RESEARCH REPORT

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Abstract. The following report is an outline of author's thesis problems as well as his approach to them and results obtained so far. It has been prepared as a part of the comprehensive examination that the author is to take.

Contents

1. Introduction

1.1. **Introductory remarks.** The main purpose of this report is to present the main streams of the research that the author has pursued over the last two years and claim some of possible directions in which his work shall be continued. An outline of his doctoral dissertation is sketched with the emphasis on what is expected to be achieved before graduation. This section introduces the reader to areas of mathematical interest of the autor and explains some of the problems which make those fields worth investigating. The section is followed by a more detailed plan of the dissertation; a general notation used in the report is introduced. Chapters 2 - 6 provide a sketch of contents of four main chapters of the thesis. The exposition here is by no means complete nor self-contained and obviously the final version of the dissertation shall vary in many points, especially chapters 5 and 6 which, up to date, contain only a plan of the work for the next year and communicate some problems or difficulties that may occur.

The main area of the author's research is in the algebraic theory of quadratic forms and in the theory of spaces of orderings. The notion of spaces of orderings has been introduced by M. Marshall in the 1970's and provides an abstract framework for studying orderings on fields and the reduced theory of quadratic forms over fields. The latter one has been vigorously developing since the late 1960's, when first papers on the subject by Pfister, Brocker, Becker and Kopping ([14], [4], [2]) were published. Numerous monographs are devoted to the subject, [8] and [11] being of frequent use. A structure of a space of orderings (X, G) is completely determined by the group structure of G and the ternary relation $a \in D(b, c)$ on G; groups arising in this way are the reduced special groups (see [6] for a more general discussion on that subject; the notation used here shall be explained in details in next sections). We are interested in the elementary language of reduced special groups L_{SG} with $\cong$ as a relational symbol, $\cdot$ as a functional symbol, two constants 1 and -1, and usual logical symbols. We assume that the atomic formulae are of the form either $t_1 = t_2$ for terms t_1, t_2 or $(t_1, t_2) \cong (t_3, t_4)$ for terms $t_1, \ldots, t_4$. Using this language we can develop the theory of special groups, for which spaces of orderings serve as models. Traditionally we shall exchange the quaternary relation $\cong$ with the ternary one $a \in D(b, c)$, according to the following rule:

$$a \in D(b, c) \text{ iff. } (b, c) \cong (a, abc).$$

Hence the atomic formulae can be (modulo some of the axioms) exchanged with the ones of the form $1 \in D(a, b)$.

Our main interest is in positive-primitive (pp for short) formulae

$$\exists v_1, \ldots, v_n \psi(v_1, \ldots, v_n, C_1, \ldots, C_k),$$

where $\psi(v_1, \ldots, v_n, C_1, \ldots, C_k)$ is a finite conjunction of atoms

$$1 \in D(a \prod_{i=1}^{n} v_i^{\epsilon_i}, b \prod_{i=1}^{n} v_i^{\delta_i}),$$

for $\epsilon_i, \delta_i \in \{0, 1\}$, $v^0 = 1$, $v^1 = v$ and a, b being products of ± 1 and a finite number of C_i's. The significance of formulae of that type in the theory of quadratic forms has been pointed out in the fundamental paper [12]; also the main questions the author tries to answer in his work have been posed there. Examples of such formulae are "two forms are isometric", "an element is represented by a form", "a form is isotropic". The following problem is known as the pp conjecture:

Open Problem: Is it true that every pp formula which holds in every finite subspace of a space of orderings holds in the whole space?

In other words, the problem poses the question of the validity of a very general and highly abstract "local-global principle". The answer to the Open Problem is affirmative for all the examples of pp formulae mentioned above and many of its generalizations, with Extended Isotropy Theorem being the deepest result (see [11], [12]). It has been shown that the class of spaces to which the conjecture is true contains spaces of orderings of finite chain length, spaces of stability index 1 (which includes spaces of orderings of curves over real closed fields) and is closed under direct sum and group extension (see [12]). For a large class of pp formulae called product-free and 1-related and for any space having finite stability index it is also proved, that the answer to the Open Problem is "yes" ([13]). However, it has always seemed unlikely that the conjecture has a positive solution in general - no examples have been known, though.

Both the space of orderings of a rational function field in two variables over a real closed field and the space of orderings of the field $\mathbb{Q}(x)$ are of stability index 2 ([9], [10]). The former has very complicated real valuations, while the latter has well-understood ones and it has been shown that the answer to the Open Problem is affirmative for such space ([5]). This suggests looking at finite extensions of $\mathbb{Q}(x)$. This is being done in the recent paper by M. Marshall and the author ([7]), where spaces of orderings of function fields of conic sections over the field $\mathbb{Q}$ are investigated. First counterexamples to the pp conjecture are given and, moreover, all function fields of rational conic sections are classified with respect to the Open Problem.

Negative answer to the Open Problem raises new, very interesting questions. First of all, one can ask about the case of more complicated extensions of $\mathbb{Q}(x)$ - the first example in the row are function fields of elliptic curves. The author wishes to investigate that matter and has already made some progress, which shall be explained later in details. Secondly, the solution of pp conjecture for conics seems to provide an answer to another long-lasting question: is pp conjecture true for the space of orderings of the field $\mathbb{Q}(x, y)$? The answer is no, however the example constructed using conics and some recently discovered model-theoretic results ([1]) is non-constructive. The author wishes to find new, "elementary" proofs of the theorems stated in [1]. A significant progress towards achieving those goals has been already made - some partial results are discussed later in chapter 6, with theorem stating that the class of spaces of orderings satisfying the pp conjecture is closed under subspaces among them. The results published in the paper [1] should develop into new methods of testing whether a given pp formula holds true in a space of orderings; elementary and constructive proofs that the author wishes to find should provide us with better understanding of the nature of some pp conjecture-related problems.

1.2. **Content of the thesis.** The author wishes to organize his dissertation as follows:

Chapter 1: *Introduction.*

Chapter 2: *Spaces of orderings.*

- **Section 1:** *Axioms for spaces of orderings.* Axioms for spaces of orderings are introduced together with basic definitions used in the theory of spaces of orderings, regarding quadratic forms, discriminants, signatures, sets of elements represented by a form, isometries, Pfister forms and Witt rings. Pfister's Local-Global Principle is proved, Harrison topology is introduced and various concepts of isometry are discussed along with alternative sets of axioms. This section briefly outlines the material covered in sections 2.1 - 2.3 in [11].
- **Section 2:** *Subspaces of spaces of orderings.* The concept of a subspace of a space of orderings is explained and the structure and basic properties of subspaces are discussed. This part complies with the material from

section 2.4 in [11]. Next, relationships between valuations, valuation rings, convex sets and orderings are presented. This part is based on the material from chapter 2 in [8]. Finallly, the Baer-Krull construction of a subspace of space of orderings containing orderings compatible with a given valuation is given. This part refers to the material from chapter 7 in [15].

Section 3: *Language L_{SG} and axioms of special groups.* Axioms for (reduced) special groups are introduced and the relationship between the theory of (reduced) special groups and the theory of spaces of orderings is explained. Abstract logical and model-theoretical framework using the language L_{SG} is introduced. This section outlines the material covered in chapter 1 of [6].

Chapter 3: *The space of orderings of the field $\mathbb{Q}(X)$.*

Section 1: *Real spectra and specializations.* The notion of orderings of fields is extended to commutative rings: prime cones, residue fields, real spectra and specializations are defined. Basic constructions in the theory of real spectra are explained. Real spectra of coordinate rings in the algebraic geometry are described in details. This section covers the material contained in sections 7.1 - 7.2 in [3].

Section 2: *Spaces of orderings of function fields.* Fundamental geometric criterion for an element to be represented by a quadratic form in a space of orderings of a function field is proved. This part is an expanded version of section 3 of the paper [5].

Section 3: *Orderings of the field $\mathbb{Q}(X)$.* A detailed description of orderings of the field $\mathbb{Q}(X)$ is given. This is a direct application of the theorems introduced in previous sections; a shortened version of this result is given in the introduction to the paper [5].

Section 4: *pp conjecture for the field $\mathbb{Q}(X)$.* The pp conjecture is proved in the case of the space of orderings of the field $\mathbb{Q}(X)$. This part is an expanded version of section 4 of the paper [5].

Chapter 4: *Spaces of orderings of rational conics.*

Section 1: *Coordinate rings and function fields of conics.* Conic sections are classified with respect to the structure of their coordinate rings. This part is an expanded version of section 1 of the paper [7] by M. Marshall and the author.

Section 2: *Spaces of orderings of function fields of elliptic conics over $\mathbb{Q}$.* The fundamental theorem stating that for a space of orderings of a function field of an irreducible conic section of elliptic type without rational points the pp conjecture fails is proved. This part is an expanded version of section 2 of the paper [7] by M. Marshall and the author.

Section 3: *Spaces of orderings of function fields of two parallel lines over $\mathbb{Q}$.* The fundamental theorem stating that for a space of orderings of a function field of two parallel lines without rational points the pp conjecture fails is proved. This part is an expanded version of section 3 of the paper [7] by M. Marshall and the author.

Chapter 5: *Spaces of orderings of elliptic curves.*

Section 1: *Coordinate rings and function fields of elliptic curves.* Elliptic curves are to be classified with respect to the structure of their coordinate rings - at least partially. This problem seems to be far more complicated than the case of conic sections. This work is yet to be done. Some partial results are given later.

Section 2: *Spaces of orderings of function fields of elliptic curves.* Depending on the results to be achieved in the previous section, some more counterexamples to the pp conjecture are expected to be given. This work is yet to be done.

Chapter 6: *Spaces of orderings of function fields in many variables.*

Section 1: *Testing pp formulas on subspaces of bounded cardinality.* Elementary proofs of the results contained in section 2.2 of [1] are to be given. New methods of veryfying pp formulas in arbitrary spaces of orderings are to be proved. This work is yet to be done.

Section 2: *Spaces of orderings of function fields in many variables.* The fundamental theorem stating that for the space of orderings of the field $\mathbb{Q}(x, y)$ the pp conjecture fails is to be proved. Further corollaries are to be given. This work is yet to be done. Some partial results are given later.

1.3. **Notation.** We shall use the standard notation for the fields and rings $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ etc. For an integral domain D we shall always denote by (D) its field of fractions; for any set S containing 0 shall denote by S^* the set $S \setminus \{0\}$. If P is a ring and R is its extension, we shall usually use small letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots, \mathfrak{p}, \mathfrak{q}, \ldots$ to denote ideals in P and capital letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots, \mathfrak{P}, \mathfrak{Q}, \ldots$ for ideals in R. For an ideal $\mathfrak{a}$ of P we shall write $\mathfrak{a}^e$ for its extension in R and for an ideal $\mathfrak{A}$ in R we shall denote by $\mathfrak{A}^c$ its contraction in P. While dealing with an algebraic curve $\mathcal{C}$ over a field K, we shall denote by $\mathcal{I}(\mathcal{C})$ its ideal, by $K[\mathcal{C}]$ its coordinate ring and by $K(\mathcal{C})$ its field of rational functions

2. Spaces of orderings

2.1. **Axioms for spaces of orderings.** Let K be a field. Since we shall mostly deal with real fields, we may assume for simplicity that K is of characteristic $\neq 2$. We call a subset T of K a **preordering** if $T + T \subset T$, $TT \subset T$ and $K^2 \subset T$ and a subset P of K an **ordering** if $P + P \subset P$, $PP \subset P$, $P \cup -P = K$ and $P \cap -P = \{0\}$. Clearly every ordering is a preordering, but the converse is not true - however, using Zorn's Lemma, every preordering might be extended to an ordering. Moreover, a slightly more subtle result could be proved by means of Zorn's lemma, which is frequently used in many proofs:

Lemma 1. *If T is a preordering in a field K of characteristic $\neq 2$ and $a \in K \setminus T$, then there exists an ordering P of K extending T and such that $a \notin P$.*

Set of all orderings extending any given preordering T shall be denoted by X_T:

$$X_T = \{P \subset K : T \subset P, P \text{ is an ordering }\}.$$

The simplest example of a preordering is the set of all sums of squares, denoted by ΣK^2. Clearly every preordering contains the set ΣK^2, so that the set of all orderings is the same as the set of all orderings extending the preordering ΣK^2 - we shall denote it simply by X_K. For a given set $S \subset K$ there always exists the smallest preordering containing it, which we shall denote by $\Sigma K^2[S]$. The set of all orderings extending such preordering shall be denoted by X_S.

Fix a preordering T of K. Denote by G_T the quotient group K^*/T^* of the subgroup $T^* = T \setminus \{0\}$ of the multiplicative group K^*. One checks that G_T is naturally identified with a subgroup of the function group $\{-1,1\}^{X_T}$: the mapping $K^* \ni a \mapsto \overline{a} \in \{-1,1\}^{X_T}$, where

$$\overline{a}(P) = \begin{cases} 1, & \text{if } a \in P \\ -1, & \text{if } a \notin P \end{cases}$$

is easily verified to be a homomorphism, whose kernel is equal to T^*. Elements of G_T viewed as functions will be thus denoted by $\overline{a}$. In the simplest case when $T = \Sigma K^2$ we shall write G_K instead of $G_{\Sigma K^2}$; also, if $T = \Sigma K^2[S]$, we will write G_S for $G_{\Sigma K^2[S]}$.

An $n-$tuple $\phi = (\overline{a}_1, \dots, \overline{a}_n)$ of elements of G_T will be called a **quadratic form** of **dimension** $\dim \phi = n$. The product $\operatorname{disc} \phi = \prod_{i=1}^n \overline{a}_i \in G_T$ shall be called the **discriminant** of ϕ and the sum $\phi(P) = \sum_{i=1}^n \overline{a}_i(P) \in \mathbb{Z}$ the **signature** of ϕ at P. We say that an element $\overline{b} \in G_T$ is **represented** by the form ϕ, if for some $t_1, \dots, t_n \in T$ $b = \sum_{i=1}^n t_i a_i$. The set of all elements represented by ϕ shall be denoted by $D(\phi)$. It is a matter of routine verification that $D(\phi)$ has the following properties:

$$D(\overline{a}) = \{\overline{a}\} \text{ and } \overline{b} \in D(\overline{a}_1, \dots, \overline{a}_n) \Leftrightarrow \overline{b} \in D(\overline{a}_1, \overline{c}) \text{ for } \overline{c} \in D(\overline{a}_2, \dots, \overline{a}_n),$$

and also

$$D(\overline{a}_1, \overline{a}_2) = \{\overline{b} : \overline{b}(P) = \overline{a}_1(P) \vee \overline{b} = \overline{a}_2(P), P \in X_T\}.$$

Quite surprisingly, every element of $D(\phi)$ has a ,,transversal" representation; the proof of the following lemma uses a nice arithmetic trick.

Lemma 2. *$\overline{b} \in D(\overline{a}_1, \dots, \overline{a}_n)$ if and only if $b = \sum_{i=1}^n t_i^* a_i$, for $t_i^* \in T^*$.*

Proof. The ,,if" part is obvious; for the remaining part of the proof suppose that $b = t_1 a_1 + \dots + t_n a_n$ for some $t_i \in T$. We obviously have

$$\frac{a_1 + \dots + a_n}{b} = r^2 - s^2, \text{ where } r = \frac{\frac{a_1 + \dots + a_n}{b} + 1}{2}, \quad s = \frac{\frac{a_1 + \dots + a_n}{b} - 1}{2}.$$

It follows that $\frac{a_1 + \dots + a_n}{b} = (r^2 + 1) - (s^2 + 1)$ and hence

$$b(r^2+1) = a_1 + \dots + a_n + b(s^2+1) = a_1 + \dots + a_n + (t_1 a_1 + \dots + t_n a_n)(s^2+1),$$

which divided by $r^2 + 1$ gives the desired representation. $\square$

The above remarks lead us to the definition of a **space of orderings**. We shall define it as a pair (X, G), where X is a nonempty set and G is a subgroup of the function group $\{-1,1\}^X$ containing the constant function -1 and satisfying certain axioms. First of all, for $x, y \in X$:

$$x \neq y \Rightarrow \exists_{a \in G} a(x) \neq b(x).$$

X naturally embeds into the group $\chi(G)$ of characters of G via the monomorphism $X \ni x \mapsto \Phi_x \in \chi(G)$, where $G \ni a \overset{\Phi_x}{\mapsto} a(x) \in \{-1,1\}$. For $a, b \in G$ we also define the set $D(a,b) = \{c \in G : a(x) = c(x) \vee b(x) = c(x), x \in X\}$. With those remarks and notation we state the second axiom:

$$\forall_{x \in \chi(G)} [x(-1) = -1 \wedge \forall_{a,b \in G} (a, b \in \ker x \Rightarrow D(a,b) \subset \ker x)] \Rightarrow x \in X$$

and the third one:

$$\forall_{a_1, a_2, a_3, b \in G} b \in D(a_1, c) \text{ for } c \in D(a_2, a_3) \Rightarrow b \in D(d, a_3) \text{ for } d \in D(a_1, a_2).$$

Not surprisingly, the pair (X_T, G_T) for any given preordering T of the field K is a space of orderings, which could be easily checked. We develop the reduced theory of quadratic forms for spaces of orderings. An $n-$tuple $\phi = (a_1, \ldots, a_n)$ of elements of G shall be called a **quadratic form** of **dimension** $\dim \phi = n$. The product $\operatorname{disc} \phi = \prod_{i=1}^n a_i \in G$ shall be called its **discriminant** and the sum $\phi(x) = \sum_{i=1}^n a_i(x) \in \mathbb{Z}$ its **signature** at $x \in X$. The set of elements **represented** by a form is defined by induction; we have already defined it for binary forms, the full definition is as follows:

$$D(a) = \{a\}; \quad D(a,b) = \{c \in G : a(x) = c(x) \vee b(x) = c(x), x \in X\};$$
$$D(a_1, \ldots, a_n) = \bigcup_{b \in D(a_2, \ldots, a_n)} D(a_1, b).$$

We also define the addition and the multiplication of quadratic forms:

$$(a_1, \ldots, a_n) \oplus (b_1, \ldots, b_m) = (a_1, \ldots, a_n, b_1, \ldots, b_m),$$
$$(a_1, \ldots, a_n) \otimes (b_1, \ldots, b_m) = (a_1 b_1, \ldots, a_1 b_m, \ldots, a_n b_1, \ldots, a_n b_m),$$

and the multiplication of a form by a scalar:

$$c(a_1, \ldots, a_n) = (ca_1, \ldots, ca_n).$$

For simplicity we shall denote by $k \times \phi$ the sum $\underbrace{\phi \oplus \ldots \oplus \phi}_{k}$ and by $((a_1, \ldots, a_n))$ the n**-fold Pfister form** $(1, a_1) \otimes \ldots \otimes (1, a_n)$. The following statements are easily verified by induction:

Lemma 3. (1) *$D(a_1, \ldots, a_n)$ does not depend on the order of entries,*
(2) *$D(c\phi) = cD(\phi)$,*
(3) *$c \in D(\phi_1 \oplus \ldots \oplus \phi_n) \Leftrightarrow c \in D(a_1, \ldots, a_n)$, $a_i \in D(\phi_i)$,*
(4) *$D(a_1, \ldots, a_n)$ is the smallest* **additively closed** *set containing $a_1, \ldots, a_n$, i.e. a set with the following property:*

$$\forall_{a,b \in G} a, b \in M \Rightarrow D(a,b) \in M,$$

(5) *$D(k \times \phi) = D(\phi)$.*

Next, we define the **isometry** of quadratic forms. We proceed by recursion:

$$(a) \cong (b) \Leftrightarrow a = b; \quad (a_1, a_2) \cong (b_1, b_2) \Leftrightarrow \forall_{x \in X} a_1(x) + a_2(x) = b_1(x) + b_2(x);$$

$$(a_1, \ldots, a_n) \cong (b_1, \ldots, b_n) \Leftrightarrow \exists_{a,b,c_3,\ldots,c_n \in G} (a_2, \ldots, a_n) \cong (a, c_3, \ldots, c_n) \wedge$$
$$\wedge (a_1, a) \cong (b_1, b) \wedge (b_2, \ldots, b_n) \cong (b, c_3, \ldots, c_n).$$

We list a few properties of this relation - they all follow by induction, however sometimes proofs are quite lengthy and involve tedious computations, due to the „artificial" nature of the above definition (this is especially true while proving the assertion (4)).

Lemma 4. (1) *$b \in D(a_1, \ldots, a_n) \Leftrightarrow \exists_{b_2,\ldots,b_n} (b, b_2, \ldots, b_n) \cong (a_1, \ldots, a_n)$,*
(2) *$(a_1, \ldots, a_n) \cong (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ for every permutation $\sigma \in S(n)$,*
(3) *if $\phi \cong \psi$ then $\dim \phi = \dim \psi$, $\operatorname{disc} \phi = \operatorname{disc} \psi$, $D(\phi) = D(\psi)$, $\phi(x) = \psi(x)$ for all $x \in X$ and $c\phi \cong c\psi$ for all $c \in G$,*
(4) *$\cong$ is an equivalence relation,*
(5) *if $\phi \cong \phi'$ and $\psi \cong \psi'$, then $\phi \oplus \psi \cong \phi' \oplus \psi'$ and $\phi \otimes \psi \cong \phi' \otimes \psi'$,*
(6) *(Witt cancelation theorem) if $\phi \cong \phi'$ and $\phi \oplus \psi \cong \phi' \oplus \psi'$, then $\psi \cong \psi'$.*

We call a pair $(-a, a)$, $a \in G$, a **hyperbolic form** or a **hyperbolic plane**; clearly $(-a, a) \cong (-1, 1)$ for $a \in G$. A form ϕ is **isotropic** if for some form ψ we have $\phi \cong (-1, 1) \oplus \psi$; otherwise it is **anisotropic**. Also, we say that a form ϕ is **universal**, if $D(\phi) = G$. Finally, we are in position to state the central definition in the reduced theory of quadratic forms, the definition of the **Witt ring** of a space of orderings. Two forms ϕ and ψ are said to be **Witt equivalent** (denoted $\phi \sin \psi$) if they are isometric modulo some hyperbolic planes, i.e. if for some integers $k, l \in \mathbb{N}$ $\phi \oplus k \times (-1, 1) \cong \psi \oplus l \times (-1, 1)$. It is easily verified that Witt equivalence is indeed an equivalence relation and that we may introduce well-defined addition and multiplication in the set of all Witt equivalence classes of a given space of orderings by extending $\oplus$ and $\otimes$ to representatives of such classes. The set of all Witt equivalence classes with such operations forms a ring, which is called the Witt ring. The following lemma (whose proof is straightforward) explains relationships between those definitions:

Lemma 5. (1) *$\phi \cong \psi \Leftrightarrow \phi \sim \psi \wedge \dim \phi = \dim \psi$,*
(2) *ϕ is isotropic $\Leftrightarrow \phi \sim \psi \wedge \dim \phi > \dim \psi$ for some ψ,*
(3) *ϕ is isotropic $\Leftrightarrow \phi$ is universal $\Leftrightarrow D(\phi) \cap -D(\phi) \neq \emptyset$,*
(4) *ϕ is anisotropic $\Leftrightarrow n \times \phi$ is anisotropic for every integer $n \in \mathbb{N}$,*
(5) *if $\phi \oplus \psi$ is isotropic, then for some $b \in G$ $b \in D(\phi)$ and $-b \in D(\psi)$.*

To show some of the methods and techniques frequently used in the discussed theory, we shall prove the following important result.

Theorem 1 (Pfister's local-global principle)**.** *$\phi \sim \psi \Leftrightarrow \forall_{x \in X} \phi(x) = \psi(x)$.*

Proof. Once we show that $\phi \sim 0 \Leftrightarrow \forall_{x \in X} \phi(x) = 0$, we are done:

$$\begin{aligned} \phi \sim \psi \quad &\Leftrightarrow \quad \phi \oplus -\psi \sim 0 \Leftrightarrow \forall_{x \in X} (\phi \oplus -\psi)(x) = 0 \\ &\Leftrightarrow \quad \forall_{x \in X} \phi(x) - \psi(x) = 0 \Leftrightarrow \forall_{x \in X} \phi(x) = \psi(x). \end{aligned}$$

The implication $(\Rightarrow)$ is trivial: if $\phi \sim 0$, then:

$$\phi \oplus (-1, 1) \oplus \ldots \oplus (-1, 1) \cong (-1, 1) \oplus \ldots \oplus (-1, 1),$$

and, by Witt cancellation theorem, $\phi \cong (-1, 1) \oplus \ldots \oplus (-1, 1)$, so obviously $\phi(x) = 0$ for all $x \in X$.

For $(\Leftarrow)$ assume that for all $x \in X$ $\phi(x) = 0$ and suppose *a contrario* that $\phi \nsim 0$. Without loss of generality we might assume that ϕ is anisotropic; if $\phi \cong (-1, 1) \oplus \psi$ for some ψ, then clearly $\phi(x) = \psi(x)$ for all $x \in X$ and $\phi \sim \psi$, so using Witt cancelation theorem respectively many times we eventually arrive to an anisotropic form. Observe, that $\phi \oplus \phi \nsim 0$ - otherwise $\phi \oplus \phi$ would be isotropic, so $D(\phi) = D(\phi \oplus \phi) = G$ and ϕ would be isotropic. Therefore $(1, 1) \otimes \phi = \phi \oplus \phi \nsim 0$ and moreover $(1, 1) \otimes (1, 1) \otimes \phi = \phi \oplus \phi \oplus \phi \oplus \phi \nsim 0$, so that the multiplicative set:

$$\{\psi : \psi \otimes \phi \nsim 0\}$$

is nonempty and contains $(1, 1)$. By Zorn's lemma we may choose the a maximal such set S.

We claim that

$$\forall_{a \in G} (1, a) \in S \vee (1, -a) \in S.$$

Suppose then that for some $a \in G$ $(1, a) \notin S \wedge (1, -a) \notin S$. Since $(1, a) \otimes (1, a) = (1, a, a, 1) = (1, 1) \otimes (1, a) \in (1, a) \otimes S$, the smallest multiplicative set containing S and $(1, a)$ is of the form $S \cup S \otimes (1, a)$. Thus we may pick $\psi_1 \in S$ such that

$(1,a)\otimes\psi_1\otimes\phi\sim 0$ (by maximality of S). Similarly, we may choose $\psi_2\in S$ such that $(1,-a)\otimes\psi_2\otimes\phi\sim 0$. Then:

$$[(1,a)\oplus(1,-a)]\otimes\psi_1\otimes\psi_2\otimes\phi\sim 0.$$

But $(1,1)\oplus(a,-a)=(1,1)\in S$, so above contradicts the definition of S.

Define the function $\chi:G\to\{-1,1\}$ by:

$$\chi(a)=\begin{cases}1, & \text{if } (1,a)\in S,\\ -1, & \text{if } (1,-a)\in S.\end{cases}$$

This function is well-defined. We shall see that it is a character of G. Indeed, fix $a,b\in G$ and let, say, $\chi(a)=\chi(b)=1$, so that $(1,-a)\notin S$ and $(1,-b)\notin S$. As before we may pick $\psi_1,\psi_2\in S$ such that $(1,-a)\otimes\psi_1\otimes\phi\sim 0$ and $(1,-b)\otimes\psi_2\otimes\phi\sim 0$. Thus $\psi_1\otimes phi\sim a\psi_1\otimes\phi$ and $\psi_2\otimes\phi\sim b\psi_2\otimes\phi$, so we get:

$$\psi_1\otimes\psi_2\otimes\phi\sim b\psi_2\otimes a\psi_1\otimes\phi\sim ab\psi_1\otimes\psi_2\otimes\phi,$$

or, in other words, $(1,-ab)\otimes\psi_1\otimes\psi_2\otimes\phi\sim 0$. Thus $\chi(ab)=1$. For different values of $\chi(a),\chi(b)$ we proceed in a similar way.

Next we shall show that

$$\forall_{a,b\in G}a,b\in\ker\chi\Rightarrow D(a,b)\subset\ker\chi.$$

Fix $a,b\in\ker\chi$ and let $c\in D(a,b)$. Then $(1,a)\in S$, $(1,b)\in S$ and by comparison of signatures $(c,abc)\cong(a,b)$. This gives:

$$(1,-a)\otimes(1,-b)\cong(1,-a,-b,ab)\cong(1,-c,ab,-cab)\cong(1,-c)\otimes(1,ab).$$

As before we may choose $\psi_1,\psi_2\in G$ such that $(1,-a)\otimes\psi_1\otimes\phi\sim 0$ and $(1,-b)\otimes\psi_2\otimes\phi\sim 0$. Suppose that $(1,c)\notin S$. Then $(1,c)\in S$ and

$$0\sim(1,-a)\otimes(1,-b)\otimes\psi_1\otimes\psi_2\otimes\phi\sim(1,-c)\otimes(1,ab)\otimes\psi_1\otimes\psi_2\otimes\phi,$$

which yields a contradiction.

By the second axiom of the theory of spaces of orderings, χ can be identified with an element x of X such that for all $a\in G$ $\chi(a)=a(x)$. We shall see that $\phi(x)\neq 0$. Let $\phi=(a_1,\ldots,a_n)$ and define $e_i=\chi(a_i)$, $i\in\{1,\ldots,n\}$. Then clearly $(1,e_ia_i)\in S$ for all $i\in\{1,\ldots,n\}$ and since $a_i(1,e_ia_i)=(a_i,e_i)=e_i(e_ia_i,1)$ we have the following contradiction:

$$\begin{aligned}0 &\nsim (1,e_1a_1)\otimes(1,e_na_n)\otimes(a_1,\ldots,a_n)=\\ &= a_1(1,e_1a_1)\otimes\ldots\otimes(1,e_na_n)\oplus\ldots\oplus a_n(1,e_1a_1)\otimes\ldots\otimes(1,e_na_n)=\\ &= e_1(1,e_1a_1)\otimes\ldots\otimes(1,e_na_n)\oplus\ldots\oplus e_n(1,e_1a_1)\otimes\ldots\otimes(1,e_na_n)=\\ &= (1,e_1a_1)\otimes\ldots\otimes(1,e_na_n)\otimes(e_1,\ldots,e_n)\sim 0\end{aligned}$$

□

By the above theorem it follows immediately that

$$\phi\cong\psi\Leftrightarrow\dim\phi=\dim\psi\wedge\forall_{x\in X}\phi(x)=\psi(x).$$

This new characterization of isometry enables us to define a new, equivalent to the old, system of axioms for a space of orderings. Before we do that, we need to introduce a topology in the group of characters of G. We define it as the weakest topology such that the mappings $f_a:\chi(G)\to\{-1,1\}$, $a\in G$, given by

$$f_a(\chi)=\chi(a)$$

are continuous; as a topology in $\{-1,1\}$ we take the discrete topology. Therefore the subbasis of our topology in $\chi(G)$ consists of sets of the forms

$$U(a) = \{\chi : \chi(a) = 1\} \text{ and } V(a) = \{\chi : \chi(a) = -1\}.$$

Clearly if we embed the set X into $\chi(G)$ in the described before manner, this topology induces a topology on X with subbasis consisting of sets of the form

$$U(a) = \{x : a(x) = 1\} \text{ and } V(a) = \{x : a(x) = -1\};$$

since for every $x \in X$ $-1(x) = -1$, we have $V(a) = U(-a)$ and it suffices to restrict ourselves to the sets of the form $U(a)$, which turn to be clopen. The basis of our topology is thus formed by the clopen sets $U(a_1, \ldots, a_n) = \bigcap_{i=1}^n U(a_i)$. The topology described above is often referred to as a **Harrison topology**. With those introductory remarks we state the following theorem, whose proof is mainly based on Pfister's local-global pronciple:

Theorem 2. *A pair (X, G) is a space of orderings if and only if:*

(1) *X is a nonempty set and G is a subgroup of $\{-1,1\}^X$ containing the function constantly equal to -1 and such that*

$$\forall_{x,y\in X} x \neq y \Rightarrow \exists_{a\in G}(a(x) \neq a(y)),$$

(2) *the image of X in $\chi(G)$ under the embedding*

$$X \ni x \mapsto (G \ni a \mapsto a(x) \in \{-1,1\}) \in \chi(G)$$

is closed with respect to the described above topology,

(3) *if $(a_1, \ldots, a_n) \cong (b_1, \ldots, b_m)$ means*

$$n = m \wedge \forall_{x\in X} a_1(x) + \ldots + a_n(x) = b_1(x) + \ldots + b_m(x)$$

and

$$D(a_1, \ldots, a_n) = \{b \in G : \exists_{b_2,\ldots,b_n\in G}(b, b_2, \ldots, b_n) \cong (a_1, \ldots, a_n)\},$$

then

$$a \in D(\phi \oplus \psi) \Leftrightarrow a \in D(b, c) \textit{ for } b \in D(\phi), c \in D(\psi).$$

2.2. **Subspaces of spaces of orderings.** For a given space of orderings (X, G) we shall define a **subspace** as a pair $(Y, G|_Y)$ consisting of a subset $Y \subset X$ of the form $Y = \bigcap_{a\in S} U(a)$ for some subset $S \subset G$ and $G|_Y = \{a|_Y : a \in G\}$. We shall be mostly interested in subspaces of spaces of the form (X_K, G_K) for some field K. In this case it is not difficult to show that:

Lemma 6. *Y is a subspace of (X_K, G_K) iff $Y = X_T$ for a preordering T.*

Proof. It is evident that this condition is sufficent: just take $(T \setminus \{0\})/(\Sigma K^2)^*$ as the corresponding set S. To prove that the condition is also necessary, suppose that $Y \subset X_K$ is such that $Y = \bigcap_{\overline{a}\in S} U(\overline{a})$ for some $S \subset G_K$. Define $T = \Sigma K^2[\{a \in K : \overline{a} \in S\}]$; a routine verification proves that $Y = X_T$. □

It is not immediate to show that a subspace of a space of orderings is a space of orderings itself in general; however, in the case of subspaces of (X_K, G_K) it is pretty straightforward: we use the alternate axioms for spaces of orderings and the ,,transversal" representation while proving (3).

Our special interest is in subspaces associated with valuations. Recall that a **Krull valuation** of a field K with values in an ordered additive group G is a mapping $v : K \to G \cup \{\infty\}$ such that:

(1) $v(a) = \infty$ if and only if $a = 0$,
(2) $v(ab) = v(a) + v(b)$ for all $a, b \in K$,
(3) $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a, b \in K$.

With a given valuation v we relate an **associated valuation ring**

$$A_v = \{a \in K : v(a) \geq 0\},$$

which indeed happens to be a valuation ring and hence a local ring. Its only maximal ideal is of the form

$$M_v = \{a \in K : v(a) > 0\}$$

and its group of units of the form

$$U_v = \{a \in K : v(a) = 0\}.$$

Clearly the quotient ring A_v/M_v is a field, which shall be denoted by K_v and called a **residue field** of v.

Next, we say that a valuation v is **compatible** with the ordering P of K if

$$(a \in P \setminus \{0\} \wedge b - a \in P) \Leftrightarrow v(a) \geq v(b).$$

We also say that a symmetric set A (that is such that if $a \in A$ then also $-a \in A$) is **convex** with respect to an ordering P if

$$(a \in P \wedge b - a \in P \wedge b \in A) \Rightarrow a \in A.$$

It is easily checked that the following conditions are equivalent:

(1) P is compatible with v,
(2) A_v is convex with respect to P,
(3) M_v is convex with respect to P,
(4) $(a \in P \wedge a \in M_v) \Rightarrow 1 - a \in P \setminus \{0\}$.

Denote by X_v the set of all orderings compatible to a given valuation v. We aim to prove that this set gives rise to a subspace of (X_K, G_K).

First, we need to learn something more about relations between a valuation and an ordering compatible with it. Say $v : K \to G \cup \{\infty\}$ is a valuation and P is an ordering of K. Denote by P_v the set $\{a + M_v : a \in P\} \subset K_v$ and by A_P the set $\{a \in K : n - a \in P \wedge n + a \in P \text{ for some } n \geq 1\}$. It is easy to check that A_P is a valuation ring with the only maximal ideal

$$M_P = \{a \in K : \frac{1}{n} - a \in P \wedge \frac{1}{n} + a \in P \text{ for all } n \geq 1\}.$$

We have the following ,,square of dependencies":

$$\begin{array}{ccc} P_v \text{ is an ordering of } K_v & \Longrightarrow & A_v \supset A_P \\ \Uparrow & & \Downarrow \\ P \text{ is compatible with } v & \Longleftarrow & 1 + M_v \subset P \end{array}$$

Indeed, suppose that P_v is an ordering of K_v. Fix an $a \in A_P$ and let $n \in \mathbb{N}$ be such that $n - a, n + a \in P$. Suppose that $a \notin A_v$. But then $a^{-1} \in A_v$ and thus $a^{-1} \in M_v$ and hence $na^{-1} \in M_v$. Without loss of generality we might assume that $a^{-1} \in P$ (otherwise take $-a^{-1}$). Therefore $na^{-1} - 1, na^{-1} + 1 \in P$. This implies $-1 + M_v = na^{-1} - 1 + M_v \in P_v$ and $1 + M_v = na^{-1} + 1 + M_v \in P_v$, but since P_v is an ordering, $1 \in M_v$ - a contradiction.

Assume that $A_v \supset A_P$. We know that $M_P = A_P \backslash U(A_P)$, so if we fix an $a \in M_v$, this implies that $a^{-1} \notin A_v$, so $a^{-1} \notin A_P$. Then $a \in A_P$ and automatically $a \in M_P$, so we showed that $M_v \subset M_P$. In particular $1 + M_v \subset 1 + M_P$.

Now assume that $1+M_v \subset P$. Fix $a \in P$ such that $a \in M_v$. Then also $-a \in M_v$, so $1 - a \in 1 + M_v \subset P$.

Finally, let us assume that P is compatible with v. We easily check that P_v is closed under addition and multiplication, that contains all squares of K_v and that $P_v \cup -P_v = K_v$. To show that $P_v \cap -P_v = \{0\}$ assume that $a+M_v, -a+M_v \in P_v$. Then $a+M_v = a'+M_v$ and $-a+M_v = a''+M_v$ for $a, a'' \in P$. Thus $a'-a, a''+a \in M_v$. Without loss of generality we may assume that $a \in P$. Since $a''+a \in M_v$ and $a''+a \in P$, $(a''+a)-a = a'' \in P$, and because M_v is convex with respect to P, this implies $a \in M_v$.

Now we can easily prove the following:

Theorem 3. *$X_v = X_T$, where $T = \Sigma K^2[1 + M_v]$*

Proof. Let $P \in X_v$. Then $1 + M_v \subset P$ and obviously $\Sigma K^2 \subset P$, so $P \in X_T$. Conversely, if $P \in X_T$, then $1 + M_v \subset P$ and P is compatible with v. □

Along with Lemma 6 this shows that $(X_v, G_K|_{X_v})$ is a subspace of (X_K, G_K). For simplicity, we shall write G_v instead of $G_K|_{X_v}$. The structure of the space (X_K, G_K) is well-understood modulo the structure of valuations of K - the celebrated result by Baer and Krull gives complete classification of all possible kinds of orderings of K. We shall briefly outline main concepts of this theory.

For a given valuation $v : K \to G \cup \{\infty\}$ a **semisection** is a mapping $s : G \to K^*$ such that

(1) $s(0) = 1$,
(2) $v(s(g)) = g$,
(3) $\frac{s(g_1+g_2)}{s(g_1)s(g_2)} \in (K^*)^2$.

The following two lemmas are the core of Baer-Krull construction:

Lemma 7. *Let $v : K \to G \cup \{\infty\}$ be a nontrivial valuation whose residue field K_v is formally real, let $s : G \to K^*$ be a semisection for v. Then every ordering $P \in X_v$ induces a pair of mappings: a constant map $\phi_P : G/2G \to X_{K_v}$ and a character $\sigma_P : G/2G \to \{-1, 1\}$ by the following means:*

$$\forall_{g\in G} \sigma_P(g + 2G)s(g) \in P$$

and

$$\forall_{g\in G}\forall_{b\in U_v}(b + M_v \in \phi_P(g + 2G)) \Leftrightarrow bs(g)\sigma_P(g + 2G) \in P.$$

Lemma 8. *Let $v : K \to G \cup \{\infty\}$ be a nontrivial valuation whose residue field K_v is formally real, let $s : G \to K^*$ be a semisection for v. Then every pair of a constant function $\phi : G/2G \to X_{K_v}$ and a character $\sigma : G/2G \to \{-1, 1\}$ induces an ordering ϕ^σ defined by the following condition:*

$$\forall_{a\in K^*}(a \in \phi^\sigma) \Leftrightarrow (\frac{a}{s(v(a))}\sigma(v(a) + 2G) + M_v \in \phi(v(a) + 2G)).$$

The Baer-Krull theorem states that the above two lemmas establish a bijective correspondence between the set X_v and the cartesian product of constant maps from $G/2G$ to orderings of the residue field and characters of $G/2G$, that is $(\phi_P)^{\sigma_P} = P$ and $(\phi_{\phi^\sigma}, \sigma_{\phi^\sigma}) = (\phi, \sigma)$. We need to clasify all orderings of K. Generally speaking all orderings might be divided into two groups - Archimedean ones and non-Archimedean ones. An ordering P is said to be **Archimedean** if the following **Archimedean certanity** holds:

$$\forall_{a\in K}\exists_{n\in\mathbb{N}} n-a\in P.$$

It is well-known from the theory of real fields, that every formally real field with Archimedean ordering is order-embeddable into the field of real numbers and thus every Archimedean ordering comes from such embedding. On the other hand it is not hard to prove (using Baer-Krull theorem) that if an ordering is compatible with a valuation then it cannot be Archimedean. Thus we arrived to the following:

Theorem 4. *The space of orderings (X_K, G_K) is a union of the Archimedean orderings coming from embeddings of K info $\mathbb{R}$ and closed subspaces (X_v, G_v) of orderings compatible with nontrivial valuations whose residue fields are formally real. The structure of (X_v, G_v) is completely described by Baer-Krull theorem.*

2.3. **Language L_{SG} and axioms of special groups.** In order to make our discussion more formal and to present some ideas in more precise manner we shall build a first order language and a theory in such language which will serve as an abstract framework for studying the theory of spaces of orderings. The **language** L_{SG} consists of a quaternary relation symbol $\cong$ called isometry, a functional symbol $\cdot$ called multiplication and two constans -1 and 1. We use the usual set of logical symbols: $\neg, \rightarrow$, a set of individual variables V, the quantifier $\forall$ and the identity symbol $=$. We define terms T by induction as the smallest set containing individual variables and constans and closed under functional symbol. For terms $t_1, \ldots, t_4 \in T$ we define atomic formulas to be either of the form $t_1 = t_2$ or $(t_1, t_2) \cong (t_3, t_4)$.

In this language we build the **theory of special groups** as the set of following sentences:

(1) $\cdot$ is a group multiplication,
(2) $\forall_a a\cdot a = 1$,
(3) $\cong$ is an equivalence relation,
(4) $\forall_{a,b}(a, b) \cong (b, a)$,
(5) $\forall_a(a, -a) \cong (-1, 1)$,
(6) $\forall_{a,b,c,d}(a, b) \cong (c, d) \rightarrow a\cdot b = c\cdot d$,
(7) $\forall_{a,b,c,d}(a, b) \cong (c, d) \rightarrow (a, -c) \cong (-b, d)$,
(8) $\forall_{a,b,c,d}(a, b) \cong (c, d) \rightarrow \forall_x(x\cdot a, x\cdot b) \cong (x\cdot c, x\cdot d)$,
(9) $\forall_a(a, a) \cong (1, 1) \leftrightarrow a = 1$,
(10) denoting by:

$$\begin{aligned}&(a_1, a_2, a_3) \cong_3 (b_1, b_2, b_3) \Leftrightarrow \\ &\quad\Leftrightarrow \quad \exists_{a,b,c_3}(a_1, a) \cong (b_1, b) \wedge (a_2, a_3) \cong (a, c_3) \wedge (b_2, b_3) \cong (b, c_3),\end{aligned}$$

$\cong_3$ is transitive.

Since we are more used to the value set notation than to the isometry relation, we shall introduce the following abbreviation:

$$a \in D(b, c) \Leftrightarrow (b, c) \cong (a, abc).$$

In view of the above axioms it is easy to check that

$$(a, b) \cong (c, d) \Leftrightarrow ab = cd \wedge ac \in D(1, cd)$$

and thus we may interchange the quaternary relation $(a, b) \cong (c, d)$ with the ternary one $a \in D(b, c)$. This, together with some other axioms, enables us to use formulae $1 \in D(a, b)$ as atomic formulae.

Clearly any model of the theory of special groups shall be called a **special group**. Since the language of special groups differs from the language of groups, we shall denote special groups by $(G, \cong -1)$. An **SG-morphism** is a group homomorphism f between two special groups $(G, \cong, -1)$ and $(H, \cong, -1)$ such that $f(-1) = -1$ and

$$\forall_{a,b,c,d \in G}((a, b) \cong (c, d) \Rightarrow (f(a), f(b)) \cong (f(c), f(d))).$$

For a special group $(G, \cong, -1)$ denote by X_G the set of all SG-morphisms of G into $\mathbb{Z}_2$. It is not hard to show, using the alternate set of axioms of spaces of orderings, that (X_G, G) is a space of orderings. Moreover, for a space of orderings (X, G), $(G, \cong, -1)$ is a special group (with the usual meaning of $\cong$ and -1). To be more precise, the two correspondences:

$$(G, \cong, -1) \mapsto (X_G, G)$$

and

$$(X, G) \mapsto (G, \cong, -1)$$

are reciprocal to each other.

3. The space of orderings of the field $\mathbb{Q}(X)$

3.1. **Real spectra and specializations.** We shall slightly generalize the notion of orderings and spaces of orderings to the ring case. Suppose that A is a commutative ring with identity. A subset $\alpha \subset A$ is called a **prime cone** if $\alpha + \alpha \subset \alpha$, $\alpha \cdot \alpha \subset \alpha$, $A^2 \subset \alpha$, $-1 \notin \alpha$ and for $ab \in \alpha$ either $a \in \alpha$ or $-b \in \alpha$. The intersection $\alpha \cap -\alpha$ of ,,positive" and ,,negative" part of a prime cone shall be denoted by $\operatorname{supp} \alpha$ and called the **support** of α. It is easy to verify that a support of a prime cone is a real prime ideal (recall that an ideal I is called **real** if $a_1^2 + \ldots + a_n^2 \in I$ implies that $a_i \in I$, $i \in \{1, \ldots, n\}$). Thus the quotient ring $A/\operatorname{supp} \alpha$ is a domain and we can build its field of fractions, which shall be denoted by $K(\operatorname{supp} \alpha)$ and called the **residue field** of A at $\operatorname{supp} \alpha$. This field can be endowed with an ordering P defined as follows: first, we define which elements of $A/\operatorname{supp} \alpha$ are ,,positive":

$$a + \operatorname{supp} \alpha \in P \Leftrightarrow a \in \alpha$$

and then extend the ordering P to the ordering $\overline{P}$ of $K(\operatorname{supp} \alpha)$:

$$\frac{a + \operatorname{supp} \alpha}{b + \operatorname{supp} \alpha} \in \overline{P} \Leftrightarrow (a + \operatorname{supp} \alpha)(b + \operatorname{supp} \alpha) \in P.$$

Finally, we embed the field $K(\operatorname{supp} \alpha)$ into its real closure $K(\alpha)$. An element $a + \operatorname{supp} \alpha \in A/\operatorname{supp} \alpha \hookrightarrow K(\operatorname{supp} \alpha) \hookrightarrow K(\alpha)$ will be simply denoted by $a(\alpha)$ (regardles of whether it belongs to $A/\operatorname{supp} \alpha$ or $K(\operatorname{supp} \alpha)$ or $K(\alpha)$). To sum up, we have the following sequence of mappings and embeddings:

$$A \xrightarrow{\text{onto}} A/\operatorname{supp} \alpha \hookrightarrow K(\operatorname{supp} \alpha) = (A/\operatorname{supp} \alpha) \hookrightarrow K(\alpha).$$

The **real spectrum** of a ring A is just the set of all prime cones - we denote it by $\operatorname{Spec} A$. We introduce a Harrison topology in $\operatorname{Spec} A$ by base clopen sets

$$U(a_1, \ldots, a_n) = \{\alpha \in SpecA : a_1(\alpha) > 0, \ldots, a_n(\alpha) > 0\}$$

(the inequalities above are meant to be with respect to the unique ordering in $K(\alpha)$). Clearly in the case when $A = K$ is a field, $\operatorname{Spec} K = X_K$. Another important example of real spectra comes from algebraic geometry: if $V \subset \mathbb{R}^n$ is an algebraic set, define its associated ideal to be

$$\mathcal{I}(V) = \{f \in \mathbb{R}[X_1, \dots, X_n] : \forall_{x \in V} f(x) = 0\}$$

and consider the ring $\mathcal{P}(V) = \mathbb{R}[X_1, \dots, X_n]/\mathcal{I}(V)$ of polynomial functions on V. The following mapping turns out to be an injective homeomorphism (with respect to the euclidean topology of V and the Harrison topology of $\operatorname{Spec} \mathcal{P}(V)$):

$$V \ni x \mapsto \alpha_x = \{f \in \mathcal{P}(V) : f(x) \geq 0\} \in \operatorname{Spec} \mathcal{P}(V).$$

Finally, it is not hard to prove, that for two cones α and β is a spectrum $\operatorname{Spec} A$ the following four conditions are equivalent:

(1) $\alpha \subset \beta$,
(2) $a(\alpha) \geq 0 \Rightarrow a(\beta) \geq 0$,
(3) $a(\alpha) > 0 \Rightarrow a(\beta) > 0$,
(4) $\beta \in \operatorname{cl}(\{\alpha\})$

If any (and hence all) of the above conditions are satisfied, we say that β is a **specialization** of α.

3.2. **Spaces of orderings of function fields.** Let K be a formally real field, uniquely ordered. Let $\mathfrak{p}$ be a prime ideal of the ring $K[X_1, \dots, X_n]$ and consider the field $F = (K[X_1, \dots, X_n]/\mathfrak{p})$. We assume that F is formally real; it follows in particular, that the ideal $\mathfrak{p}$ is real. Let R be a real closure of K and let V denote the zero set of $\mathfrak{p}$ in R^n:

$$V = \{(a_1, \dots, a_n) \in R^n : f(a_1, \dots, a_n) = 0, f \in \mathfrak{p}\}.$$

We want to learn something about the nature of the space (X_F, G_F). To be more precise, we need to give a geometric meaning to the formula $\overline{f} \in D(\overline{g}, \overline{h})$ in a subspace (X_T, G_T) of (X_F, G_F), where T is a finitely generated preordering. In order to do that we first prove two important lemmas, which establish a ,,traingle of dependencies" that might be viewed as a generalization of Hilbert's 17th problem: for fixed $f, g_1, \dots, g_s \in F$

$$\begin{tikzcd}
\overline{f} \in D((\overline{g}_1, \dots, \overline{g}_s)) \arrow[r, Leftrightarrow] \arrow[dr, Leftrightarrow] & \forall_{P \in X_F}(g_1, \dots, g_s \in P \Rightarrow f \in P) \arrow[d, Leftrightarrow] \\
 & \forall_{W \text{ - irreducible component of } V \text{ of maximal dim}} \ \forall_{a \in \operatorname{reg} W} \begin{array}{l} (g_1(a) > 0, \dots, g_s(a) > 0 \\ \Rightarrow f(a) \geq 0) \end{array}
\end{tikzcd}$$

Lemma 9. $\overline{f} \in D((\overline{g}_1, \dots, \overline{g}_s)) \Leftrightarrow \forall_{P \in X_F}(g_1, \dots, g_s \in P \Rightarrow f \in P)$

Proof. If $\overline{f}$ is represented by the Pfister form $((\overline{g}_1, \dots, \overline{g}_s))$, we may just take a ,,transversal" representation:

$$f = \sigma_1 + \sigma_2 \sum g_i + \sigma_3 \sum g_i g_j + \ldots + \sigma_k \sum g_1 \cdot \ldots \cdot g_s, \quad \sigma_i \in \Sigma F^2$$

and observe that if $g_1, \dots, g_s \in P$, then also $f \in P$.

Conversely, suppose that for all $P \in X_F$ if $g_1, \dots, g_s \in P$ then also $f \in P$. Observe that

$$((\overline{g}_1, \dots, \overline{g}_s)) \cong \overline{f}((\overline{g}_1, \dots, \overline{g}_s));$$

indeed, both forms are of the same dimension and comparing signatures we see, that if all $g_1, \ldots, g_s \in P$ for some ordering P, then the signatures of both the left and the right hand sides are equal to 2^s. If for some $i \in \{1, \ldots, s\}$ $g_i \notin P$ for an ordering P, then both signatures are equal to 0. Now, since $1 \in D((\overline{g}_1, \ldots, \overline{g}_s))$, it is evident that $\overline{f} \in D((\overline{g}_1, \ldots, \overline{g}_s))$. $\square$

The second equivalence is much more difficult to prove and we need some preliminaries. We shall write $X = (X_1, \ldots, X_n)$ for simplicity.

Lemma 10. *Every $\alpha \in \operatorname{Spec} K[X]$ extends uniquely to $\beta \in \operatorname{Spec} R[X]$ such that* $\operatorname{supp} \alpha = K[X] \cap \operatorname{supp} \beta$.

Proof. In order to show that the desired cone exists, consider the sequence:

$$\begin{aligned} (K, Q) \quad &\xrightarrow{\iota_1} \quad (K[X], \alpha) \xrightarrow{\kappa} (K[X]/\operatorname{supp} \alpha, P) \xrightarrow{q} \\ &\xrightarrow{q} \quad ((K[X]/\operatorname{supp} \alpha), \overline{P}) \xrightarrow{r} (K(\alpha), \Sigma K(\alpha)^2), \end{aligned}$$

where Q is the unique ordering of K. It obviously gives a rise to the embedding:

$$K \overset{r \circ q \circ \kappa \circ \iota_1}{\hookrightarrow} K(\alpha)$$

and since Q is unique, we must have $Q = (r \circ q \circ \kappa \circ \iota_1)^{-1}(\Sigma K(\alpha)^2)$. On the other hand there exists an embedding $(K, Q) \overset{\iota_2}{\hookrightarrow} (R, \Sigma R^2)$, so there also exists an order-preserving K-embedding $\Phi : R \to K(\alpha)$. This embedding can be easily extended to the embedding $\tilde{\Phi} : R[X] \to K(\alpha)$ by setting:

$$\tilde{\Phi}(\Sigma a_{i_1 \ldots i_n} X_1^{i_1} \ldots X_n^{i_n}) = \Sigma \Phi(a_{i_1 \ldots i_n}) r \circ q \circ \kappa (X_1^{i_1} \ldots X_n^{i_n}).$$

Now we define β to be $\tilde{\Phi}^{-1}(\Sigma K(\alpha)^2)$ and without difficulty we verify that $\alpha = \beta \cap K[X]$.

Now suppose that we have two cones β_1 and β_2 such that $\operatorname{supp} \alpha = K[X] \cap \operatorname{supp} \beta_i$. Consider the following two diagrams, $i = \{1, 2\}$:

$$\begin{tikzcd} K[X] \arrow[r, "\kappa"] \arrow[dr, "\kappa_i"'] & K[X]/\operatorname{supp} \alpha \arrow[r, "\phi"] \arrow[d, "\Phi_i"'] & (K[X]/\operatorname{supp} \alpha) \arrow[d, "\tilde{\Phi}_i"'] \\ & (R[X]/\operatorname{supp} \beta_i, P_i) \arrow[r, "\phi_i"] & ((R[X]/\operatorname{supp} \beta_i), \tilde{P}_i) \arrow[d, hook, "r_i"'] \\ & & (R[X](\beta_i), \Sigma R[X](\beta_i)^2) \end{tikzcd}$$

In the above diagrams κ is onto and $\ker \kappa_i = \ker \kappa$, so there exists exactly one injective homomorphism $\Phi_i : K[X]/\operatorname{supp} \alpha \to R[X]/\operatorname{supp} \beta_i$ per each diagram. Those Φ_i's are easily extendable to $\tilde{\Phi}_i : (K[X]/\operatorname{supp} \alpha) \to (R[X]/\operatorname{supp} \beta_i)$ given by

$$\tilde{\Phi}_i(\frac{\overline{f}}{\overline{g}}) = \frac{\Phi_i(\overline{f})}{\Phi_i(\overline{g})}.$$

It is a matter of routine verification that $\tilde{\Phi}_i$'s are well-defined homomorphism, so they have to be embeddings. It is also easy to check that the extensions $(K[X]/\operatorname{supp} \alpha) \subset (R[X]/\operatorname{supp} \beta_i)$ are algebraic, so both $R[X](\beta_i)$ are algebraic closures of $(K[X]/\operatorname{supp} \alpha)$ and there is an order-preserving $(K[X]/\operatorname{supp} \alpha)$-isomorphism $\Psi : R[X](\beta_1) \to R[X](\beta_2)$ This, in turn, gives $\beta_1 = \beta_2$. $\square$

Now we are in position to prove the second fundamental lemma:

Lemma 11. $\forall_{P\in X_F}(g_1,\dots,g_s\in P\Rightarrow f\in P)$ *if and only if for every irreducible component* W *of* V *of maximal dimension and for every regular point* $a\in W$ $(g_1(a)>0,\dots,g_s(a)>0\Rightarrow f(a)\geq 0)$.

Proof. ($\Leftarrow$) Let $P\in X_F$ and define $\alpha\in\operatorname{Spec}K[X]$ by

$$f\in\alpha\Leftrightarrow f+\mathfrak{p}\in P,$$

where $\mathfrak{p}=\operatorname{supp}\alpha$. Let $\beta\in\operatorname{Spec}R[X]$ be the unique cone extending α such that $\operatorname{supp}\alpha=K[X]\cap\operatorname{supp}\beta$. Consider the set

$$W=\mathcal{Z}(\operatorname{supp}\beta)=\{(a_1,\dots,a_n)\in R^n: f(a_1,\dots,a_n)=0, f\in\operatorname{supp}\beta\}.$$

This is clearly an irreducible component of V. We shall see that W is of maximal dimension.

Irreducible sets of V are in bijective correspondence with prime ideals, so suppose that there exists a prime ideal $\mathfrak{q}\rhd R[X]$ such that $\mathcal{Z}(\mathfrak{q})\subset\mathcal{Z}(\operatorname{supp}\alpha)=V$. Then clearly $\mathfrak{q}\supset\operatorname{supp}\alpha$ and $\mathfrak{q}\cap K[X]\supset\operatorname{supp}\alpha$. Consider the diagram

$$\begin{tikzcd} K[X] \arrow[r, "\kappa", "onto"'] \arrow[dr, "\overline{\kappa}"'] & K[X]/\operatorname{supp}\alpha \arrow[d, dashed, "\Phi"] \\ & K[X]/\mathfrak{q}\cap K[X] \end{tikzcd}$$

Clearly $\ker\kappa=\mathfrak{q}\cap K[X]\supset\operatorname{supp}\alpha=\ker\kappa$ and $\overline{\kappa}$ is onto, so there exists exactly one epimorphism $\Phi: K[X]/\operatorname{supp}\alpha\to K[X]/\mathfrak{q}\cap K[X]$. Now, if $\mathfrak{p}_1\subsetneq\mathfrak{p}_2\subsetneq\dots\subsetneq\mathfrak{p}_k$ is a chain of prime ideals in $K[X]/\mathfrak{q}\cap K[X]$ of maximal length, then

$$\Phi^{-1}(\mathfrak{p}_1)\subset\Phi^{-1}(\mathfrak{p}_2)\subset\dots\subset\Phi^{-1}(\mathfrak{p})_k;$$

in fact all those inclusions are strong, that is, we shall replace $\subset$ with $\subsetneq$. This shows that $\dim K[X]/\operatorname{supp}\alpha\geq\dim K[X]/\mathfrak{q}\cap K[X]$. Using the alternate definition of the dimension we see, that since $(K[X]/\operatorname{supp}\alpha)\subset(R[X]/\operatorname{supp}\beta)$ is algebraic (which we have already observed in the proof of the previous lemma), $\dim K[X]/\operatorname{supp}\alpha=\dim R[X]/\operatorname{supp}\beta=\dim W$. Similarly, $(K[X]/\mathfrak{q}\cap K[X])\subset(R[X]/\mathfrak{q})$ is algebraic, so

$$\dim K[X]/\mathfrak{q}\cap K[X]=\dim R[X]/\mathfrak{q}=\dim\mathcal{Z}(\mathfrak{q}).$$

The statement ,,a is a regular point of W such that if $g_1(a)>0,\dots,g_s(a)>0$, then $f(a)\geq 0$" is expressible as the following formula in the field R:

$$f_i(a)=0\wedge o[\frac{\partial f_i}{\partial X_j}(a)]=n-\dim W\wedge g_i(a)>0\Rightarrow f(a)\geq 0.$$

By Tarski transfer principle, this implies that the following formula holds in the reald closure of the field $(R[X]/\operatorname{supp}\beta)$:

$$f_i(\beta)=0\wedge o[\frac{\partial f_i}{\partial X_j}(\beta)]=n-\dim W\wedge g_i(\beta)>0\Rightarrow f(\beta)\geq 0.$$

The first part of the formula, that is $f_i(\beta)=0\wedge o[\frac{\partial f_i}{\partial X_j}(\beta)]=n-\dim W$, is trivially satisfied; $g_i(\beta)>0$ obviously implies $g_i(\alpha)>0$, which in turns translates as $g_i\in P$ - similarly for $f(\beta)\geq 0$.

($\Rightarrow$) Let W be an irreducible component of V of maximal dimension and let $\mathfrak{q}=\mathcal{Z}(W)$. $\mathfrak{q}$ is a real prime ideal: since $\mathcal{Z}(\mathfrak{q})=\mathcal{Z}(\mathcal{I}(W))=W\subset V=\mathcal{Z}(\mathcal{I}(V))$, by the real nullstelensatz we have $\mathfrak{q} = \mathcal{I}(\mathcal{Z}(\mathfrak{q})) \supset \mathcal{I}(\mathcal{Z}(\mathcal{I}(V))) \supset \mathcal{I}(V) \supset \mathfrak{p}$. In particular $\mathfrak{q} \cap K[X] \supset \mathfrak{p}$ - we want to show that those two ideals are indeed equal. Observe that since $(K[X]/\mathfrak{p}) \subset (R[X]/\mathcal{I}(V))$ is algebraic, $\dim K[X]/\mathfrak{p} = \dim R[X]/\mathcal{I}(V) = \dim V$ - similarly $\dim K[X]/\mathfrak{q} \cap K[X] = \dim W$. Consider the diagram

$$\begin{tikzcd} K[X] \arrow[r, "\kappa", "onto"'] \arrow[dr, "\overline{\kappa}"'] & K[X]/\mathfrak{q} \arrow[d, dashed, "\Phi"] \\ & K[X]/\mathfrak{q} \cap K[X] \end{tikzcd}$$

in which $\ker \overline{\kappa} = \mathfrak{q} \cap K[X] \supset \mathfrak{p} = \ker \kappa$ and $\overline{\kappa}$ is onto, so that there exists a unique epimorphism $\Phi : K[X]/\mathfrak{p} \to K[X]/\mathfrak{q} \cap K[X]$. Suppose that Φ is not injective; then a chain of prime ideals of $K[X]/\mathfrak{q} \cap K[X]$ of maximal length would give a rise to a longer chain of prime ideals in $K[X]/\mathfrak{p}$, which is impossible. Thus Φ is one-to-one and that implies $\mathfrak{q} \cap K[X] = \mathfrak{p}$.

Now let $a \in W$ be a regular point. This point induces a prime cone $\overline{\beta}_a = \{f + \mathfrak{q} : f(a) \geq 0\} \in \operatorname{Spec}(R[X]/\mathfrak{q})$. $\overline{\beta}_a$ is a specialization of a prime cone, whose support is of dimension $\dim W$. As before we chheck that $\operatorname{supp} \overline{\beta}_a = \mathfrak{q}$ - and define $\beta_a \in \operatorname{Spec} R[X]$ by

$$f \in \beta_a \Leftrightarrow f + \mathfrak{q} \in \overline{\beta}_a.$$

Clearly $\operatorname{supp} \beta_a = \mathfrak{q}$. Finally, let $\alpha \in \operatorname{Spec} K[X]$ be the cone induced by β_a; we see that $\operatorname{supp} \alpha = \mathfrak{p}$. If we define $P \in X_F$ in a natural way as an extension of α, we can easily check that the assumption $g_1(a) > 0, \ldots, g_s(a) > 0$ translates into $g_1 + \mathfrak{p} \in P, \ldots, g_s + \mathfrak{p} \in P$, which, in turn, gives $f + \mathfrak{p} \in P$ and $f(a) \geq 0$. $\square$

As a special case when $s = 1$ we obtain a concrete geometric meaning of the atomic formula $\overline{f} \in D(1, \overline{g})$. By use of a very simple quadratic form-algebra this can be generalized as follows:

Theorem 5. *Let T be a preordering generated by $g_1, \ldots, g_s$, let $f, g, h \in F$. Then $\overline{f} \in D(\overline{g}, \overline{h})$ in the space (X_T, G_T) iff for every irreducible component of V of maximal dimension and for every regular point $a \in W$ if $g_1(a) > 0, \ldots, g_s(a) > 0$, then $f(a)g(a) \geq 0$ or $f(a)h(a) \geq 0$.*

3.3. **Orderings of the field $\mathbb{Q}(X)$.** We shall describe all orderings of the field $\mathbb{Q}(X)$, starting from the description of its valuations and then applying Baer-Krull theorem. The following result is well-known:

Theorem 6. *Let $\mathbb{P}$ be the set of all irreducible polynomials in the ring $\mathbb{Q}[X]$. For an arbitrary $\pi \in \mathbb{P}$ define the mapping $v_{X,\pi} : \mathbb{Q}(X) \to \mathbb{R} \cup \{\infty\}$ by:*

$$v_{X,\pi}(a) = \begin{cases} \infty & \text{if } a = 0, \\ n_\pi & \text{if } a = u \prod_{\rho \in \mathbb{P}} Q(X)^{n_\rho}, n_\rho \in \mathbb{Z}, u \in \mathrm{U}(\mathbb{Q}[X]). \end{cases}$$

Define also the function $v_{X,\infty} : \mathbb{Q}(X) \to \mathbb{R} \cup \{\infty\}$ by:

$$v_{X,\infty}(a) = \begin{cases} \infty, & \text{if } a = 0, \\ \deg g - \deg f, & \text{if } a = \frac{f(X)}{g(X)}, f, g \in \mathbb{Q}[X] \setminus \{0\}. \end{cases}$$

We set $\deg \infty = 1$.

(1) *The mapping*

$$\pi \mapsto v_{X,\pi}$$

establishes a bijection between the set $\mathbb{P} \cup \{\infty\}$ *and the set of all normalized exponential valuations of the field* $\mathbb{Q}(X)$ *such that* $v(a) = 0$ *for* $a \in \mathbb{Q}$*. In particular, every non-trivial exponential valuation in* $\mathbb{Q}(X)$ *is discrete.*

(2) $\bigcap_{\pi \in \mathbb{P} \cup \{\infty\}} A_{v_{X,\pi}} = \mathbb{Q}$.

(3) *For every* $\pi \in \mathbb{P} \cup \{\infty\}$ *the residue field* $\mathbb{Q}_{v_{X,\pi}}$ *of the valuation* $v_{X,\pi}$ *is a simple extension of the field* $\mathbb{Q}$ *(more precisely - a simple extension of an isomorphic image of the field* $\mathbb{Q}$*,* $\kappa_{v_{X,\pi}}(\mathbb{Q})$*, where* $\kappa_{v_{z,P}} : A_{v_{X,\pi}} \to \mathbb{Q}_{v_{X,\pi}}$ *is the cannonical epimorphism). Moreover* $[Q_{v_{X,\pi}} : \kappa_{v_{X,\pi}}(\mathbb{Q})] = \deg \pi$.

It is possible to show (by means of Krull intersection theorem) that all valuations in $\mathbb{Q}(X)$ constant on $\mathbb{Q}$ are exponential and thus the above theorem gives a complete description of all valuations that are of our interest. By the Baer-Krull theorem we know that orderings in $\mathbb{Q}(X)$ arise from the orderings in the residue fields associated with valuations on $\mathbb{Q}(X)$. Thhe above theorem states that the residue fields associated with valuations on $\mathbb{Q}(X)$ are just the algebraic number fields $\mathbb{Q}(\alpha)$. Next, by the Artin-Schreier theorem, $\mathbb{Q}(\alpha)$ is an ordered field if and only if $\mathbb{Q}(\alpha) \subset \mathbb{R}$, that is if $\alpha \in \mathbb{R}$. We shall describe orderings in $\mathbb{Q}(\alpha)$ in more details.

Orderings on $\mathbb{Q}(\alpha)$ are in bijective correspondence with $\mathbb{Q}$-embeddings of $\mathbb{Q}(\alpha)$ into $\mathbb{R}$. Moreover, it is not difficult to check, that embeddings of $\mathbb{Q}(\alpha)$ into $\mathbb{R}$ are in bijective correspondence with real roots of the minimal polynomial π of α. On the other hand, valuations that are of our interest are in bijective correspondence with irreducible polynomials in $\mathbb{Q}[X]$ and the element ∞. Let π be a monic, irreducible polynomial and let $\alpha_1, \dots, \alpha_m$ be all its real roots. Residue fields of $v_{X,\pi}$ are isomorphic to $\mathbb{Q}(\alpha_j)$, each of which has m orderings corresponding to various embeddings of $\mathbb{Q}(\alpha_j)$ into $\mathbb{R}$; to avoid considering too many isomorphic cases we shall simply assume that each $\mathbb{Q}(\alpha_j)$ has only one ordering, coming from the natural embedding into the reals. Next, the semisection $s : \mathbb{Z} \to \mathbb{Q}(X) \setminus \{\infty\}$ of $v_{X,\pi}$ is clearly given by $s(n) = \pi^n$ and there are two characters $\sigma : \mathbb{Z}/2\mathbb{Z} \to \{-1, 1\}$, one mapping -1 to -1 and the other to 1. Applying directly Baer-Krull theorem we obtain $2m$ orderings corresponding to polynomial π:

$$\phi_1^{\sigma_1}, \dots, \phi_m^{\sigma_1}, \phi_1^{\sigma_2}, \dots, \phi_m^{\sigma_2},$$

which, in terms of polynomials, are described as follows: if $g \in \mathbb{Q}[X]$ and $g = \pi^{v(g)} \cdot h$, then

$$g \in \phi_i^{\sigma_1} \Leftrightarrow (h(\alpha_i) \in \mathbb{R}^2 \wedge v(g) \text{ - even }) \vee (-h(\alpha_i) \in \mathbb{R}^2 \wedge v(g) \text{ - odd })$$

and

$$g \in \phi_i^{\sigma_1} \Leftrightarrow h(\alpha_i) \in \mathbb{R}^2.$$

Similarly, we can check that the remaining valuation $v_{X,\infty} : \mathbb{Q}(X) \to \mathbb{Z} \cup \{\infty\}$ be the remaining valuation of $\mathbb{Q}(X)$ given by:

$$v_{X,\infty}(g) = \begin{cases} \infty, & \text{if } g = 0, \\ \deg q - \deg p, & \text{if } g = \frac{p(X)}{q(X)}, p, q \in \mathbb{Q}[X] \setminus \{0\}. \end{cases}$$

induces two orderings

$$\phi^{\sigma_1} \text{ and } \phi^{\sigma_2}$$

defined, for a polynomial $g = a_n X^n + \ldots + a_0 \in \mathbb{Q}[X]$, as follows:

$$g \in \phi^{\sigma_1} \Leftrightarrow (a_n > 0 \wedge n \text{ - even }) \vee (a_n < 0 \wedge n \text{ - odd })$$

and

$$g \in \phi^{\sigma_2} \Leftrightarrow a_n > 0.$$

For our needs it will be much more convenient to use a slightly different description of those orderings. Namely, for real roots $\alpha_1, \ldots, \alpha_m$ of a monic, irreducible polynomial π, we define orderings α_i^+ and α_i^- by the rules

$$g \in \alpha_i^+ \Leftrightarrow \exists_{\epsilon>0} g > 0 \text{ on } (\alpha_i, \alpha_i + \epsilon)$$

and

$$g \in \alpha_i^- \Leftrightarrow \exists_{\epsilon>0} g > 0 \text{ on } (\alpha_i - \epsilon, \alpha_i),$$

for $g \in \mathbb{Q}[X]$ and $i \in \{1, \ldots, m\}$. Now, since π has no multiple roots, it is easy to check that

$$\{\alpha_i^+, \alpha_i^-\} = \{\phi_i^{\sigma_1}, \phi_i^{\sigma_2}\}.$$

We have thus provided the full description of the space of orderings $(X_{v_{X,\pi}}, G_{v_{X,\pi}})$ and $(X_{v_\infty}, G_{v_\infty})$. We shall denote this space by (X_π, G_π) and (X_∞, G_∞), respectively.

3.4. **pp-conjecture for the field $\mathbb{Q}(X)$.** Let $f, p \in \mathbb{Q}[X]$ be two non-zero, square-free polynomials, let S denote the set of monic irreducible divisors of f and p having at least one real root. For $\pi \in S$ we denote by $\alpha_1, \ldots, \alpha_m$ all real roots of π. Denote by g_π the mapping $g_\pi : G_{\mathbb{Q}(X)} \to G_\pi$ given by

$$g_\pi(\overline{f}) = \overline{f}|_{X_\pi}.$$

The following two conditions are equivalent:

(1) $1 \in D(\overline{f}, \overline{p})$ in the space $(X_{\mathbb{Q}(X)}, G_{\mathbb{Q}(X)})$,

(2) $1 \in D(g_\pi(\overline{f}), g_\pi(\overline{p}))$ in the space (X_π, G_π), for all $\pi \in S \cup \{\infty\}$.

The implication (1) $\Rightarrow$ (2) is obvious and for the other, by Theorem 5 it suffices to show that for all $x \in \mathbb{R}$ $f(x) \geq 0$ or $g(x) \geq 0$. Sort all real roots of all $\pi \in S$ in such a way that $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. Set $\alpha_0 = -\infty$ and $\alpha_{n+1} = +\infty$. Clearly $f(\alpha_i) = 0$ or $p(\alpha_i) = 0$. Fix $i \in \{1, \ldots, n\}$ and take π such that $\pi(\alpha_i) = 0$. By Theorem 5 $1 \in D(g_\pi(\overline{f}), g_\pi(\overline{p}))$ means that for every real root α of π $f \in \alpha^+$ or $p \in \alpha^+$ and $f \in \alpha^+$ or $p \in \alpha^-$. If one of f, p belongs to α_i^+, then it is positive on $(\alpha_i, \alpha_i + \epsilon)$ for some $\epsilon > 0$. Otherwise f or p is positive on (α_i, α_{i+1}). In case $i = 0$ we proceed in a similar manner.

The above equivalence enables us to prove the following fundamental result:

Theorem 7. *Let ϕ be a pp formula with parameters $\overline{c}_1, \ldots, \overline{c}_k$, where c_i's are sqyare-free:*

$$\phi = \exists_{\overline{v}_1, \ldots, \overline{v}_n} \psi(\overline{v}_1, \ldots, \overline{v}_n, \overline{c}_1, \ldots, \overline{c}_k),$$

where ψ is a finite conjunction of atomic formulae:

$$\overline{a} \cdot \prod_{i=1}^{n} \overline{v}_i^{\epsilon_i} = 1$$

or

$$1 \in D(\overline{a} \prod_{i=1}^{n} \overline{v}_i^{\epsilon_i}, \overline{b} \prod_{i=1}^{n} \overline{v}_i^{\mu_i}),$$

where $\epsilon_i, \mu_i \in \{0,1\}$, $\overline{v}^0 = 1$, $\overline{v}^1 = \overline{v}$ and $\overline{a}, \overline{b}$ are products of ± 1 and some of $\overline{c}_i$'s. If S denotes the set of all monic irreducible divisors of $c_1, \dots, c_k$ having at least one real root, then the following two conditions are equivalent:

(1) *ϕ holds in $(X_{\mathbb{Q}(X)}, G_{\mathbb{Q}(X)})$,*
(2) *ϕ holds in (X_π, G_π) for every $\pi \in S \cup \{\infty\}$.*

Proof. One implication is trivial and for the other denote by (Y, H) the subspace generated by all subspaces (X_π, G_π), $\pi \in S \cup \{\infty\}$. H is a quotient of G for and denote the appropriate surjective mapping by $f : G_{\mathbb{Q}(X)} \to H$. It follows that $H = \prod_{\pi \in S \cup \{\infty\}} G_\pi$ and that

$$H \models \exists_{\overline{v}_1, \dots, \overline{v}_n} \psi(\overline{v}_1, \dots, \overline{v}_n, f(\overline{c}_1), \dots, f(\overline{c}_k)).$$

Thus there exist square-free $t_1, \dots, t_n \in \mathbb{Q}[X] \setminus \{0\}$ such that

$$H \models \psi(f(\overline{t}_1), \dots, f(\overline{v}_n), f(\overline{c}_1), \dots, f(\overline{c}_k))$$

and it follows that

$$G_\pi \models \psi(g_\pi(\overline{t}_1), \dots, g_\pi(\overline{v}_n), g_\pi(\overline{c}_1), \dots, g_\pi(\overline{c}_k)).$$

Now decompose t_i as $t_i = t_{i0} \cdot t_{i1}$, where $t_{i0} = \prod\{\pi \in S : \pi | t_i\}$. Since π's are monic, t_i and t_{i0} have the same leading coefficient. If we order all real roots of all $\pi \in S$ so that $\alpha_1 < \alpha_2 < \ldots < \alpha_m$ and set $\alpha_0 = -\infty$, $\alpha_{m+1} = +\infty$, denote $I_l = (\alpha_l, \alpha_{l+1})$ and $\delta_{il}^- = \operatorname{sgn}(t_{il}(\alpha_l))$, $\delta_{il}^+ = \operatorname{sgn}(t_{il}(\alpha_{l+1}))$, then we see, that since no $\pi \in S$ divides t_{il}, t_{il} doesn't vanish on α_l and $\delta_{il}^-, \delta_{il}^+ \in \{\pm 1\}$. Let

$$p_{il} = \begin{cases} 0 & \text{if } \delta_{il}^- = \delta_{il}^+, \\ 1 & \text{if } \delta_{il}^- \neq \delta_{il}^+ \end{cases},$$

denote by τ_i the leading coefficient of t_i and pick any rational number $r_l \in I_l$. Finally, define

$$t'_{il} = \tau_i \prod_{l=0}^{m} (X - r_l)^{p_{il}} \text{ and } t'_i = t_{i0} \cdot t'_{il}.$$

We shall show that

$$G_{\mathbb{Q}(X)} \models \psi(\overline{t}'_1, \dots, \overline{t}'_n, \overline{c}_1, \dots, \overline{c}_k).$$

According to the remarks preceeding the theorem, it suffices to show that

$$G_\pi \models \psi(g_\pi(\overline{t}'_1), \dots, g_\pi(\overline{t}'_n), g_\pi(\overline{c}_1), \dots, g_\pi(\overline{c}_k)),$$

for all $\pi \in S \cup \{\infty\}$ together with π being monic irreducible divisors of t'_i's. First, consider the case when $\pi \in S \cup \{\infty\}$. If $\pi \in S$, then we shall first show that

$$\operatorname{sgn}(t_{i1}(\alpha_l)) = \operatorname{sgn}(t'_{i1}(\alpha_l)), \quad l \in \{0, \dots, m+1\}.$$

Since t_{i1} changes sign on I_l if and only if t'_{i1} does and $\operatorname{sgn}(t_{i1}(\alpha_{m+1})) = \operatorname{sgn}(t'_{i1}(\alpha_{m+1})) = \operatorname{sgn}(\tau_i)$ it follows by descending induction that our claim is true for $\alpha_m, \alpha_{m-1}, \dots, \alpha_0$. Now, given the fact that we know that $X_\pi = \{\alpha_+, \alpha_- : \alpha \text{ is a real root of } \pi\}$, we see that $1 \in D(\overline{a} \prod_{i=1}^n \overline{t}_i^\epsilon, \overline{b} \prod_{i=1}^n \overline{t}_i^{\mu_i})$ holds in (X_π, G_π) if and only if for every real root of π:

$$(a \prod_{i=1}^n t_i^{\epsilon_i} \in \alpha^+ \vee b \prod_{i=1}^n t_i^{\mu_i} \in \alpha^+) \wedge (a \prod_{i=1}^n t_i^{\epsilon_i} \in \alpha^- \vee b \prod_{i=1}^n t_i^{\mu_i} \in \alpha^-),$$

which we have just proved. If $\pi = \infty$, then the validation of our claim follows from the fact that (just proved) that t_{i1} and t'_{i1} have the same sign at $\pm infty$ and thus their degrees are congruent modulo 2.

If $\pi = X - r_l$, then t'_{i1} doesn't change sign on (α_l, r_l) and (r_l, α_{l+1}) and t_{i0}, c_j don't change sign on (α_l, α_{l+1}). Let $1 \in D(\overline{f}_1, \overline{f}_2)$ be an atomic formula appearing in ψ. f_1, f_2 don't change sign on (α_l, r_l) and (r_l, α_{l+1}), so if one of them is positive at α_{l+1}^+ then so is at r_l^- and if one of them is positive at α_{l+1}^- then so is at r_l^+. Since $X_\pi = \{r_l^-, r_l^+\}$, that completes the proof. □

The above theorem gives an affirmative solution to the pp conjecture in the case of the field $\mathbb{Q}(X)$.

4. Spaces of orderings of rational conics

4.1. **Coordinate rings and function fields of conics.** Our main goal in this paragraph is to show that either a function field of an irreducible conic section is a purely transcendental extension of a ground field of degree 1 or a coordinate ring of such curve is a PID (both possibilities may occur). We give elementary proofs of those facts.

We shall investigate all non-isomorphic classes of coordinate rings of conics. Let K be a fixed field, $\operatorname{char} K \neq 2$. The following lemma states a well-known fact:

Lemma 12. *Let $F \in K[X, Y]$ be a polynomial of degree 2. The curve*

$$\mathcal{C} : F(X, Y) = 0, \tag{1}$$

is affine isomorphic to a curve of parabolic type:

$$aX^2 + Y = 0, \qquad a \in K^*, \tag{2}$$

or to a curve of parallel type:

$$aX^2 + c = 0, \qquad a \in K^*, c \in K, \tag{3}$$

or to a curve of elliptic (hyperbolic) type:

$$aX^2 + bY^2 + c = 0, \qquad a, b \in K^*, c \in K. \tag{4}$$

Lemma 13. *If the irreducible curve (1) is affine isomorphic to the curve (2), then its function field $K(\mathcal{C})$ is a purely transcendental extension of K of degree 1.*

Proof. $\mathcal{C}$ is affine isomorphic to $\mathcal{C}' : aX^2 + 2Y = 0$, hence

$$K(\mathcal{C}) \cong (K[X, Y]/(aX^2 + Y)) = K(x, y),$$

where x, y are elements transcendental over K such that $ax^2 + y = 0$. Then $K(x, y) = K(x, ax^2) = K(x)$. □

Among all non-parabolic conics we shall distinguish between curves having K-rational points and curves without such points.

Lemma 14. *If the irreducible curve (1) has a K-rational point, then it is affine isomorphic to a curve of type either (2) or (4).*

Proof. Suppose that $\mathcal{C}$ is affine isomorphic to $\mathcal{C}' : aX^2 + c = 0$ and has a K-rational point. Thus $\mathcal{C}'$ has a K-rational point (q, r) and

$$aX^2 + c = aX^2 - aq^2 = a(X - q)(X + q),$$

so that $\mathcal{C}'$ is reducible and so is $\mathcal{C}$ - a contradiction. □

Next, consider the special case of the irreducible curve

$$aX^2 + bY^2 = 0, \qquad a, b \in K^*. \tag{5}$$

This curve is birationally isomorphis to the curve (3) without K-ational points. Indeed, the mapping $(X, Y) \mapsto (\frac{X}{Y}, 1)$ maps (5) onto $\mathcal{C}' : a(\frac{X}{Y})^2 + b = 0$. Clearly, if (p, q) is a K-rational point on $\mathcal{C}'$, then $b = -ap^2$ and $aX^2 + bY^2 = a(X - pY)(X + pY)$, so that (5) is reducible.

Lemma 15. *If the irreducible curve (1) is affine isomorphic to*

$$aX^2 + bY^2 + c = 0, \qquad a, b, c \in K^*. \tag{6}$$

and has a K-rational point, then $K(\mathcal{C}) \cong K(z)$ for a z transcendental over K.

Proof. By assumption $K(\mathcal{C}) \cong K(x, y)$, where $ax^2 + by^2 + c = 0$ and x, y are transcendental over K. Moreover $aq^2 + br^2 + c = 0$ for some $(q, r) \in K^2$. Thus $ax^2 - aq^2 = br^2 - by^2$. Let $z = \frac{x-q}{y-r}$. Hence $K(z) \subset K(x, y)$ - conversely, we have:

$$\begin{aligned} az(x+q) &= a\frac{x-q}{y-r}(x+q) = \frac{ax^2 - aq^2}{y-r} = \\ &= -\frac{by^2 - br^2}{y-r} = -b(y+r) \end{aligned}$$

and after rearranging:

$$azx + by = -azq - br. \tag{7}$$

On the other hand the equation $z = \frac{x-q}{y-r}$ gives:

$$x - zy = q - zr. \tag{8}$$

The determinant $-az^2 - b$ of the system of equations (7) and (8) is nonzero; if it was zero, then $a(x - q)^2 + b(y - r)^2 = 0$, which, since $ax^2 + by^2 + c = 0$ and $aq^2 + br^2 + c = 0$, would imply that $2c + 2axq + 2byr = 0$. Since $c \neq 0$, at least one of q and r is nonzero, so that we may express either x or y as a linear function of y or x, respectively. Thus (6) shows that either y or x is algebraic over K and, consequently, both x and y are algebraic over K - a contradiction. Therefore we may express both x and y as rational functions of z. $\square$

In view of the above lemmas we shall restrict our research to coordinate rings of type (3) and (6) conics without K-rational points: we aim to prove that such rings are PID. For a curve $\mathcal{C}$ affine isomorphic to (3) we have $K[\mathcal{C}] \cong K[x, y]$, where x and y are transcendental over K and $ax^2 + c = 0$, so that $K[\mathcal{C}] = K[\sqrt{-\frac{c}{a}}][y] = K(\sqrt{-\frac{c}{a}})[y]$ is a PID. The elliptic (hyperbolic) case is more complicated; we start with a lemma.

Lemma 16. *Let R be a PID, let $\Delta \in R$ be a square-free element and let $2 \in R^*$. Then $R[\sqrt{\Delta}]$ is a Dedekind domain.*

Proof. Since the integral closure of a Dedekind domain in a finite extension of its quotient field is Dedekind, it suffices to show that $R[\sqrt{\Delta}]$ is the integral closure of R. An element $\alpha + \beta\sqrt{\Delta} \in R[\sqrt{\Delta}]$ is a root of the polynomial $T^2 - 2\alpha T + \alpha^2 - \Delta\beta^2$ in $R[T]$ and hence is integral. Conversely, fix an integral element $g = \alpha + \beta\sqrt{\Delta} \in (R[\sqrt{\Delta}])$, $\alpha, \beta \in (R)$. Since the mapping:

$$(R[\sqrt{\Delta}]) \ni \varphi + \psi\sqrt{\Delta} = h \mapsto \overline{h} = \varphi - \psi\sqrt{\Delta} \in (R[\sqrt{\Delta}])$$

is an R-automorphism, $\overline{g}$ is integral and so is $g + \overline{g} = 2\alpha$. Since 2 is invertible in R, also α is integral, which means that $\alpha \in R$.

Now $\beta\sqrt{\Delta} = g - \alpha$ is integral and, consequently, $\beta^2\Delta$ is integral. But $\beta^2\Delta \in (R)$, implying that $\beta^2\Delta \in R$. Let $\varphi, \psi \in R$ be such that $\beta = \frac{\varphi}{\psi}$, $\gcd(\varphi, \psi) = 1$. Then $(\frac{\varphi}{\psi})^2\Delta = \eta$ for some $\eta \in R$, which gives $\varphi^2\Delta = \psi^2\eta$. But Δ is sqare-free and φ, ψ are coprime, so, since R is a PID, ψ^2 has to be a unit in R and, consequently, ψ is also a unit. Therefore $\beta = \frac{\varphi}{\psi} \in R$ proving that $g \in R[\sqrt{\Delta}]$. □

This lemma applies to the coordinate ring of the irreducible curve $\mathcal{C}$ of type (6) with no K-rational points: since $c \neq 0$, $K[\mathcal{C}] \cong R[\sqrt{\Delta}]$, where $R = K[x]$ and $\Delta = -\frac{c}{b} - \frac{a}{b}x^2$ is irreducible in R and hence square-free (if $\Delta = (x - q)(x - r)$, $q, r \in K$, then $(q, 0), (r, 0) \in K^2$ would be K-rational points on $\mathcal{C}$).

Theorem 8. *The coordinate ring $K[\mathcal{C}]$ of the irreducible curve (3) or (6) with no K-rational points is a PID.*

Proof. We have already discussed type (3) curves. Let $K[\mathcal{C}] \cong R[\sqrt{\Delta}]$, $\Delta = -\frac{c}{b} - \frac{a}{b}x^2$, where x is an element transcendental over K, be the coordinate ring of an irreducible curve (6) without K-rational points. For all prime $\rho \in R$, $\deg_x\rho > 1$ we shall show:

$$[\exists_{\alpha\in R}(\deg_x\alpha < \deg_x\rho) \wedge (\rho|\alpha^2 - \Delta)] \Rightarrow [\exists_{u\in K^*}\exists_{h\in R[\sqrt{\Delta}]}u\rho = h\overline{h}]$$

where $\overline{h}$ is the image of h under the conjugate authomorphism:

$$R[\sqrt{\Delta}] \ni \varphi + \psi\sqrt{\Delta} = h \mapsto \overline{h} = \varphi - \psi\sqrt{\Delta} \in R[\sqrt{\Delta}].$$

We proceed by induction on $\deg_x\rho$: if $\deg_x\rho = 1$, then for some $q \in K$ $\rho(q) = 0$ and hence $0 = \alpha^2(q) - \Delta(q) = \alpha^2(q) + \frac{c}{b} + \frac{a}{b}q^2$, contradicting the fact that $\mathcal{C}$ has no K-rational points.

If $\deg_x\rho = 2$, then $\deg_x\alpha \leq 1$ and since $\deg_x\Delta = 2$, $\deg_x(\alpha^2 - \Delta) \leq 2$. By assumption $\rho\gamma = \alpha^2 - \Delta$ for some $\gamma \in R$; since Δ is square-free, $\gamma \neq 0$. But $\deg_x\gamma = 0$ and thus $\gamma = u \in K^*$.

Assume that $\deg_x\rho > 2$ and for $\gamma, \alpha \in R$ such that $\deg_x\alpha < \deg_x\rho$ we have $\gamma\rho = \alpha^2 - \Delta$. If γ is constant, then, because Δ is square-free, $\gamma = u \in K^*$. Otherwise let $\gamma = \rho_1 \cdot \ldots \cdot \rho_s$, $\rho_1, \ldots, \rho_s \in R$, be the factorization of γ into irreducibles (and hence primes, as R is PID). Fix an arbitrary $i \in \{1, \ldots, s\}$. Since $\deg_x\alpha < \deg_x\rho$ it follows that $\deg_x\gamma < \deg_x\rho$ and, consequently, $\deg_x\rho_i < \deg_x\rho$. Clearly $\rho_i|\alpha^2 - \Delta$ so if β_i is the remainder of the division of α by ρ_i, then $\rho_i|\beta_i^2 - \Delta$ and $\deg_x\beta_i < \deg_x\rho_i$. By hypothesis $u_i\rho_i = h_i\overline{h_i}$ for $u_i \in K^*$, $h_i \in R[\sqrt{\Delta}]$, and hence $u\rho\prod_{i=1}^s h_i\overline{h_i} = h_0\overline{h_0}$ for some $u \in K^*$, where $h_0 = \alpha - \sqrt{\Delta}$.

For every $i \in \{1, \ldots, s\}$ (h_i) and $(\overline{h_i})$ are prime ideals of $R[\sqrt{\Delta}]$. Indeed, out of two possible decompositions of $(\rho_i)^e$ into prime ideals in $R[\sqrt{\Delta}]$:

$$(\rho_i)^e \text{ prime}, \ [R[\sqrt{\Delta}]/(\rho_i)^e : R/(\rho_i)] = 2,$$

$$(\rho_i)^e = \mathfrak{A}_1\mathfrak{A}_2, \ \mathfrak{A}_i \text{ prime}, \ [R[\sqrt{\Delta}]/\mathfrak{A}_i : R/(\rho_i)] = 1, \ \ i \in \{1, 2\},$$

the first one cannot occur: since $h_i\overline{h_i} \in (\rho_i)^e$, we may assume that $h_i \in (\rho_i)^e$, so that $1 + (\rho_i)^e$ and $\sqrt{\Delta} + (\rho_i)^e$ are linearly dependent over $R/(\rho_i)$. Thus $(h_i)(\overline{h_i}) = (\rho_i)^e = \mathfrak{A}_1\mathfrak{A}_2$ and $\{(h_i), (\overline{h_i})\} = \{\mathfrak{A}_1, \mathfrak{A}_2\}$.

Using the same arguments we can show that $(\rho)^e = \mathfrak{B}_1\mathfrak{B}_2$, $\mathfrak{B}_i$ prime, $[R[\sqrt{\Delta}]/\mathfrak{B}_i : R/(\rho_i)] = 1$, $i \in \{1,2\}$. Thus:

$$\mathfrak{B}_1\mathfrak{B}_2(h_1)\dots(h_s)(\overline{h_1})\dots(\overline{h_s}) = (h_0)(\overline{h_0}) = \mathfrak{P}_1\dots\mathfrak{P}_k\mathfrak{Q}_1\dots\mathfrak{Q}_l,$$

where $\mathfrak{P}_1\dots\mathfrak{P}_k$, $\mathfrak{Q}_1\dots\mathfrak{Q}_l$ are decompositions of (h_0), $(\overline{h_0})$ into prime ideals. By the uniqueness of such decomposition, either (h_i) or $(\overline{h_i})$ appears in the decomposition of (h_0) - we may assume it is (h_i), $i \in \{1,\dots,s\}$. Thus $(h_1\dots h_s)$ is a factor of (h_0) and for some $h \in R[\sqrt{\Delta}]$ $h_0 = h_1\dots h_s h$, so that $u\rho = h\overline{h}$.

It remains to show that every prime ideal $\mathfrak{P}$ of $R[\sqrt{\Delta}]$ is principal. Indeed, $\mathfrak{P}\cap R$ is prime and since R is a PID, $\mathfrak{P}\cap R = (\pi)$ for some prime $\pi \in R$. There are two possible decompositions of $(\pi)^e$ in $R[\sqrt{\Delta}]$:

$$(\pi)^e \text{ prime}, \quad [R[\sqrt{\Delta}]/(\pi)^e : R/(\pi)] = 2, \tag{9}$$

$$(\pi) = \mathfrak{P}_1\mathfrak{P}_2, \quad \mathfrak{P}_i \text{ prime}, \quad [R[\sqrt{\Delta}]/\mathfrak{P}_i : R/(\pi)] = 1, \tag{10}$$

for $i \in \{1,2\}$ ($\mathfrak{P}_1$ and $\mathfrak{P}_2$ might be equal). If (9) holds, then $\mathfrak{P} = (\pi)^e$, since $(\pi)^e \subset \mathfrak{P}$ and the Krull dimension $\dim R[\sqrt{\tilde{d}}] = 1$.

If (10) holds, then, since $(\pi)^e \subset \mathfrak{P}\cap\overline{\mathfrak{P}}$, where $\overline{\mathfrak{P}}$ denotes the image of $\mathfrak{P}$ under the conjugate automorphism and $\operatorname{rad}\mathfrak{P}_i = \mathfrak{P}_i$, $i \in \{1,2\}$, it follows that $\mathfrak{P}_1 \subset \mathfrak{P}$ and $\mathfrak{P}_2 \subset \overline{\mathfrak{P}}$ (we rearrange indices, if neccesary). An isomorphic image of a prime ideal is prime and $\dim R[\sqrt{\Delta}] = 1$, so $\mathfrak{P}_1 = \mathfrak{P}$ and $\mathfrak{P}_2 = \overline{\mathfrak{P}}$.

Because $[R[\sqrt{\Delta}]/\mathfrak{P} : R/(\pi)] = 1$ and $R/(\pi)$ is a field, there exists $\alpha \in R\setminus(\pi)$ such that $\alpha + \sqrt{\Delta} \in \mathfrak{P}$. Then $\alpha^2 - \Delta \in \mathfrak{P}\cap R = (\pi)$, that is $\pi|\alpha^2-\Delta$. Replacing α by the remainder of the division of α by π we may assume that $\deg_x \alpha < \deg_x \pi$. By previous remarks $u\pi = h\overline{h}$ for some $u \in K^*$, $h \in R[\sqrt{\Delta}]$. Thus $\mathfrak{P}\overline{\mathfrak{P}} = (\pi)^e = (h)(\overline{h})$ and h or $\overline{h}$ generate $\mathfrak{P}$. $\square$

4.2. **Spaces of orderings of function fields of elliptic conics over $\mathbb{Q}$.** We focus on curves over a fixed ordered field K and investigate some of their properties in the real closure $\tilde{K}$. In view of the previous remarks we can restrict our considerations to the curves of type (6). To avoid trivial cases we shall assume existence of some $\tilde{K}$-rational points. The curve (6) clearly satisfies either:

$$a > 0, \quad b > 0, \quad c < 0, \qquad \text{(elliptic type)}, \qquad \text{or} \tag{11}$$

$$a > 0, \quad b < 0, \quad c < 0, \qquad \text{(hyperbolic type)}. \tag{12}$$

Lemma 17. *The curve (6) of type (12) without K-rational points is birationally isomorphic to the curve (6) satisfying (11) with no K-rational points.*

Proof. The birational isomorphism between the curves (6) satisfying (12) and (6) satisfying (11) is given by $(X,Y) \mapsto (\frac{Y}{X}, \frac{1}{X})$ and maps $\mathcal{C} : aX^2 + bY^2 + c = 0$ onto $\mathcal{C}' : b(\frac{Y}{X})^2 + c(\frac{1}{X})^2 + a = 0$. Suppose that the resulting curve has a K-rational point (q,r). If $r \neq 0$ then $(\frac{1}{r}, \frac{q}{r})$ is a K-rational point on $\mathcal{C}$, which yields a contradiction. If $r = 0$, then we can parametrize K-rational points (q', r') on $\mathcal{C}'$ for which $r' \neq 0$ by lines with K-rational slopes passing through $(q,0)$; the set of such points is clearly nonempty, which again is a contradiction. $\square$

¿From now on let $K = \mathbb{Q}$. We shall therefore consider only the irreducible curves (6) without $\mathbb{Q}$-rational points for which (11) holds.

Lemma 18. *Let $\mathbb{P}$ denote the set of irreducibles of $\mathbb{Q}[\mathcal{C}]$, let $p \in \mathbb{P}$, $p = P + \mathcal{I}(\mathcal{C})$, $P \in \mathbb{Q}[X,Y]$, let $\xi^{(1)}, \ldots, \xi^{(m)}$ be all real points of intersection of $P(X,Y) = 0$ with $\mathcal{C}$. Each point $\xi^{(i)}$, $i \in \{1, \ldots, m\}$, induces two orderings $Q^+_{\xi^{(i)}}$ and $Q^-_{\xi^{(i)}}$ of the function field F of $\mathcal{C}$:*

$$g \in Q^+_{\xi^{(i)}} \Leftrightarrow (H(\xi^{(i)}) > 0 \wedge n \text{ even}) \vee (H(\xi^{(i)}) < 0 \wedge n \text{ odd}), \tag{13}$$

$$g \in Q^-_{\xi^{(i)}} \Leftrightarrow H(\xi^{(i)}) > 0, \tag{14}$$

for $g = p^n \cdot h \in \mathbb{Q}[\mathcal{C}]$, $h = H + \mathcal{I}(\mathcal{C})$, $H \in \mathbb{Q}[X,Y]$. Conversely, every ordering of F is of the form (13) or (14) for some $p \in \mathbb{P}$ and a real point of intersection $\xi^{(i)}$.

Proof. As before $\mathbb{Q}[\mathcal{C}] \cong R[\sqrt{\Delta}]$, with $R = \mathbb{Q}[x]$ and $\Delta = -\frac{c}{b} - \frac{a}{b}x^2$, x being transcendental over $\mathbb{Q}$. Since $R[\sqrt{\Delta}]$ is a PID, for $p \in \mathbb{P}$ the function $v_p : F \to \mathbb{Z} \cup \{\infty\}$ given by

$$v_p(g) = \begin{cases} \infty & \text{if } g = 0, \\ n_p & \text{if } g = u \cdot \prod_{q \in \mathbb{P}} q^{n_q}, n_q \in \mathbb{Z}, u \in R[\sqrt{\Delta}]^*, \end{cases} \tag{15}$$

is a discrete valuation such that:

$$M_{v_p} = (p) \text{ in } A_{v_p}, \quad (p) = M_{v_p} \cap R[\sqrt{\Delta}] \text{ in } R[\sqrt{\Delta}],$$

$$A_{v_p} = R[\sqrt{\Delta}]_{(p)} \supset R[\sqrt{\Delta}], \;\; F_{v_p} = A_{v_p}/M_{v_p} = R[\sqrt{\Delta}]/(p).$$

Conversely, every discrete valuation $v : F \to \mathbb{Z} \cup \{\infty\}$ such that $R[\sqrt{\Delta}] \subset A_v$ is of the form (15).

We are interested in valuations corresponding to orderings on F, that is the ones having formally real residue fields. For such $v : F \to \mathbb{Z} \cup \{\infty\}$ constant on $\mathbb{Q}$, $F = \mathbb{Q}(x,y)$, where x, y are transcendental over $\mathbb{Q}$ and $ax^2 + by^2 + c = 0$, we have $R[\sqrt{\Delta}] \subset A_v$; indeed, since (11) holds, $a = q_1^2 + \ldots + q_k^2$, $b = r_1^2 + \ldots + r_l^2$, $q_i, r_j \in \mathbb{Q}$ for $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, l\}$ and thus:

$$-c = ax^2 + by^2 = (q_1x)^2 + \ldots + (q_kx)^2 + (r_1y)^2 + \ldots + (r_ly)^2,$$

which follows that

$$\begin{aligned} 0 = v(-c) &= 2\min\{v(q_1x), \ldots, v(q_kx), v(r_1y), \ldots, v(r_ly)\} \\ &= 2\min\{v(x), v(y)\}, \end{aligned}$$

proving that $x, y \in A_v$, so $R[\sqrt{\Delta}] = \mathbb{Q}[x,y] \subset A_v$.

Every valuation ring $\mathbb{Q} \subset B \subset F$ is a PID and a valuation ring which is a PID is a discrete valuation ring, so every valuation in F constant on $\mathbb{Q}$ is of the form (15). A valuation in F with formally real residue field induces a valuation in $\mathbb{Q}$ compatible to some ordering of $\mathbb{Q}$. Every ordering compatible to a non-trivial valuation is non-archimedean, but the unique ordering of $\mathbb{Q}$ is archimedean, hence every valuation in F is trivial on $\mathbb{Q}$ and (15) describes all valuations of our interest.

For an arbitrary $p \in \mathbb{P}$ the extension $F_{v_p} = R[\sqrt{\Delta}]/(p) \supset \mathbb{Q}$ is finite. By the Artin-Schreier theorem, F_{v_p} is formally real if and only if $F_{v_p} \subset \mathbb{R}$. Orderings of F_{v_p} are in bijection with $\mathbb{Q}$-embeddings of F_{v_p} into $\mathbb{R}$: each embedding $\iota : F_{v_p} \hookrightarrow \mathbb{R}$ defines an ordering $Q_\iota = \iota^{-1}(\mathbb{R}^2)$. Furthermore, if $p = P + \mathcal{I}(\mathcal{C})$, $P \in \mathbb{Q}[X,Y]$, then $\mathbb{Q}$-embeddings of F_{v_p} into $\mathbb{R}$ are in bijective correspondence with real points of intersection of $P(X,Y) = 0$ with $\mathcal{C}$. Indeed, the mapping $\xi \mapsto \iota_\xi$, where

$\iota_\xi(G + (aX^2 + bY^2 + c)) = G(\xi)$, is the required bijection, for an arbitrary point of intersection ξ.

Instead of considering F_{v_p} with m orderings $Q_{\iota_{\xi^{(i)}}}$, $i \in \{1, \ldots, m\}$, we shall look at its isomorphic images $\iota_{\xi^{(i)}}(F_{v_p})$ with the ordering from $\mathbb{R}$. The semisection $s_p : \mathbb{Z} \to F_{v_p} \setminus \{0\}$ of v_p is given by $s_p(n) = p^n$ and the two characters of $\mathbb{Z}/2\mathbb{Z}$, one mapping $1 + 2\mathbb{Z}$ onto -1 and the other $1 + 2\mathbb{Z}$ onto 1, shall be denoted by σ_1, σ_2; hence all orderings compatible with v_p are $Q^+_{\xi^{(1)}}, \ldots, Q^+_{\xi^{(m)}}, Q^-_{\xi^{(1)}}, \ldots, Q^-_{\xi^{(m)}}$, where for $g = p^n \cdot h \in \mathbb{Q}[\mathcal{C}]$, $h = H + \mathcal{I}(\mathcal{C})$, $H \in \mathbb{Q}[X, Y]$:

$$\begin{aligned} g \in Q^+_{\xi^{(i)}} &\Leftrightarrow \frac{g}{s_p(n)}\sigma_1(n + 2\mathbb{Z}) + (p) \in Q_{\iota_{\xi^{(i)}}} \Leftrightarrow \\ &\Leftrightarrow \quad (H(\xi^{(i)}) > 0 \wedge n \text{ even}) \vee (H(\xi^{(i)}) < 0 \wedge n \text{ odd}). \end{aligned}$$

$Q^-_{\xi^{(i)}}$ is described analogously. □

Lemma 19. *Let $p = P + \mathcal{I}(\mathcal{C})$, $P \in \mathbb{Q}[X, Y]$, be prime in the coordinate ring $\mathbb{Q}[\mathcal{C}]$ of the irreducible curve (6) without $\mathbb{Q}$-points satisfying (11), let ξ be a real point of intersection of $P(X, Y) = 0$ with $\mathcal{C}$. Then P changes sign on $\mathcal{C}$ at ξ and P intersects $\mathcal{C}$ in an even number of points.*

Proof. Let $\xi = (\xi_1, \xi_2)$. For a fixed $\epsilon > 0$ and $\mathcal{C}$ viewed as a curve over the field $\mathbb{R}$ we shall show that

$$S^+_\epsilon = \{\zeta \in \mathcal{C} : U(\zeta) > 0, P(\zeta) > 0\} \neq \emptyset,$$

where $U(X, Y) = \epsilon^2 - (X - \xi_1)^2 - (Y - \xi_2)^2$. We identify $\mathcal{C}$ with its image under the embedding $\mathcal{C} \hookrightarrow \mathrm{Spec}\,(\mathbb{Q}[\mathcal{C}])$:

$$\zeta \mapsto Q_\zeta = \{G + (aX^2 + bY^2 + c) \in \mathbb{Q}[\mathcal{C}] : G(\zeta) > 0\}.$$

The unique constructible subset $\widehat{S^+_\epsilon}$ of the real spectrum $\mathrm{Spec}\,(\mathbb{Q}[\mathcal{C}])$ such that $\widehat{S^+_\epsilon} \cap \mathcal{C} = S^+_\epsilon$ is given by:

$$\widehat{S^+_\epsilon} = \{\wp \in \mathrm{Spec}\,(\mathbb{Q}[\mathcal{C}]) : u(\wp) > 0, p(\wp) > 0\},$$

where $a(\wp)$ denotes the image of a under the canonical embedding of $\mathbb{Q}[\mathcal{C}]$ into the real closure of $(\mathbb{Q}[\mathcal{C}]/\wp \cap -\wp)$ and $a(\wp) > 0$ means $a \in \wp$. Since $Q^+_\xi \in \widehat{S^+_\epsilon}$, we have that $\widehat{S^+_\epsilon} \neq \emptyset$ and thus $S^+_\epsilon = \widehat{S^+_\epsilon} \cap \mathcal{C} \neq \emptyset$.

Hence there are real points on $\mathcal{C}$ arbitrarily close to ξ where P is positive. Similarly there are points on $\mathcal{C}$ close to ξ where P is negative. Since P has only finitely many points of intersection with $\mathcal{C}$, P is positive on one side of ξ and negative on the other. □

Lemma 20. *Let $P(X, Y) = mX + nY + k \in \mathbb{Q}[X, Y]$, $m \neq 0$, $p = P + \mathcal{I}(\mathcal{C})$. Then p is prime in the coordinate ring $\mathbb{Q}[\mathcal{C}]$ of the irreducible curve (6) without $\mathbb{Q}$-rational points satisfying (11).*

Proof. Clearly $p = (mx + k) + n\sqrt{\Delta} \in R[\sqrt{\Delta}]$, where $R = \mathbb{Q}[x]$ and $\Delta = -\frac{c}{b} - \frac{a}{b}x^2$. Let $\overline{p}$ denote the image of p under the conjugate automorphism. Then $p \cdot \overline{p} = (mx + k)^2 - n^2\Delta \in R$ is irreducible; if it was reducible, then $p \cdot \overline{p}(q) = 0$ for some $q \in \mathbb{Q}$. Hence $\Delta(q) = (\frac{mq+k}{n})^2$ and $a(\frac{mx+k}{n})^2 + bq^2 + c = 0$, but $\mathcal{C}$ has no $\mathbb{Q}$-rational points. Thus p is irreducible: if $p = p_1 p_2$, then $p\overline{p} = p_1\overline{p_1} \cdot p_2\overline{p_2}$, $p_1\overline{p_1}, p_2\overline{p_2} \in R$. □

Now observe that for two real points $\xi^{(1)}, \xi^{(2)}$ on the irreducible curve (6) satisfying (11) without $\mathbb{Q}$-rational points we may pick rational points $q^{(1)}, q^{(2)} \in \mathbb{Q}^2$ lying arbitrarily close to $\xi^{(1)}, \xi^{(2)}$. If $P(X,Y) = 0$ is the line intersecting $q^{(1)}$ and $q^{(2)}$, then $P \in \mathbb{Q}[X,Y]$, $P(X,Y) = 0$ lies arbitrarily close to $\xi^{(1)}, \xi^{(2)}$ and $p = P + \mathcal{I}(\mathcal{C}) \in \mathbb{Q}[\mathcal{C}]$ is irreducible.

Next we shall state a general theorem concerning pp formulae, which is a slight modification of the main theorem proved in Section 3.

Theorem 9. *Let $\mathcal{C}$ be the irreducible curve (6) without $\mathbb{Q}$-rational points satisfying (11), let F denote its function field. For a given pp-formula*

$$\phi = \exists \overline{t_1} \dots \exists \overline{t_n} \psi(\overline{t_1}, \dots, \overline{t_n}, \overline{f_1}, \dots, \overline{f_k}),$$

$f_1, \dots, f_k \in \mathbb{Q}[\mathcal{C}]$, *$\overline{f_i} \in G_{\Sigma F^2}$ being images of $f_i \cdot \Sigma F^2$ under the embedding $G_{\Sigma F^2} \ni a \cdot \Sigma F^2 \mapsto \overline{a} \in \{-1,1\}^{X_F}$, let Σ denote the set of all irreducible factors of f_i, $i \in \{1, \dots, k\}$. The following conditions are equivalent:*

(1) *$G_p \models \phi_p$, for every $p \in \Sigma$.*
(2) *$G_{\Sigma F^2[S]} \models \phi_S$, for every proper subspace $(X_{\Sigma F^2[S]}, G_{\Sigma F^2[S]})$.*
(3) *$G_{\Sigma F^2[S]} \models \phi_S$, for every finite subspace $(X_{\Sigma F^2[S]}, G_{\Sigma F^2[S]})$.*

Proof. The only nontrivial part is (1) $\Rightarrow$ (2). Let $(X_{\Sigma F^2[S]}, G_{\Sigma F^2[S]})$ be a proper subspace for some $S \subset F$, let $0 \neq d \in \Sigma F^2[S] \setminus \Sigma F^2$ - we may assume that $d \in \mathbb{Q}[\mathcal{C}]$, $d = D + \mathcal{I}(\mathcal{C})$. Clearly $X_{\Sigma F^2[S]} \subset X_{\Sigma F^2[d]}$. There exists $Q \in X_F$ such that $d \notin Q$, so $X_{\Sigma F^2[d]}$ is proper and it suffices to show that $G_{\Sigma F^2[d]} \models \phi_{\{d\}}$. It is equivalent to show that there exist $t_1', \dots, t_n' \in \mathbb{Q}[\mathcal{C}] \setminus \{0\}$ such that for each atom $1 \in D(\overline{g} \prod_{i=1}^n \overline{t_i}^{\epsilon_i}, \overline{h} \prod_{i=1}^n \overline{t_i}^{\delta_i})$ of the formula ϕ, $\epsilon_i, \delta_i \in \{0,1\}$, $\overline{t_i}^0 = 0$, $\overline{t_i}^1 = \overline{t_i}$, $i \in \{1, \dots, n\}$ and $\overline{g}, \overline{h}$ being products of ± 1 and a finite number of $\overline{f_i}$:

$$1 \in D_{G_{\Sigma F^2[d]}}(\tau_{\{d\}}(\overline{g}) \prod_{i=1}^n \overline{t_i'}^{\epsilon_i}, \tau_{\{d\}}(\overline{h}) \prod_{i=1}^n \overline{t_i'}^{\delta_i})$$

holds true, where $\tau_{\{d\}} : G_{\Sigma F^2} \to G_{\Sigma F^2[d]}$ is given by $\tau_{\{d\}}(\overline{g}) = \overline{g}|_{X_{\Sigma F^2[d]}}$ and $\overline{t_i'}$ is the image of $t_i' \cdot \Sigma F^2[d]$ under the embedding:

$$G_{\Sigma F^2[d]} \ni a \cdot \Sigma F^2[d] \mapsto \overline{a} \in \{-1,1\}^{X_{\Sigma F^2[d]}}.$$

Let $\overline{g}$ and $\overline{h}$ be images of $g \cdot \Sigma F^2[d]$ and $h \cdot \Sigma F^2[d]$ under the above embedding, let $g = G + \mathcal{I}(\mathcal{C})$, $h = H + \mathcal{I}(\mathcal{C})$, $G, H \in \mathbb{Q}[X,Y]$. It suffices to show that there exist $T_1', \dots, T_n' \in \mathbb{Q}[X,Y] \setminus \{0\}$ such that for each atom $1 \in D(\overline{g} \prod_{i=1}^n \overline{t_i}^{\epsilon_i}, \overline{h} \prod_{i=1}^n \overline{t_i}^{\delta_i})$ of the formula ϕ and for each point $\zeta \in \mathcal{C}$ such that $D(\zeta) > 0$:

$$G \prod_{i=1}^n T_i'^{\epsilon_i}(\zeta) \geq 0 \text{ or } H \prod_{i=1}^n T_i'^{\delta_i}(\zeta) \geq 0.$$

Since $d \notin Q$, there exists $\wp \in \operatorname{Spec}(\mathbb{Q}[\mathcal{C}])$ such that $d(\wp) < 0$. Thus:

$$\widehat{S^-} = \{\wp \in \operatorname{Spec}(\mathbb{Q}[\mathcal{C}]) : d(\wp) < 0\} \neq \emptyset$$

implying that $S^- = \{\zeta \in \mathcal{C} : D(\zeta) < 0\} \neq \emptyset$. Since $D \in \mathbb{Q}[X,Y]$ is continuous, $D^{-1}((-\infty, 0))$ is open and hence S^- is open in the topology of $\mathcal{C}$ inherited from $\mathbb{R}^2$. Every component J of S^- is an open arc; replacing it by a smaller arc, if necessary, we may assume that J does not contain any points of intersection of $P(X,Y) = 0$ with $\mathcal{C}$, $p = P + \mathcal{I}(\mathcal{C})$, $P \in \mathfrak{Q}[X,Y]$, $p \in \Sigma$. Now it suffices to show that for any such arc J there exist $T_1' \dots T_n' \in \mathbb{Q}[X,Y] \setminus \{0\}$ such that for each atom $1 \in D(\overline{g} \prod_{i=1}^n \overline{t_i}^{\epsilon_i}, \overline{h} \prod_{i=1}^n \overline{t_i}^{\delta_i})$ and for $\zeta \in \mathcal{C} \setminus J$:

$$G \prod_{i=1}^n T_i'^{\epsilon_i}(\zeta) \geq 0 \text{ or } H \prod_{i=1}^n T_i'^{\delta_i}(\zeta) \geq 0.$$

Fix an arc J as required. The points of intersection divide $\mathcal{C}$ into disjoint arcs, exactly one of them containing J. Let $\mathcal{A} = \{I_1, \dots, I_m\}$ be the set of remaining arcs. For $I_j \in \mathcal{A}$ let $p_{I_j} \in \mathbb{Q}[\mathcal{C}]$ be a linear irreducible intersecting $\mathcal{C}$ in two points: one lying in I_j and the other in J.

Let (Y, H) be the subspace of $(X_F, G_{\Sigma F^2})$ generated by the subspaces $(X_F^{v_p}, G_p)$ for $p \in \Sigma$. By duality H is a quotient of G, with some surjective $\tau : G_{\Sigma F^2} \to H$ being a quotient map, and by the approximation theorem for independent valuations, (Y, H) is a direct sum of the $(X_F^{v_p}, G_p)$, $p \in \Sigma$. By our assumptions and by Łoś theorem for ultraproducts:

$$H \models \exists \overline{t_1} \dots \exists \overline{t_n} \psi(\overline{t_1}, \dots, \overline{t_n}, \tau(\overline{f_1}), \dots, \tau(\overline{f_k})).$$

Thus there exist square-free $t_1, \dots, t_n \in \mathbb{Q}[\mathcal{C}]$ such that

$$H \models \psi(\tau(\overline{t_1}), \dots, \tau(\overline{t_n}), \tau(\overline{f_1}), \dots, \tau(\overline{f_k}))$$

and - by Łoś theorem - for all $p \in \Sigma$:

$$G_p \models \psi(\tau_p(\overline{t_1}), \dots, \tau_p(\overline{t_n}), \tau_p(\overline{f_1}), \dots, \tau_p(\overline{f_k})),$$

where $\tau_p : G_{\Sigma F^2} \to G_p$ is given by $\tau_p(\overline{g}) = \overline{g}|_{X_F^{v_p}}$. Factor each t_i as $t_i = t_{i0} \cdot t_{i1}$, where t_{i0} is the product of those $p \in \Sigma$ which divide t_i, $i \in \{1, \dots, n\}$. Fix $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$ and denote by $\overline{\overline{(\zeta^{(j)}; \zeta^{(j)'})}}$ the arc I_j with endpoints $\zeta^{(j)}$ and $\zeta^{(j)'}$. Define $\mu_{ij}^- = \operatorname{sgn} T_{i1}(\zeta^{(j)})$ and $\mu_{ij}^+ = \operatorname{sgn} T_{i1}(\zeta^{(j)'})$. Since no $p \in \Sigma$ divides t_{i1}, T_{i1} does not vanish at $\zeta^{(j)}, \zeta^{(j)'}$, and therefore $\mu_{ij}^-, \mu_{ij}^+ \in \{\pm 1\}$. Define $t_{i1}' = \vartheta_i \cdot \prod_{j=1}^m p_{I_j}^{\theta_{ij}}$ and $t_i' = t_{i0} \cdot t_{i1}'$, where ϑ_i is the leading coefficient of T_{i1} and

$$\theta_{ij} = \begin{cases} 0 & \text{if } \mu_{ij}^- = \mu_{ij}^+ \\ 1 & \text{if } \mu_{ij}^- \neq \mu_{ij}^+. \end{cases}$$

Then for each $\zeta \in \mathcal{C} \setminus J$ $G \prod_{i=1}^n T_i'^{\epsilon_i}(\zeta) \geq 0$ or $H \prod_{i=1}^n T_i'^{\epsilon_i}(\zeta) \geq 0$. □

Now we are in a position to state and prove the main theorem.

Theorem 10. *Let $(X_F, G_{\Sigma F^2})$ be the space of orderings of the function field F of the irreducible curve (4) without $\mathbb{Q}$-rational points satisfying (11). Then there exists a pp-formula ϕ such that $G_{\Sigma F^2} \models \neg\phi$, but $G_{\Sigma F^2[S]} \models \phi_S$ for every proper subspace $(X_{\Sigma F^2[S]}, G_{\Sigma F^2[S]})$.*

Proof. Let $p_1, \dots, p_6 \in \mathbb{Q}[\mathcal{C}]$ be linear irreducibles, let $\xi^{(1i)}, xi^{(2i)}$ be points of intersection of P_i with $\mathcal{C}$, $i \in \{1, \dots, 6\}$, arranged as follows:

$$\xi^{(11)}, \xi^{(22)}, \xi^{(13)}, \xi^{(21)}, \xi^{(14)}, \xi^{(23)}, \xi^{(15)}, \xi^{(24)}, \xi^{(16)}, \xi^{(25)}, \xi^{(12)}, \xi^{(26)}$$

Let $f_1 = p_1 p_6$, $f_2 = p_1 p_4$, $f_3 = -p_1 p_2 p_3 p_5$. Consider the formula

$$\phi = \exists t_1 \exists t_2 \left(t_1 \in D(1, f_1) \wedge t_2 \in D(1, f_2) \wedge f_3 t_1 t_2 \in D(1, f_1 f_2) \right).$$

We shall show that $G_{\Sigma F^2} \models \neg\phi$. Suppose the contrary: let $G_{\Sigma F^2} \models \phi$ for some square-free $t_1, t_2 \in \mathbb{Q}[\mathcal{C}]$. Let $t_i = T_i + \mathcal{I}(\mathcal{C})$, $T_i \in \mathbb{Q}[X,Y]$, $f_j = F_j + \mathcal{I}(\mathcal{C})$,

$F_j \in \mathbb{Q}[X,Y]$, for $i \in \{1,2\}$, $j \in \{1,2,3\}$. The signs of F_1, F_2 and F_3 on the arcs between $\xi^{(ki)}$, $k \in \{1,2\}$, $i \in \{1,\dots,6\}$, are as follows:

	$(\xi^{(11)};\xi^{(22)})$	$(\xi^{(22)};\xi^{(13)})$	$(\xi^{(13)};\xi^{(21)})$	$(\xi^{(21)};\xi^{(14)})$	$(\xi^{(14)};\xi^{(23)})$	$(\xi^{(23)};\xi^{(15)})$	$(\xi^{(15)};\xi^{(24)})$	$(\xi^{(24)};\xi^{(16)})$	$(\xi^{(16)};\xi^{(25)})$	$(\xi^{(25)};\xi^{(12)})$	$(\xi^{(12)};\xi^{(26)})$	$(\xi^{(26)};\xi^{(11)})$
f_1	-	-	-	+	+	+	+	+	-	-	-	+
f_2	-	-	-	+	-	-	-	+	+	+	+	+
f_3	-	+	-	+	+	-	+	+	+	-	+	+

On the arcs $\overline{\overline{(\xi^{(21)};\xi^{(14)})}}$, $\overline{\overline{(\xi^{(24)};\xi^{(16)})}}$ and $\overline{\overline{(\xi^{(26)};\xi^{(11)})}}$ F_1 and F_2 are positive, T_1 and T_2 are nonnegative. Near $\xi^{(23)}$ F_1 is positive and so is T_1. There is only one irreducible $p \in \mathbb{Q}[\mathcal{C}]$ such that P intersects $\mathcal{C}$ at $\xi^{(23)}$, namely p_3; indeed, the kernel of the evaluation homomorphism $\mathbb{Q}[\mathcal{C}] \ni g \mapsto G(\xi^{(23)}) \in \mathbb{R}$ is generated by p_3. Thus $v_{p_3}(t_1)$ is even and T_1 does not change sign at $\xi^{(13)}$. Near $\xi^{(13)}$ F_1F_2 is positive, so $F_3T_1T_2$ is positive and F_3 changes the sign, so that T_2 changes sign. That means that $v_{p_3}(t_2)$ is odd and hence T_2 changes sign at $\xi^{(23)}$. To sum up: T_2 changes signs at $\xi^{(23)}$ and $\xi^{(13)}$, but T_1 does not.

Near $\xi^{(12)}$ F_1 is positive and so is T_2. Thus T_2 does not change sign at $\xi^{(22)}$. Near $\xi^{(22)}$ F_1F_2 is positive and so is $F_3T_1T_2$ and F_3 changes sign, so T_1 must change sign. Thus T_1 changes signs at $\xi^{(12)}$ and $\xi^{(22)}$, but T_2 does not. Near $\xi^{(11)}$ F_1F_2 is positive and so is $F_3T_1T_2$. F_3 changes sign and so does T_1T_2. Thus one of T_1 and T_2 changes sign, but not both. Thus at $\xi^{(11)}$ and $\xi^{(21)}$ either T_1 changes sign (at both points) or T_2 changes sign, but not both.

On the arc $\overline{\overline{(\xi^{(11)};\xi^{(22)})}}$ F_1F_2 is positive and F_3 is negative, so T_1T_2 is negative or zero. Hence at any point of this arc if T_1 changes sign, then so does T_2 (and vice versa) - say there are m_1 such simultaneous sign changes. Similarly, there are m_3 simultaneous sign changes of T_1 and T_2 on the arc $\overline{\overline{(\xi^{(13)};\xi^{(21)})}}$. On $\overline{\overline{(\xi^{(22)};\xi^{(13)})}}$ both F_1F_2 and F_3 are positive, so T_1T_2 is positive or zero. Thus at any if T_1 changes sign, then so does T_2 - say there are m_2 such sign changes.

On $\overline{\overline{(\xi^{(11)};\xi^{(21)})}}$ T_1 and T_2 each change sign $m_1+m_2+m_3+1$ times. The signs of T_1 and T_2 at $\xi^{(11)}$ are the same as at $\xi^{(21)}$, so $m_1+m_2+m_3$ is odd. On all the other arcs at least one of F_1 and F_2 is positive, so at least one of T_1 and T_2 is nonnegative - thus the simultaneous sign changes of T_1 and T_2 occur only at the indicated $m_1+m_2+m_3$ points.

The units of $\mathbb{Q}[\mathcal{C}]$ are the elements of $\mathbb{Q}^*$: indeed, if $g = \alpha + \beta\sqrt{\Delta}$, $\Delta = -\frac{c}{b} - \frac{c}{a}x^2$, $\alpha, \beta \in \mathbb{Q}[x]$, is an irreducible and $\overline{g} = \alpha - \beta\sqrt{\Delta}$, then $g \cdot \overline{g} = \alpha^2 - \beta^2(-\frac{c}{b} - \frac{c}{a}x^2)$ is a unit of $(\mathbb{Q}[x]$. Thus $\deg_x g\overline{g} = 0$, so that $\beta = 0$ and $\deg_x \alpha = 0$. Now let

$$t_1 = u_1 q_1 \dots q_k r_1 \dots r_l \text{ and } t_2 = u_2 q_1 \dots q_k r_1' \dots r_m'$$

be factorizations of t_1 and t_2 into irreducibles, $u_1, u_2 \in \mathbb{Q}^*$. Let $q_i = Q_i + \mathcal{I}(\mathcal{C})$, $Q_i \in \mathbb{Q}[X,Y]$. The simultaneous sign changes occur at the points of intersection of Q_i with $\mathcal{C}$. But since for each Q_i there is an even number of such points and for $i \neq j$ Q_i and Q_j intersect $\mathcal{C}$ in different points, $m_1+m_2+m_3$ must be even.

Finally, $G_{p_i} \models \phi_{p_i}$ for $i \in \{1, \dots, 6\}$ by the substitutions

	p_1	p_2	p_3	p_4	p_5	p_6
t_1	1	f_3	1	1	1	1
t_2	f_3	1	f_3	1	1	1

,

and by Theorem 9 $G_{\Sigma F^2[S]} \models \phi_S$ for every submodel $G_{\Sigma F^2[S]}$. □

4.3. **Spaces of orderings of function fields of two parallel lines over $\mathbb{Q}$.** In this section we complete our analysis by considering the case of a real irreducible two paralles lines, i. e. $aX^2 + c = 0$, $a > 0$, $c < 0$. We might as well assume $a = 1$. This case is similar to the elliptic case, and the main arguments and results from the previous paragraph carry over, with a bit of modification here and there.

The coordinate ring $\mathbb{Q}[\mathcal{C}]$ can be identified with $\mathbb{Q}(\sqrt{-c})[y]$, the polynomial ring in one variable y with coefficients in the field $\mathbb{Q}(\sqrt{-c})$. The valuations that are of interest to us are also easy to describe. Units are identified with non-zero elements of $\mathbb{Q}(\sqrt{d})$. Unlike what happens in the elliptic case, units no longer necessarily have constant sign on $\mathcal{C}$.

We still have the linear irreducibles $\pi = rx+sy+t$, $r, s, t \in \mathbb{Q}$, $s \neq 0$, but these no longer suffice. To copy certain of the constructions used in the proofs of Theorems 9 and 10, we also use the fact that there are enough quadratic irreducibles in $\mathbb{Q}[\mathcal{C}]$ of the form

$$\pi = x \pm (r(y+s)^2 + t), \ r, s, t \in \mathbb{Q}, \ r > 0, \ |t| < \sqrt{-c}.$$

Lemma 21. *For given real r, s, t satisfying $r > 0$, $|t| < \sqrt{-c}$, there exist rationals r', s' and t' arbitrarily close to r, s and t respectively, such that $x + (r'(y+s')^2 + t')$ and $x - (r'(y+s')^2 + t')$ are irreducible in $\mathbb{Q}[\mathcal{C}]$.*

Proof. The discriminant of $\sqrt{-c} \pm (r'(y+s')^2 + t') \in \mathbb{Q}(\sqrt{-c})[y]$ is $-4r'(t' \pm \sqrt{-c})$. We want this to be not a square in $\mathbb{Q}(\sqrt{-c})$. Proceed as follows: choose r' to be any rational square close to r, choose s' close to s, choose t' close to t and such that ${t'}^2 + c$ is not a rational square (so then $-t' - \sqrt{-c}$ and $-t' + \sqrt{-c}$ are not squares in $\mathbb{Q}(\sqrt{-c})$). We can, for example, choose t' of the form $t' = p^k t_1$ where p is a prime such that the value of $-c$ at p is odd, $2k > v_p(-c)$ and $v_p(t_1) \geq 0$. Then $v_p({t'}^2 + c) = v_p(-c)$ is odd, so ${t'}^2 + c$ is not a square in $\mathbb{Q}$. □ □

The correspondence between points on $\mathcal{C}$ and orderings on $\mathbb{Q}(\mathcal{C})$ is the same as before, but now there are additional orderings corresponding to the four half-branches of $\mathcal{C}$ at ∞. These are precisely the orderings compatible with the real valuation v_∞ on $\mathbb{Q}(\mathcal{C})$ defined by $v_\infty(f) = -\deg_y(f)$.

Lemma 19 carries over with the same proof. Using this, we see that an irreducible π has an even (resp., odd) number of roots on the line $x = -\sqrt{-c}$, and also on the line $x = \sqrt{-c}$, if $\deg_y(\pi)$ is even (resp., if $\deg_y(\pi)$ is odd).

Lemma 20 also carries over without change but, regarding part (2) of Lemma 20, there is also a similar result for the point at infinity: Suppose f, g, h are non-zero elements of A and $\overline{f}, \overline{g}, \overline{h}$ denote the associated elements of G_{v_∞}. Then $\overline{f} \in D(\overline{g}, \overline{h})$ holds in $(X_{v_\infty}, G_{v_\infty})$ iff $fg \geq 0$ at p or $fh \geq 0$ at p holds for all real points $p = (\pm\sqrt{-c}, p_2)$ of $\mathcal{C}$ with $|p_2|$ sufficiently large.

With these preliminary remarks out of the way, we are now in a position to state the main results of this section:

Theorem 11. *Let $\mathbb{Q}(\mathcal{C})$ be the function field of a rational conic $x^2 + c = 0$, where c is a negative and not a square. For a given pp-formula*

$$\phi = \exists t_1 \dots \exists t_n \psi(t_1, \dots, t_n, \overline{f_1}, \dots, \overline{f_k}),$$

$\overline{f_i}$ denoting the image of $f_i \in \mathbb{Q}[\mathcal{C}]^*$ under the homomorphism $f \mapsto \overline{f}$ from F^* to G, let Σ denote the set of all irreducible factors of the f_i, $i \in \{1, \dots, k\}$. The following conditions are equivalent:

1. *$G_{v_\pi} \models \phi_{v_\pi}$, for each $\pi \in \Sigma \cup \{\infty\}$.*
2. *$G_S \models \phi_S$, for each proper subspace (X_S, G_S) of (X, G).*
3. *$G_S \models \phi_S$, for each finite subspace (X_S, G_S) of (X, G).*

The proof of Theorem 11 is the same as the proof of Theorem 9, with minor modifications to allow for the fact that we are now dealing with two parallel lines. In defining the π_I we allow not only linear irreducibles, but also quadratics irreducibles as well (to take care of the case where the intervals I and J are both on the same component of $\mathcal{C}$). In the last step, in the definition of the $\tilde{t_i}$, we define $\tilde{t_i} = \mu_i \prod_{I \in T} {\pi_I}^{s_{iI}}$, where $s_{iI} = 0$ or 1 depending on whether $\overline{t_i}$ has the same sign or opposite sign at the opposite ends of the open interval I, and where $\mu_i \in \{1, -1, x, -x\}$ is chosen so that $\tilde{t_i}$ has the same sign as $\overline{t_i}$ at the ends of each of the intervals $I \in T$.

Theorem 12. *Let $\mathbb{Q}(\mathcal{C})$ be the function field of a rational conic $x^2 + c = 0$, where c is a negative and not a square. Then there exists a pp-formula ϕ with parameters in G such that $G \models \neg\phi$, but $G_S \models \phi_S$ for each proper subspace (X_S, G_S) of (X, G).*

Again, the proof of Theorem 12 is analogous to the proof of Theorem 10, but instead of using just linear irreducibles we also allow suitably chosen quadratic irreducibles. We arrange the zeros p_{1i}, p_{2i}, $i = 1, \dots, 6$ of these six irreducibles (for example), so that the first six points

$$p_{11}, p_{22}, p_{13}, p_{21}, p_{14}, p_{23}$$

are on the line $x = -\sqrt{-c}$, in upward order, and the next six points

$$p_{15}, p_{24}, p_{16}, p_{25}, p_{12}, p_{26}$$

are on the line $x = \sqrt{-c}$, in the downward order. The reader may check that this particular arrangement uses two linear irreducibles and four quadratic irreducibles (two opening to the left, and two opening to the right).

5. Spaces of orderings of elliptic curves

5.1. **Coordinate rings and function fields of elliptic curves.** We shall try to mimic main results from the first section of the previous chapter. First problem that we have to solve is the fact that - unlike conic sections - elliptic curves divide into two groups: singular and nonsingular curves. Fortunately, this problem is quite easy to resolve. The following sequel of three lemmas completely classifies singular elliptic curves over the fields $\mathbb{Q}$ with respect to the pp conjecture.

Lemma 22. *Singular point on an elliptic curve with rational coefficients has rational coordinates.*

Proof. It is well-known, that any elliptic curve is birationally equivalent to a Weierstrass curve:

$$y^2 = x^3 + ax^2 + bx + c.$$

Let (x_0, y_0) be a singular point, i. e. a point such that:

$$\frac{\partial f}{\partial x}(x_0, y_0) = 3x_0^2 + 2ax_0 + b = 0,$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = -2y_0 = 0,$$

where $f(x, y) = x^3 + ax^2 + bx + c - y^2$. Thus clearly $y_0 = 0 \in \mathbb{Q}$. As a solution of the equation $3x_0^2 + 2ax_0 + b = 0$, x_0 is of the form $x_0 = p + q\sqrt{r}$, $p, q, r \in \mathbb{Q}$. Moreover, $x_0^3 + ax_0^2 + bx_0 + c = y_0^2 = 0$, so x_0 is a root of multiplicity 2 of $x_0^3 + ax_0^2 + bx_0 + c = 0$ and thus:

$$x_0^3 + ax_0^2 + bx_0 + c = (x - x_0)^2(x - x_1).$$

By comparison of coefficients we get $x_0^2 x_1 = -c$, $x_0^2 + 2x_0x_1 = b$ and $2x_0 + x_1 = -a$. ¿From the first equation we infer, that x_1 is a conjugate of x_0^2, that is $x_1 = p^2 + q^2r - 2pq\sqrt{r}$. On the other hand, from the last equation we have $x_1 = -a - 2p - 2q\sqrt{r}$. Thus, in particular, $2pq\sqrt{r} = 2q\sqrt{r}$. Suppose that both $q \neq 0$ and $r \neq 0$. Then $p = 1$ and, consequently, $x_0 = 1 + q\sqrt{r}$, $x_1 = 1 + q^2r - 2q\sqrt{r}$. Hence:

$$b = x_0^2 + 2x_0x_1 = 3 - q^2r + 2q^3r\sqrt{r},$$

which implies that $2q^3r\sqrt{r} = 0$, that is either $q = 0$ or $r = 0$ - a contradiction. Therefore $q = 0$ or $r = 0$ and in both cases $x_0 = p \in \mathbb{Q}$. □

Lemma 23. *Singular curves are birationally equivalent to curves of the form $Y^2 = X^3 + AX^2$, $A \in \mathbb{Q}$.*

Proof. Let $(x_0, y_0) \in \mathbb{Q}^2$ be a singular point. The Weierstrass curve:

$$y^2 = x^3 + ax^2 + bx + c$$

is then affine isomorphic to a desired curve by the substitution $X := x - x_0$, $Y := y$; indeed, we have:

$$\begin{aligned} x^3 + ax^2 + bx + c &= X^3 + 3X^2x_0 + 3Xx_0^2 + x_0^3 + \\ &+ \quad aX^2 + 2aXx_0 + ax_0^2 + bX + bx_0 + c = \\ &= \quad X^3 + X^2(3x_0 + a) + X(3x_0^2 + 2ax_0 + b) = X^3 + AX^2 \end{aligned}$$

□

Lemma 24. $K(X, Y) \cong K(T)$ *if* $Y^2 = X^3 + AX^2$

Proof. Let $T = \frac{X}{Y}$. Then clearly $K(T) \subset K(X, Y)$. Conversely, $Y = \frac{X}{T}$, so $\frac{X^2}{T^2} = X^2(X + A)$ which implies that $\frac{1}{T^2} = X + A$. Therefore $X = \frac{1}{T^2} - A$ and $Y = \frac{1}{T^3} - \frac{A}{T}$. □

Therefore function fields of singular elliptic curves are isomorphic to $\mathbb{Q}(X)$ and the pp conjecture holds for spaces of orderings of such fields by the results of chapter 3. Thus we can restrict ourselves to the case of nonsingular elliptic curves.

We have shown that every singular elliptic curve over $\mathbb{Q}$ has a rational parametrization. Unfortunately, there is no hope for nonsingular curves to have such property.

Lemma 25. *No nonsingular elliptic curve over $\mathbb{Q}$ admits a rational parametrization.*

Proof. Suppose that E is a nonsingular elliptic curve and that $(x(t), y(t))$ is a parametrization of E. We may consider it as a parametrization of a complex elliptic curve with complex numbers. Thus we have a holomorphic function $f : \mathbb{C} \to E(\mathbb{C})$ from complex numbers to complex points on $E(\mathbb{C})$. We can extend this function to the meromorphic function $F : \mathbb{C} \cup \{\infty\} \to \mathbb{P}(\mathbb{C})$ from the punctured complex plane to the complex projective extension of $E(\mathbb{C})$: we map point "at infinity" onto a suitable limiting value and points for which denominators of $x(t), y(t)$ vanish onto the point "at infinity". Such function is onto.

Now $\mathbb{P}(\mathbb{C})$ is topologically homeomorphic to a complex torus (the proof of this fact uses the assumption that E is nonsingular) and $\mathbb{C} \cup \{\infty\}$ is homeomorphic to a real 2-sphere. We have thus constructed a covering of a torus by a sphere, which is not possible - a contradiction. □

Thus we will not be able to reduce some of the cases of function fields of nonsingular elliptic curves to the case of the field $\mathbb{Q}(X)$, as we did with conic sections.

Next, in the case of conic sections we have distinguished between curves having rational points and curves without such points. The case of nonsingular elliptic curves is much more complicated; apparently it is hard even to name an example on a nonsingular elliptic curve without rational points. The simplest examples of such curves are found among Mordell curves, that is the curves of the form:

$$y^2 = x^3 + k, k \in \mathbb{Z}.$$

Clearly every Mordell curve is nonsingular: the only point for which both partial derivatives vanish is $(0, 0)$ and it doesn't belong to the curve. There are certain values of k for which the curve $y^2 = x^3 + k$ has no integer solutions, for example if $k = 6, 7, 11, 13, 14, 20, 21, 23, 29, 32, \dots$ (proofs of these facts are quite elementary and can be found in number theory books). How to choose those curves among them, which not only have no integer points, but no rational points as well? In this case the celebrated Nagell-Lutz theorem comes handy: for a nonsingular curve with integer coefficients and discriminant D, if a rational point (x, y) is of finite order in the group $E(\mathbb{Q})$, then x and y are integers and either $y = 0$, in which case (x, y) has order two, or else y divides D. Using this theorem we see that there are no rational points of finite order.

What about the ones of infinite order? The group of rational points $E(\mathbb{Q})$ is a finitely generated abelian group and therefore $E(\mathbb{Q}) = \mathbb{Z}^r \oplus Tors$; we already know that $Tors = 0$ and we can compute the rank r using some software, for example GP pari:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
r	0	1	1	0	1	0	0	1	1	1	1	1	0	0	2

The common values are, for example, 6 and 7 - therefore there are no rational points on curves

$$y^2 = x^3 + 6 \text{ and } y^2 = x^3 + 7$$

(elementary proofs of these facts are also available).

The above examples show that it seems incredibly difficult to classify all the elliptic curves with respect to the pp conjecture. However, the author hopes that he will be able at least to find some more counterexamples to the pp conjecture. As an object of further examination, Mordell curves without rational points look very promising - for example curves $y^2 = x^3 + 6$ or $y^2 = x^3 + 7$. A sllightly modified argument from chapter 4 allows us to show, that coordinate rings of those curves are Dedekind domains. There is certain hope that they will be also principal ideal domains and that we will be able to easily describe valuations of functions fields of those curves. These issues are currently under investigation.

6. Spaces of orderings of function fields in many variables

6.1. **Spaces of orderings of function fields in many variables.** In this section we shall prove that the pp conjecture fails for the space of orderings of the field $\mathbb{Q}(x, y)$. The strategy is as follows: we shall prove that the class of spaces of orderings for which pp conjecture holds is closed under subspaces and group extensions and then consider the diagram:

$$\begin{tikzcd} \mathbb{Q}[x,y]_{(f)} \arrow[r] \arrow[dr] & (\mathbb{Q}[x,y]/(f)) , \\ & \mathbb{Q}(x,y) \end{tikzcd}$$

where f is such polynomial that the pp conjecture fails in the space of orderings of $(\mathbb{Q}[x, y]/(f))$. The space of orderings of the field of fractions of the local ring $\mathbb{Q}[x, y]_{(f)}$ is a subspace of the space of orderings of the field $\mathbb{Q}(x, y)$ and the space of orderings of the field $(\mathbb{Q}[x, y]/(f))$ is its group extension. Suppose that the pp conjecture holds for the space of orderings of $\mathbb{Q}(x, y)$. Then it also holds for $\mathbb{Q}[x, y]_{(f)}$, as for a subspace, and thus it holds for $(\mathbb{Q}[x, y]/(f))$, as for a group extension - a contradiction. By induction the result follows for a space of orderings of any rational function field over $\mathbb{Q}$ in a finite number of variables. It remains to prove the two key results: that the class of spaces of orderings for which pp conjecture holds is closed under subspaces and group extensions.

Theorem 13. *Let $(\overline{X}, \overline{G})$ be a space of orderings and (X, G) its group extension. Then pp conjecture holds in (X, G) if and only if pp conjecture holds in $(\overline{X}, \overline{G})$.*

Proof. Assume that pp conjecture holds in the space $(\overline{X}, \overline{G})$ and suppose that

$$P(\underline{a}) = \exists \underline{t} \bigwedge_{j=1}^{m} p_j(\underline{t}, \underline{a}) \in D(1, q_j(\underline{t}, \underline{a})),$$

where $\underline{t} = (t_1, \ldots, t_n)$, $\underline{a} = (a_1, \ldots, a_k)$ and $p_j(\underline{t}, \underline{a})$, $q_j(\underline{t}, \underline{a})$ are products of $\pm$ some of the t_i's, $i \in \{1, \ldots, n\}$, and some of the a_l's, $l \in \{1, \ldots, k\}$, is a pp formula which holds in every finite subspace of (X, G) but does not hold in (X, G). Clearly $P(\underline{a})$ will not hold in a space generated by the group extension $\overline{G}[\underline{a}] \supset \overline{G}$, so that we may assume that the extension $G \supset \overline{G}$ is finite. Taking instead of $\overline{G}$ the biggest intermediate group for which $P(\underline{a})$ will hold and instead of G an extension of such group by one element, we may also assume that $G = \overline{G}[c]$, that is $G = \overline{G} \cup \overline{G}c$.

Fix an arbitrary $\delta \in \{0, 1\}^n$, $\delta = (\delta_1, \ldots, \delta_n)$ and let

$$\underline{u} = (u_1, \ldots, u_n),\ u_i \in \overline{G},\ i \in \{1, \ldots, n\},$$

where $\underline{t} = (u_1 c^{\delta_1}, \ldots, u_n c^{\delta_n})$. Moreover, let

$$\underline{b} = (b_1, \ldots, b_k),\ b_l \in \overline{G},\ l \in \{1, \ldots, k\},$$

where $\underline{a} = (b_1 c^{\epsilon_1}, \ldots, b_k c^{\epsilon_k})$ for certain $\epsilon_l \in \{0, 1\}$, $l \in \{1, \ldots, k\}$. Denote by

$$p_j^\delta(\underline{u},\underline{b}) = p_j(\underline{a},\underline{t}),\ j \in \{1,\dots,m\},$$
$$q_j^\delta(\underline{u},\underline{b}) = q_j(\underline{a},\underline{t}),\ j \in \{1,\dots,m\}.$$

and

$$P^\delta(\underline{b}) = \exists \underline{u} \bigwedge_{j=1}^m p_j^\delta(\underline{u},\underline{b}) \in D(1, p_j^\delta(\underline{u},\underline{b})).$$

We claim that $P^\delta(\underline{b})$ is a formula in $(\overline{X},\overline{G})$. Indeed, fix an $j \in \{1,\dots,m\}$ and consider $p_j^\delta(\underline{u},\underline{b})$ and $q_j^\delta(\underline{u},\underline{b})$. There are four following cases to consider:

(1) $p_j^\delta(\underline{u},\underline{b}) = p_j^\delta c$ and $q_j^\delta(\underline{u},\underline{b}) = q_j^\delta c$ for some $p_j^\delta, q_j^\delta \in \overline{G}$,
(2) $p_j^\delta(\underline{u},\underline{b}) = p_j^\delta c$ and $q_j^\delta(\underline{u},\underline{b}) = q_j^\delta$ for some $p_j^\delta, q_j^\delta \in \overline{G}$,
(3) $p_j^\delta(\underline{u},\underline{b}) = p_j^\delta$ and $q_j^\delta(\underline{u},\underline{b}) = q_j^\delta c$ for some $p_j^\delta, q_j^\delta \in \overline{G}$,
(4) $p_j^\delta(\underline{u},\underline{b}) = p_j^\delta$ and $q_j^\delta(\underline{u},\underline{b}) = q_j^\delta$ for some $p_j^\delta, q_j^\delta \in \overline{G}$.

Case (4) is immediate. Case (3) is reducible to case (1) by means of the identity

$$p_j^\delta \in D(1, q_j^\delta c) \Leftrightarrow p_j^\delta q_j^\delta c \in D(1, q_j^\delta c),$$

and case (2) is reducible to case (1) since

$$p_j^\delta c \in D(1, q_j^\delta) \Leftrightarrow p_j^\delta c \in D(1, -p_j^\delta q_j^\delta c).$$

Finally, since $D(1, q_j^\delta c) = \{1, q_j^\delta c\}$ and $p_j^\delta c \neq 1$, in case (1) the atomic formula

$$p_j^\delta c \in D(1, q_j^\delta c)$$

reduces to $p_j^\delta c = q_j^\delta c$, which is equivalent to $p_j^\delta = q_j^\delta$, the latter being a statement in $(\overline{X},\overline{G})$.

Clearly $P^\delta(\underline{b})$ can not hold in $(\overline{X},\overline{G})$. Thus there exists a finite subspace $(\overline{Y}^\delta, \overline{G}|_{\overline{Y}^\delta})$ of $(\overline{X},\overline{G})$ for which $P^\delta(\underline{b})$ fails to hold. Let $(\overline{Y}, \overline{G}|_{\overline{Y}}) = \bigcap_{g \in S \subset \overline{G}} U_{\overline{X}}(g)$ be a subspace of $(\overline{X},\overline{G})$ generated by the set $\bigcup_{\delta \in \{0,1\}^n} \overline{Y}^\delta$. As a space generated by finite set it is finite itself and gives a rise to three finite subspaces of (X,G):

$$Y_1 = \bigcap_{g \in S \subset \overline{G}} U_X(g), \qquad Y_2 = \bigcap_{g \in S \subset \overline{G}} U_X(g) \cap U_X(c)$$

and

$$Y_3 = \bigcap_{g \in S \subset \overline{G}} U_X(g) \cap U_X(-c).$$

Since none of $P^\delta(\underline{b})$ holds in $(\overline{Y}, \overline{G}|_{\overline{Y}})$, $P(\overline{a})$ holds in neither of $(Y_s, G|_{Y_s})$, $s \in \{1,2,3\}$, all of them being finite subspaces of (X,G) - a contradiction.

Now assume that the pp conjecture holds in (X,G) and suppose that

$$P(\underline{a}) = \exists \underline{t} \bigwedge_{j=1}^m p_j(\underline{t},\underline{a}) \in D(1, q_j(\underline{t},\underline{a})),$$

where $\underline{t} = (t_1,\dots,t_n)$, $\underline{a} = (a_1,\dots,a_k)$ and $p_j(\underline{t},\underline{a})$, $q_j(\underline{t},\underline{a})$ are as before is a pp formula which holds in every finite subspace of $(\overline{X},\overline{G})$ but does not hold in $(\overline{X},\overline{G})$. Similarly, we may assume that $G = \overline{G}[c]$. Clearly $P(\underline{a})$ is also a formula in (X,G). As before, every finite subspace $(\overline{Y}, \overline{G}|_{\overline{Y}})$ of $(\overline{X},\overline{G})$ gives a rise to three finite subspaces $(Y_1, G|_{Y_1})$, $(Y_2, G|_{Y_2})$ and $(Y_3, G|_{Y_3})$ of (X,G) and if $P(\underline{a})$ holds in $(\overline{Y}, \overline{G}|_{\overline{Y}})$ then it also holds in all of the three mentioned spaces. Conversely, if $(Y, G|_Y) = \bigcap_{gc^\epsilon \in Sc^{\{0,1\}} \subset \overline{G}c^{\{0,1\}} = G} U_X(gc^\epsilon)$ is a finite subspace of (X,G) then so is $\bigcap_{gc^\epsilon \in Sc^{\{0,1\}} \subset \overline{G}c^{\{0,1\}} = G} U_X(gc^\epsilon) \cap U(c)$, which, in turn, induces a finite subspace $(\overline{Y}, \overline{G}|_{\overline{Y}}) = \bigcap_{g\in S\subset \overline{G}} U_{\overline{X}}(g)$ of $(\overline{X}, \overline{G})$. Therefore if $P(\underline{a})$ holds true in every finite subspace of $(\overline{X}, \overline{G})$ then it also holds true in every finite subspace of (X, G). By our assumptions this implies that $P(\underline{a})$ holds in (X, G), that is for some $\underline{t} = (u_1 c^{\epsilon_1}, \dots, u_n c^{\epsilon_n})$, $u_i \in \overline{G}$, $\epsilon_i \in \{0, 1\}$, $i \in \{1, \dots, n\}$ we have

$$p_j(\underline{t}, \underline{a}) \in D(1, q_j(\underline{t}, \underline{a})), \text{ for } j \in \{1, \dots, m\}.$$

In each $p_j(\underline{t}, \underline{a})$ and $q_j(\underline{t}, \underline{a})$, $j \in \{1, \dots, m\}$ replace $\underline{t}$ with $\underline{u}$, $\underline{u} = (u_1, \dots, u_n)$. We thus obtain elements $p_j(\underline{u}, \underline{a})$ and $q_j(\underline{u}, \underline{a})$ of the group $\overline{G}$ such that

$$p_j(\underline{u}, \underline{a})c^{\mu_j} = p_j(\underline{t}, \underline{a}) \text{ and } q_j(\underline{u}, \underline{a})c^{\nu_j} = q_j(\underline{t}, \underline{a})$$

for some $\mu_j, \nu_j \in \{0, 1\}$, $j \in \{1, \dots, m\}$. We shall show that

$$p_j(\underline{u}, \underline{a}) \in D_{\overline{X}}(1, q_j(\underline{u}, \underline{a})) \text{ for all } j \in \{1, \dots, m\}.$$

Fix a $j \in \{1, \dots, m\}$. As before, we shall consider four distinct cases.

(1) $p_j(\underline{u}, \underline{a})c = p_j(\underline{t}, \underline{a})$ and $q_j(\underline{u}, \underline{a})c = q_j(\underline{t}, \underline{a})$,
(2) $p_j(\underline{u}, \underline{a})c = p_j(\underline{t}, \underline{a})$ and $q_j(\underline{u}, \underline{a}) = q_j(\underline{t}, \underline{a})$,
(3) $p_j(\underline{u}, \underline{a}) = p_j(\underline{t}, \underline{a})$ and $q_j(\underline{u}, \underline{a})c = q_j(\underline{t}, \underline{a})$,
(4) $p_j(\underline{u}, \underline{a}) = p_j(\underline{t}, \underline{a})$ and $q_j(\underline{u}, \underline{a}) = q_j(\underline{t}, \underline{a})$.

Case (4) is obvious. Case (1) just means that $p_j(\underline{u}, \underline{a}) = q_j(\underline{u}, \underline{a})$, so trivially $p_j(\underline{u}, \underline{a}) \in D_{\overline{X}}(1, q_j(\underline{u}, \underline{a}))$. In case (3) the formula $p_j(\underline{u}, \underline{a}) \in D_X(1, q_j(\underline{u}, \underline{a})c)$ is equivalent to $p_j(\underline{u}, \underline{a})q_j(\underline{u}, \underline{a})c \in D_X(1, q_j(\underline{u}, \underline{a})c)$, so again $p_j(\underline{u}, \underline{a})q_j(\underline{u}, \underline{a}) = q_j(\underline{u}, \underline{a})$ and thus $p_j(\underline{u}, \underline{a}) = 1$ and $p_j(\underline{u}, \underline{a}) \in D_{\overline{X}}(1, q_j(\underline{u}, \underline{a}))$ is satisfied. In a similar way in case (2) we get $q_j(\underline{u}, \underline{a}) = -1$, so that the claimed statement holds true.

We have thus proved that $P(\underline{a})$ holds in $(\overline{X}, \overline{G})$, which is a contradiction. □

Theorem 14. *Let (X, G) be a space of orderings and let $(Y, G|_Y)$ be its subspace. If pp conjecture holds in (X, G), then it also holds in $(Y, G|_Y)$.*

Proof. **Step 1:** Let $Y = U(a_1, \dots, a_n)$. We shall show that if pp conjecture holds in (X, G), then it also holds in $(Y, G|_Y)$.

Let $P(\underline{a}) = \exists \underline{t} \bigwedge_{j=1}^m p_j(\underline{a}, \underline{t}) \in D(1, q_j(\underline{a}, \underline{t}))$ be a pp formula such that $P(\underline{a})$ holds for every finite subspace of $(Y, G|_Y)$. We shall show that $P(\underline{a})$ holds true for $(Y, G|_Y)$. Let $Z = \bigcap_{a\in S} U(a)$ be a finite subspace of (X, G). Then $Z \cap Y$ is a finite subspace of $(Y, G|_Y)$, so $P(\underline{a})$ holds true for $(Z \cap Y, G|_{Z\cap Y})$. Let

$$P(\underline{a}, Y) = \exists \underline{t} \bigwedge_{j=1}^m p_j(\underline{a}, \underline{t}) \in D((1, q_j(\underline{a}, \underline{t})) \otimes ((a_1, \dots, a_n))).$$

By [11, Theorem 2.4.4] $P(\underline{a}, Y)$ holds for Z if and only if $P(\underline{a})$ holds for $Z \cap Y$. Thus $P(\underline{a}, Y)$ holds true for every finite subspace $(Z, G|_Z)$ and since the pp conjecture remains valid in (X, G), $P(\underline{a}, Y)$ holds true in (X, G), so that $P(\underline{a})$ holds true in $(Y, G|_Y)$.

Step 2: Let $Y = \bigcap_{a\in S} U(a)$. As before, we shall show that if pp conjecture holds in (X, G), then it also holds in $(Y, G|_Y)$.

For let $P(\underline{a}) = \exists \underline{t} \bigwedge_{j=1}^m p_j(\underline{a}, \underline{t}) \in D(1, q_j(\underline{a}, \underline{t}))$ be a pp formula such that $P(\underline{a})$ holds for every finite subspace of $(Y, G|_Y)$. We shall show that $P(\underline{a})$ holds true for $(Y, G|_Y)$. In order to do that it suffices to show that for some finite subset $T \subset S$ $P(\underline{a})$ holds in $\bigcap_{a\in T} U(a)$. Suppose, *a contrario*, that for every finite subset $T \subset S$ $P(\underline{a})$ fails in $\bigcap_{a\in T} U(a)$. By **step 1** it follows that for every finite subset

$T \subset S$ there exists a finite subspace Z_T of $\bigcap_{a\in T} U(a)$ such that $P(\underline{a})$ fails in Z_T. By [1, Lemma 4] for every finite subset $T \subset S$ there exists a finite subspace Y_T of $\bigcap_{a\in T} U(a)$ of cardinality at most B, where B is some integer, such that $P(\underline{a})$ fails in Y_T. Let $Y_T = \{x_1^T, x_2^T, \ldots, x_B^T\}$. For $i \in \{1, \ldots, B\}$ $\{x_i^T : T \in 2^S, T$ - finite$\}$ is a net with entries directed according to the rule

$$x_i^T \geq x_i^{T'} \text{ if and only if } T \supseteq T'.$$

Since X is compact, $\{x_1^T : T \in 2^S, T$ - finite$\}$ has a cluster point x_1. Let $\{x_1^{T_1} : T_1 \in \Sigma_1\}$ be a net finer than $\{x_i^T : T \in 2^S, T$ - finite$\}$ which converges to x_1, where $\Sigma_1 \subset \{T \in 2^S : T$ - finite$\}$. Next, $\{x_2^{T_1} : T_1 \in \Sigma_1\}$ has a cluster point x_2, so let $\{x_2^{T_{12}} : T_{12} \in \Sigma_{12}\}$ be a net finer than $\{x_2^{T_1} : T_1 \in \Sigma_1\}$ which converges to x_2, where $\Sigma_{12} \subset \Sigma_1$. By induction we will eventually construct the net $\{x_B^{T_{12\ldots B}} : T_{12\ldots B} \in \Sigma_{12\ldots B}\}$ finer than $\{x_B^{T_{12\ldots B-1}} : T_{12\ldots B-1} \in \Sigma_{12\ldots B-1}\}$ which converges to a cluster point x_B of the net $\{x_B^{T_{12\ldots B-1}} : T_{12\ldots B-1} \in \Sigma_{12\ldots B-1}\}$, where $\Sigma_{12\ldots B} \subset \Sigma_{12\ldots B-1}$. Clearly the net

$$\{x_i^{T_{12\ldots B}} : T_{12\ldots B} \in \Sigma_{12\ldots B}\}$$

is finer than $\{x_i^T : T \in 2^S, T$ - finite$\}$ and converges to x_i, $i \in \{1, \ldots, B\}$. Let Z be a space generated by $x_1, \ldots, x_B$.

We shall show that Z is a finite subspace of Y; indeed, it suffices to show that all the generators $x_1, \ldots, x_B$ are elements of Y. Fix an arbitrary $i \in \{1, \ldots, B\}$ and $a \in S$ - we shall show that $x_i \in U(a)$. Suppose that $x_i \notin U(a)$. Since X is compact and hence regular, there is an open set V such that $x_i \in V$ and $V \cap U(a) = \emptyset$. But x_i is a cluster point of $\{x_i^{T_{12\ldots B}} : T_{12\ldots B} \in \Sigma_{12\ldots B}\}$ and hence of $\{x_i^T : T \in 2^S, T$ - finite$\}$, so there exists an element x_i^T such that $x_i^T \geq x_i^{\{a\}}$ and $x_i^T \in V$. Then $x_i^T \in Y_T \subset \bigcap_{a\in T} U(a) \subset U(a)$ - a contradiction.

Finally, we shall show that $P(\underline{a})$ fails in Z. Suppose that the converse is true and that $P(\underline{a})$ holds true in Z. Let $\underline{t}$ be such that $p_j(\underline{a}, \underline{t}) \in D(1, q_j(\underline{a}, \underline{t}))$ in Z, $j \in \{1, \ldots, m\}$. Clearly

$$x_1, x_2, \ldots, x_B \in U = \bigcap_{j=1}^{m} \{U(p_j(\underline{a}, \underline{t})) \cup [U(-p_j(\underline{a}, \underline{t})) \cap U(-q_j(\underline{a}, \underline{t}))]\}.$$

Since x_i is a limit of the net $\{x_i^{T_{12\ldots B}} : T_{12\ldots B} \in \Sigma_{12\ldots B}\}$, there exists a $T_i \in \Sigma_{12\ldots B}$ such that $x_i^{T_{12\ldots B}} \in U$ for all $x_i^{T_{12\ldots B}} \geq x_i^{T_i}$, where $i \in \{1, \ldots, B\}$. Let $T_0 \in \Sigma_{12\ldots B}$ be such that $T_0 \supseteq T_i$, $i \in \{1, \ldots, B\}$. Then $x_1^{T_0}, x_2^{T_0}, \ldots, x_B^{T_0} \in U$, so $P(\underline{a})$ holds true in Y_{T_0} - a contradiction. $\square$

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