligned}$$

which shows that $\mathcal{I}c \in L^\Phi$ and that $\|\mathcal{I}c\|_\Phi \leqslant \|c\|_\infty$. On the other hand, given any $\lambda \in (0,1)$ one can find $m \in \mathbb{N}$ such that $\frac{|c_m|}{\lambda\|c\|_\infty} > 1$, whence

$$I_\Phi\left(\frac{\mathcal{I}c}{\lambda\|c\|_\infty}\right) = \sum_{n=1}^{\infty} I_\Phi\left(\frac{c_n x_n}{\lambda\|c\|_\infty}\right) \geqslant I_\Phi\left(\frac{|c_m|}{\lambda\|c\|_\infty} x_m\right) = \infty.$$

In consequence,

$$\left\| \frac{\mathcal{I}c}{\lambda \|c\|_\infty} \right\|_\Phi \geqslant 1,$$

that is, $\|\mathcal{I}c\|_\Phi \geqslant \lambda \|c\|_\infty$ and, by the arbitrariness of $\lambda \in (0,1)$,

$$\|\mathcal{I}c\|_\Phi \geqslant \|c\|_\infty.$$

In such a way we proved that $\mathcal{I}$ is an isometry. It is obvious that the operator $\mathcal{I}$ is linear and, since the functions x_n that were used to the construction of the operator $\mathcal{I}$ are non-negative, the operator $\mathcal{I}$ is also non-negative.

In consequence, if $c = (c_n), d = (d_n) \in \ell^\infty$ and $c \leqslant d$, that is, $c_n \leqslant d_n$ for any $n \in \mathbb{N}$, then $\mathcal{I}(d - c) \geqslant 0$, whence $\mathcal{I}(d) - \mathcal{I}(c) = \mathcal{I}(d - c) \geqslant 0$. We proved in such a way that the linear isometry $\mathcal{I}$ is an order isometry.

$(e) \Rightarrow (a)$

Now we will show that if $\Phi \in \Delta_2$, then $(L^\Phi)_a = L^\Phi$. Under the assumption $\Phi \in \Delta_2$ take any $x \in L^\Phi$ and any sequence $(x_n)_{n=1}^\infty$ in L^Φ such that $0 \leqslant x_n \leqslant |x|$ for $\mu-$a.e. $t \in \Omega$ and any $n \in \mathbb{N}$ as well as that $x_n \to 0$ $\mu-$a.e. in Ω. Let us take any $\lambda > 0$. Since $x \in L^\Phi$ and $\Phi \in \Delta_2$, we have that $I_\Phi(\lambda x) < \infty$. Moreover,

$$0 \leqslant \Phi(x_n(t)) \leqslant \Phi(x(t))$$

for $\mu-$a.e. $t \in \Omega$ and all $n \in \mathbb{N}$ and $\Phi(x_n(t)) \to 0$ as $n \to \infty$ for $\mu-$a.e. $t \in \Omega$. Therefore, by the Lebesgue dominated convergence theorem,

$$I_\Phi(\lambda x_n) \to 0 \text{ as } n \to \infty \; (\forall\, \lambda > 0),$$

which means that

$$\|x_n\|_\Phi \to 0 \text{ as } n \to \infty,$$

so the proof of the implication $(e) \Rightarrow (a)$ is finished.

Lozanovsky proved that any Banach lattice which is order continuous does not contain an isomorphic copy of ℓ^∞, so the implication $(a) \Rightarrow (b)$ is obvious.

Let us prove now that $(b) \Rightarrow (c)$. We know that $(b) \Rightarrow (e) \Rightarrow (a)$. Therefore, $I_\Phi(\lambda x) < \infty$ for any $\lambda > 0$ and any $x \in L^\Phi$ whenever (b) holds. Since $I_\Phi(x) \leqslant \|x\|_\Phi$ for any $x \in \overline{B_{\|.\|_\Phi}}$, so $B_{\|.\|_\Phi} \subseteq B_{m,\Phi}$. Let us take any $x \in B_{m,\Phi}$ and assume for the contrary that $x \notin B_{\|.\|_\Phi}$, that is, $\|x\|_\Phi = 1$. By $\Phi \in \Delta_2$, we know that $I_\Phi(2x) < \infty$. Let us define the function $f : [0,2] \to \mathbb{R}_+$ by

$$f(\lambda) := I_\Phi(\lambda x).$$

We have by the assumptions that $f(1) < 1$ and $f(2) < \infty$. Since f is convex on $[0,2]$, we know that f is continuous on $(0,2)$, whence we deduce that $f(1+\varepsilon) < 1$ for some $\varepsilon > 0$. This means that $I_\Phi((1+\varepsilon)x) < 1$, that is, $\|(1+\varepsilon)x\|_\Phi \leqslant 1$, whence $\|x\|_\Phi \leqslant \frac{1}{1+\varepsilon} < 1$, a contradiction, which shows that $x \in B_{\|.\|_\Phi}$, and the inclusion $B_{m,\Phi} \subseteq B_{\|.\|_\Phi}$ is proved.

Since the set $B_{\|.\|_\Phi}$ is open in the $\|.\|_\Phi-$topology, so the implication $(c) \Rightarrow (d)$ is obvious.

Let us prove now that $(d) \Rightarrow (e)$ or, equivalently, that $\neg(e) \Rightarrow \neg(d)$. So assume that $\Phi \notin \Delta_2$ and let

$$x = \sup_n x_n = \sum_{n=1}^{\infty} x_n,$$

where $(x_n)_{n=1}^{\infty}$ is the sequence constructed in the proof of the implication $(b) \Rightarrow (e)$. Then, we have $I_\Phi(x) = \sum_{n=1}^{\infty} I_\Phi(x_n) < \sum_{n=1}^{\infty} 2^{-n} = 1$. Since $I_\Phi(\lambda x_n) = \infty$ for any $n \in \mathbb{N}$ and $\lambda > 1$, we also have that $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$.

We claim that $x \in \operatorname{Int} B_{m,\|.\|_\Phi}$ in the norm topology generated by the norm $\|.\|_\Phi$, which means that there is $\varepsilon > 0$ such that $x + \varepsilon B_{\|.\|_\Phi} \subseteq B_{m,\Phi}$. Since $\|x\|_\Phi = 1$, so $\frac{\varepsilon}{2}x \in \varepsilon B_{\|.\|_\Phi}$ and by $x + \varepsilon B_{\|.\|_\Phi} \subseteq B_{m,\Phi}$ it must be $x + \frac{\varepsilon}{2}x \in B_{m,\Phi}$, that is, $I_\Phi\left(\left(1+\frac{\varepsilon}{2}\right)x\right) < 1$, which contradicts to the condition $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$.

Remark. Let us note that $x \in \mathrm{Int}(B_{m,\Phi})$ if and only if $I_\Phi(\lambda x) < \infty$ for some $\lambda > 1$.

Indeed. We have just proved that if $x \in B_{m,\Phi}$ and $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$, then $x \notin \mathrm{Int}(B_{m,\Phi})$. However, if $x \in B_{m,\Phi}$ and $I_\Phi(\lambda x) < \infty$ for some $\lambda > 1$, then $I_\Phi(\alpha x) < 1$ for some $\alpha > 1$.

Really, the function $f(\beta) = I_\Phi(\beta x)$ is convex and finite on the interval $[0, \lambda]$. So, it is continuous on the interval $(0, \lambda)$. Since $I_\Phi(x) < 1$, by continuity of f on the interval $(0, \lambda)$ with $\lambda > 1$, there exists $\alpha > 1$ such that $f(\alpha) = I_\Phi(\alpha x) < 1$. Denoting $\varepsilon = 1 - \frac{1}{\alpha}$ and assuming that $y \in x + \varepsilon B_{\|\cdot\|_\Phi}$, we have that there exists $z \in B_{\|\cdot\|_\Phi}$ such that

$$y = x + \varepsilon z = \frac{1}{\alpha}(\alpha x) + \left(1 - \frac{1}{\alpha}\right) z,$$

whence

$$I_\Phi(y) \leqslant \frac{1}{\alpha} I_\Phi(\alpha x) + \left(1 - \frac{1}{\alpha}\right) I_\Phi(z) < \frac{1}{\alpha} + \left(1 - \frac{1}{\alpha}\right) = 1.$$

This means that $y \in B_{m,\Phi}$. By the arbitrariness of $y \in x + \varepsilon B_{\|\cdot\|_\Phi}$, we proved that $x + \varepsilon B_{\|\cdot\|_\Phi} \subseteq B_{m,\Phi}$, which means that $x \in \mathrm{Int}(B_{m,\Phi})$.

Let us note that we proved the following sequence of implications:

$$(b) \Rightarrow (e) \Rightarrow (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e),$$

so also the subsequence of implications:

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a),$$

which finishes the proof of the equivalence of these conditions.

Thank you very much!