Summary of professional accomplishments

by Paweł Gładki

1. Name and surname: Paweł Gładki.

2. Scientific degrees:

a) Ph.D. in Mathematics, September 15, 2007, Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Canada. Title of the Dissertation: *The pp Conjecture in the Theory of Spaces of Orderings*. Adviser: Prof. Murray Marshall.

b) M.Sc. in Mathematics, June 2, 2002, Institute of Mathematics, University of Silesia, Katowice, Poland. Title of the Master's Thesis: *Riemann Hypothesis for Algebraic Function Fields*. Adviser: Prof. Kazimierz Szymiczek.

3. Academic appointments:

A) Permanent

a) Department of Computer Science, AGH University of Science and Technology, Kraków, Poland; Assistant Professor; 1.X.2010 – present.

b) Institute of Mathematics, University of Silesia, Katowice, Poland; Assistant Professor; 1.X.2009 – present.

B) Visiting

a) Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Canada; Visiting Assistant Professor; 1.IX.2014 – 31.XII.2014.

b) Centre International de Rencontres Mathématiques, Luminy, France; Research in Pairs Scholar; 12.XI.2012 – 25.XI.2012.

c) Mathematisches Forschungsinstitut Oberwolfach, Oberwolfach, Germany; Research in Pairs Scholar; 30.X.2011 – 13.XI.2011.

d) Laboratoire de Mathématiques, Université Savoie Mont Blanc, Chambéry, France; Visiting Scholar; 1.IX.2011 – 30.IX.2011.

e) Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Canada; Visiting Assistant Professor; 1.VII.2010 – 30.IX.2010.

f) Department of Mathematics and Statistics, University of California Santa Barbara, Santa Barbara, USA; Visiting Assistant Professor; 1.X.2007 – 30.IX.2009.

g) Fields Institute for Research in Mathematical Sciences, Toronto, Canada; Stipendee; 1.I.2007 – 30.IV.2007.

4. **Indication of the achievement according to Article 16 Paragraph 2 of the Act of March 14, 2003 on scientific degrees and scientific title and on degrees and title in the field of art (Dz. U. 2016 r. poz. 882 ze zm. w Dz. U. z 2016 r. poz. 1311):**

The indicated scientific achievement consists of a series of seven publications entitled:

Selected applications of hyperalgebras in the algebraic theory of quadratic forms.

4a. List of publications constituting the indicated scientific achievement:

[E1]. P. Gładki, M. Marshall, *Witt equivalence of function fields over global fields,* Trans. Amer. Math. Soc. **369** (2017), 7861 – 7881.

[E2]. P. Gładki, M. Marshall, *Witt equivalence of function fields of curves over local fields,* Comm. Algebra **45** (2017), 5002 – 5013.

[E3]. P. Gładki, *Witt equivalence of fields: a survey with a special emphasis on applications of hyperfields* in: *Ordered Algebraic Structures and Related Topics,* 169 – 185, Contemp. Math. **697**, Amer. Math. Soc., Providence, RI, 2017.

[O1]. P. Gładki, *Orderings of higher level in multifields and multirings*, Ann. Math. Silesianae **24** (2010), 15 – 25.

[O2]. P. Gładki, M. Marshall, *Orderings and signatures of higher level on multirings and hyperfields*, J. K-theory **10** (2012), 489 – 518.

[O3]. P. Gładki, *Root selections and 2^p-th root selections in hyperfields*, Discuss. Math., Gen. Algebra Appl., accepted.

[P1]. P. Gładki, K. Worytkiewicz, *Witt rings of quadratically presentable fields,* Categ. Gen. Algebr. Struct. Appl., accepted.

4b. Description of the abovementioned papers and obtained results:

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1 Introduction and background.

Quadratic forms constitute a large domain of research with roots in classical mathematics and truly remarkable developments over the past few decades. Its origins go back to Euler and Fermat, and at the times of Gauss there already existed a deep theory of quadratic forms with integer coefficients. A new stimulus was provided at the beginning of the 20th century by celebrated 11th and 17th Hilbert's problems announced at the International Congress of Mathematicians in Paris, that were completely resolved by Hasse, Artin and Schreier in the 1920s. Modern theory goes back to the pioneering work of Witt [63], who introduced the notion of what is now called the Witt ring of a field, and by Pfister [49] and Cassels [10], who identified first significant properties of Witt rings: roughly speaking, a Witt ring encodes the theory of symmetric bilinear forms over a given field, therefore explaining the behaviour of the orthogonal geometry build over such a field.

The main tools used to study quadratic forms in this summary are hyperfields, that is algebras resembling fields but with addition allowed to take multiple values: the detailed definitions will be provided below. It is difficult to point to the exact reference of who formally introduced hyperfields to mathematics, but at least in the sense that they are used here, they appeared for the first time in 1956 in the works of Krasner [34] on approximations of valued fields. For the decades that followed, structures with multivalued addition have been better known to computer scientists, due to their applications to fuzzy logic, automata, cryptography, coding theory and hypergraphs (see [16], [17] and [64]), as well as, to some extent, to mathematicians with expertise in harmonic analysis (see [38]). Recently, the hyperstructure theory has witnessed a certain revival in connection with various fields: in a series of papers by Connes and Consani [11], [12], [13], with applications to number theory, incidence geometry, and geometry in characteristic one, in works by Viro [60], [59], with applications to tropical geometry, by Izhakian and Rowen [25] and Izhakian, Knebusch and Rowen [24], with applications to recently introduced algebraic objects such as supertropical algebras, or by Lorscheid [39], [40] to blueprints – these are algebraic objects which aim to provide a firm algebraic foundation to tropical geometry. Jun applied the idea of hyperstructures to generalise the definition of valuations and developed the basic notions of algebraic geometry over hyperrings [27], [28], [29].

Very natural examples of hyperfields are also found in the algebraic theory of quadratic forms. This was first observed by Marshall [43] – his paper, together with some open questions that it contained, sparked the author's interest in hyperalgebras and motivated much of the research discussed here. The seven papers constituting the scientific achievement under consideration illustrate three applications of hyperfields in quadratic forms: the papers **[E1]**, **[E2]** and **[E3]** are concerned with Witt equivalence of fields, the papers **[O1]**, **[O2]** and **[O3]** with higher ordering theory and related concepts in hyperfields and multirings, and the paper **[P1]** with axiomatic theories of quadratic forms. In what follows, we shall discuss them in detail.

Let F be a field of characteristic $\neq 2$ and let V be a finitely dimensional vector space over F. A *quadratic form* q on V is a function $q\colon V \to F$ such that the associated function $b_q\colon V \times V \to F$ defined by

$$b_q(u,v) = \frac{1}{2}[q(u+v) - q(u) - q(v)]$$

is *bilinear*, i.e. linear with respect to each of the two variables, and that

$$q(av) = a^2 q(v),$$

for all $a \in F$, $v \in V$. The pair (V, q) shall be then called a *quadratic space* and the pair (V, b_q) a *bilinear space.* Two vectors $u, v \in V$ are *orthogonal* if $b_q(u, v) = 0$.

Two quadratic spaces (V_1, q_1) and (V_2, q_2) over the same field F are *isometric* provided there exists an isomorphism of vector spaces $\phi: V_1 \to V_2$ such that

$$q_2(\phi(v)) = q_1(v),$$

for all $v \in V_1$, and the two quadratic forms q_1 and q_2 are then called *equivalent*, denoted $q_1 \cong q_2$. For a quadratic form q over V elements of the set $D_F(q)$ of nonzero values of q:

$$D_F(q) = \{a \in F^\times | \exists v \in V\ [a = q(v)]\}$$

are said to be *represented* by q over F. Since $q(av) = a^2 q(v)$, for $a \in F$, $v \in V$, it follows that $D_F(q)$ consists of whole cosets of the multiplicative group $F^\times$ modulo the subgroup $F^{\times 2}$ of nonzero squares. Therefore, $D_F(q)$ can be perceived as a subset of the group $F^\times / F^{\times 2}$ of square classes of F.

For a quadratic space (V, q) the dimension of V is called the *dimension* of q, written $\dim q$. If $\mathcal{B} = (u_1, \ldots, u_n)$ is a basis for V, the matrix $B = [b_q(u_i, u_j)] \in F_n^n$ shall be called the *matrix* of q with respect to $\mathcal{B}$. If B_1 and B_2 are two matrices of q with respect to distinct bases $\mathcal{B}_1$ and $\mathcal{B}_2$, then B_1 and B_2 are necessarily *congruent*, i.e. $B_1 = P B_2 P^T$, where P is a nonsingular matrix – thus, $\det B_1$ and $\det B_2$, if nonzero, lie in the same coset $(\det B) F^{\times 2}$, which is then called the *determinant* of q, written $\det q$. If $\det B = 0$ for some basis $\mathcal{B}$, we take $\det q$ to be 0. A form q is *nonsingular* if $\det q \neq 0$.

For every quadratic form q over a field F with $\operatorname{char} F \neq 2$ there exists a basis $\mathcal{B}$ such that the matrix B of q with respect to $\mathcal{B}$ is diagonal, that is the form q can be *diagonalized*. Such a $\mathcal{B}$ consists of vectors that are pairwise orthogonal. One easily checks that if $v = (x_1, \ldots, x_n)$ is a vector whose coordinates are taken with respect to the basis $\mathcal{B}$, and if $a_1, \ldots, a_n$ are the diagonal entries of the matrix B of q with respect to $\mathcal{B}$, then

$$q(v) = a_1 x_1^2 + \ldots + a_n x_n^2.$$

If q' is a quadratic form equivalent to q, $q' \cong q$, and if q' is diagonalized so that $a_1', \ldots, a_n'$ are the diagonal entries of the matrix B' of q' with respect to a certain basis $\mathcal{B}'$, then, as $B = P B' P^T$, for some $P \in F_n^n$, $\det P \neq 0$, one also readily verifies that a_i and a_i' lie in the same coset modulo $F^{\times 2}$. For these reasons we shall identify the quadratic form q (or, for that matter, the class of quadratic forms equivalent to q) with the formal n-tuple $\langle \overline{a}_1, \ldots, \overline{a}_n \rangle$, where $\overline{a}_i = a_i F^{\times 2}$.

Consider a binary quadratic form $q = \langle \overline{a}_1, \overline{a}_2 \rangle$. In view of the above,

$$D_F(q) = \{\overline{a} \in F^\times / F^{\times 2} | \exists x_1, x_2 \in F\ [a = a_1 x_1^2 + a_2 x_2^2]\},$$

which indicates that the multiplicative group $F^\times / F^{\times 2}$ can be endowed with a certain multivalued additive structure closely related to the theory of quadratic forms. This is, indeed, the case: if $\operatorname{char} F \neq 2$ and $F \neq \mathbb{F}_3, \mathbb{F}_5$, one defines

$$\overline{a}_1 + \overline{a}_2 = D_F(\langle \overline{a}_1, \overline{a}_2 \rangle),$$

for all $\overline{a}_1, \overline{a}_2 \in F^\times / F^{\times 2}$, and if $\operatorname{char} F = 2$ or $F = \mathbb{F}_3$ or $\mathbb{F}_5$, one defines

$$\overline{a}_1 + \overline{a}_2 = \begin{cases} D_F(\langle \overline{a}_1, \overline{a}_2 \rangle) \cup \{\overline{a}_1, \overline{a}_2\}, & \text{if } \overline{a}_1 \neq -\overline{a}_2, \\ F^\times / F^{\times 2}, & \text{if } \overline{a}_1 = -\overline{a}_2, \end{cases}$$

for all $\overline{a}_1$, $\overline{a}_2 \in F^\times / F^{\times 2}$. Denote by $Q(F)$ the group $F^\times / F^{\times 2}$ with the element $\overline{0}$ adjoined, multivalued addition $+$ defined as above for nonzero classes $\overline{a}_1$, $\overline{a}_2 \in F^\times / F^{\times 2}$ and extended naturally to $\overline{0}$ by setting $\overline{0} + \overline{a} = \overline{a}$, and usual multiplication $\cdot$ extended naturally to $\overline{0}$ by $\overline{a} \cdot \overline{0} = \overline{0} \cdot \overline{a} = \overline{0}$. It can be easily verified (**[E1]**, Proposition 2.1) that $Q(F)$ with such operations has the following properties:

(QH1). $(\overline{a} + \overline{b}) + \overline{c} = \overline{a} + (\overline{b} + \overline{c})$, for all $\overline{a}, \overline{b}, \overline{c} \in Q(F)$;

(QH2). $\overline{a} + \overline{b} = \overline{b} + \overline{a}$, for all $\overline{a}, \overline{b} \in Q(F)$;

(QH3). $(\overline{a} \in \overline{b} + \overline{c}) \Rightarrow (\overline{b} \in \overline{a} + (-\overline{c}))$, for all $\overline{a}, \overline{b}, \overline{c} \in Q(F)$;

(QH4). $\overline{a} + \overline{0} = \overline{a}$, for all $\overline{a} \in Q(F)$;

(QH5). $(Q(F) \setminus \{\overline{0}\}, \cdot, \overline{1})$ is a commutative monoid;

(QH6). $\overline{a} \cdot \overline{0} = \overline{0}$, for all $\overline{a} \in Q(F)$;

(QH7). $\overline{a} \cdot (\overline{b} + \overline{c}) \subseteq \overline{a} \cdot \overline{b} + \overline{a} \cdot \overline{b}$, for all $\overline{a}, \overline{b}, \overline{c} \in Q(F)$;

(QH8). $\overline{1} \neq \overline{0}$;

(QH9). every $\neq \overline{0}$ element of $Q(F)$ has a multiplicative inverse.

$Q(F)$ is then called a *quadratic hyperfield* of F and, as the name suggests, is a special example of a *hyperfield*, that is an algebra with multivalued addition $(H, +, -, \cdot, 0, 1)$, where $H \neq \emptyset$, $0, 1 \in H$ and $+ : H \times H \to 2^H$, $- : H \to H$, $\cdot : H \times H \to H$ are functions such that

(H1). $\forall a, b, c \in H\, [(a + b) + c = a + (b + c)]$;

(H2). $\forall a, b \in H\, [a + b = b + a]$;

(H3). $\forall a, b, c \in H\, [(a \in b + c) \Rightarrow (b \in a + (-c))]$;

(H4). $\forall a \in H\, [a + 0 = a]$;

(H5). $(H \setminus \{0\}, \cdot, 1)$ is a commutative monoid;

(H6). $\forall a \in H\, [a \cdot 0 = 0]$;

(H7). $\forall a, b, c \in H\, [a(b + c) \subseteq ab + ac]$;

(H8). $0 \neq 1$;

(H9). $\forall a \in H \setminus \{0\} \exists a^{-1} \in H\, [a \cdot a^{-1} = 1]$.

Note that $a + (b + c) = \bigcup_{x \in b + c} a + x$. As with fields, we shall write $H^\times$ to denote $H \setminus \{0\}$. Following [45], an algebra $(H, +, -, 0)$ satisfying **(H1)** – **(H4)** will be called a (canonical) *hypergroup*, an algebra $(H, +, -, \cdot, 0, 1)$ satisfying **(H1)** – **(H8)** a *multiring*, and an algebra $(H, +, -, \cdot, 0, 1)$ satisfying **(H1)** – **(H6)**, **(H8)** and

(H7'). $\forall a, b, c \in H\, [a(b + c) = ab + ac]$

a *hyperring*. Observe that, by **(H7)** and **(H9)**, every hyperfield satisfies **(H7')**.

Hyperfields form a category with morphisms between H_1 and H_2 defined to be functions $f : H_1 \to H_2$ such that

(M1). $\forall a, b \in H_1\, [f(a + b) \subseteq f(a) + f(b)]$,

(M2). $\forall a, b \in H_1\, [f(ab) = f(a) f(b)]$,

(M3). $\forall a \in H_1\, [f(-a) = -f(a)]$,

(M4). $f(0) = 0$,

(M5). $f(1)=1$.

Hyperfields, although at a first glance a bit exotic, are, in fact, very natural objects that surface already in elementary school mathematics: indeed, consider the hyperfield $Q_2=\{-1,0,1\}$ with usual multiplication, where 0 is the neutral element of commutative addition, and

$$1+1=\{1\},\ \ (-1)+(-1)=\{-1\},\ \ 1+(-1)=\{-1,0,-1\};$$

here "1" can be interpreted as positive reals, "-1" as negative reals, "0" as the number 0, and $+$ as an outcome of addition of two reals with possibly different signs.

With these introductory remarks and definitions out of our way, we can now proceed to discuss the main results the papers constituting the scientific achievement indicated in this summary.

2 Hyperfields and Witt equivalence of fields.

If (V_1,q_1) and (V_2,q_2) are two quadratic spaces, then $(V_1\oplus V_2, q_1\perp q_2)$ with

$$(q_1\perp q_2)(v_1,v_2)=q_1(v_1)+q_2(v_2)$$

is a quadratic space as well, called the *orthogonal sum* of q_1 and q_2. Likewise, $(V_1\otimes V_2,\, q)$ is a quadratic space called the *tensor product* of q_1 and q_2, denoted $q_1\otimes q_2$, where the associated bilinear form b_q is given by

$$b_q(v_1\otimes v_2, w_1\otimes w_2)=b_{q_1}(v_1,w_1)\cdot b_{q_2}(v_2,w_2),$$

for all simple tensors $v_1\otimes v_2,\, w_1\otimes w_2\in V_1\otimes V_2$. If $q_1=\langle\overline{a}_1,\ldots,\overline{a}_n\rangle$ and $q_2=\langle\overline{b}_1,\ldots,\overline{b}_m\rangle$ are diagonalized forms, then

$$q_1\perp q_2=\langle\overline{a}_1,\ldots,\overline{a}_n,\overline{b}_1,\ldots,\overline{b}_m\rangle \text{ and } q_1\otimes q_2=\langle\overline{a}_1\overline{b}_1,\ldots,\overline{a}_1\overline{b}_m,\ldots,\overline{a}_n\overline{b}_1,\ldots,\overline{a}_n\overline{b}_m\rangle.$$

Orthogonal sum and tensor product of nonsingular quadratic forms are nonsingular. A form q is called *isotropic*, if, for some nonzero vector $v\in V$, $q(v)=0$. A simple, yet important, example of a nonsingular isotropic form is the *hyperbolic plane*, that is the 2-dimensional form whose diagonalization is equal to $\langle\overline{1},-\overline{1}\rangle$. If a form q is isotropic, then, for a hyperbolic form h_1 and some quadratic form q_1, $q\cong h_1\perp q_1$; proceeding by induction, we eventually arrive at a decomposition

$$q\cong h_1\perp\ldots\perp h_i\perp q_a,$$

where $h_1,\ldots,h_i$ are hyperbolic planes, and q_a is *anisotropic*, i.e. not isotropic. It turns out that the number i is uniquely defined, and the form q_a is defined uniquely up to isometry – it is thus called the *anisotropic part* of q. If $q_a=0$, the form q is called *hyperbolic*.

Two quadratic forms q and q' are *Witt equivalent*, denoted $q\sim q'$, if their anisotropic parts q_a and q'_a are isometric, $q_a\cong q'_a$. As expected, Witt equivalence is, in fact, an equivalence relation, which turns out to be compatible with orthogonal sum and tensor product, i.e. if $q\sim q'$ and $r\sim r'$, then

$$q\perp r\sim q'\perp r' \text{ and } q\otimes r\sim q'\otimes r'.$$

If $\operatorname{char} F\neq 2$, then classes of Witt equivalence of nonsingular quadratic forms over F with addition and multiplication induced by $\perp$ and $\otimes$ form a commutative ring with identity called *Witt ring* of F and denoted by $W(F)$. If $\operatorname{char} F=2$, a similar construction leads to the notion of *Witt ring of nonsingular symmetric bilinear forms* of F, also denoted by $W(F)$. In this case classes of Witt equivalence of nonsingular quadratic forms fail to form a ring, yet they form an Abelian group denoted $W_q(F)$ which is a $W(F)$-module [6].

A quadratic form $\langle\overline{1},\overline{a}\rangle$, $\overline{a}\in F^{\times}/F^{\times 2}$, is called a *1-fold Pfister form*, and a tensor product of n 1-fold Pfister forms $\langle\overline{1},\overline{a}_1\rangle\otimes\ldots\otimes\langle\overline{1},\overline{a}_n\rangle$, $\overline{a}_1,\ldots,\overline{a}_n\in F^{\times}/F^{\times 2}$, is called a *n-fold Pfister forms*. The Abelian group $I(F)$ generated by Witt equivalence classes of 1-fold Pfister forms is an ideal of the Witt ring $W(F)$, called the *fundamental ideal*. The n-th power $I^n(F)$ of $I(F)$ is generated as an Abelian group by Witt equivalence classes of n-fold Pfister forms.

We say two fields F and E are *Witt equivalent*, denoted $F\sim E$, if $W(F)$ and $W(E)$ are isomorphic as rings. We shall explain in some detail what are the implications of Witt equivalence. Firstly, the situation where quadratic forms over two fields behave in exactly the same way is captured by the following definition:

Definition 2.1. *Two fields F and E of characteristic $\neq 2$ are said to be* equivalent with respect to quadratic forms*, if there exists a pair of bijections $t: F^{\times}/F^{\times 2}\to E^{\times}/E^{\times 2}$ and $T:\mathfrak{C}(F)\to\mathfrak{C}(E)$, where $\mathfrak{C}(F)$ and $\mathfrak{C}(E)$ are sets of equivalence classes of nonsingular quadratic forms over F and E, such that the following four conditions are satisfied:*

i. $T(\langle\overline{a}_1,\ldots,\overline{a}_n\rangle)=\langle t(\overline{a}_1),\ldots,t(\overline{a}_n)\rangle$*, for all* $\overline{a}_1,\ldots,\overline{a}_n\in F^{\times}/F^{\times 2}$,

ii. $\det T(q)=t(\det q)$*, for every nonsingular quadratic form q over F,*

iii. $D_E(T(q))=t(D_F(q))$*, for every nonsingular quadratic form q over F,*

iv. $t(\overline{1})=\overline{1}$ *and* $t(-\overline{1})=-\overline{1}$.

The classical criterion for Witt equivalence by Harrison [23] combined together with a theorem due to Cordes [14] gives the following result:

Theorem 2.2. *For two fields F and E of characteristic $\neq 2$ the following conditions are equivalent:*

1. *F and E are equivalent with respect to quadratic forms,*
2. *there exists a group isomorphism $t: F^{\times}/F^{\times 2}\to E^{\times}/E^{\times 2}$ such that $t(\overline{1})=\overline{1}$, and for all $\overline{a}$, $\overline{b}\in F^{\times}/F^{\times 2}$*
$$\overline{1}\in D_F(\overline{a},\overline{b})\Leftrightarrow\overline{1}\in D_E(t(\overline{a}),t(\overline{b})),$$
3. *$F\sim E$,*
4. *$W(F)/I^3(F)\cong W(E)/I^3(E)$.*

A version of this criterion for the characteristic 2 case is due to Baeza and Moresi [6], where the main argument relies on the observation that the Arason-Pfister Hauptsatz [2] holds in every characteristic.

It follows that Witt equivalent fields can be understood to be fields having the same quadratic form theory. Observe, however, that in view of what has been remarked here about quadratic hyperfields and morphisms of hyperfields, a much simpler formulation of the Harrison-Cordes criterion is possible (**[E1]**, Proposition 3.2):

Theorem 2.3. *Let F and E be any fields. Then $F\sim E$ if and only if their quadratic hyperfields $Q(F)$ and $Q(E)$ are isomorphic as hyperfields.*

The quadratic hyperfield $Q(F)$ thus encodes exactly the same information as the Witt ring $W(F)$. At the same time, it appears to be a much simpler and easier object to understand.

The problem of determining which fields are Witt equivalent turns out to be quite challenging and manageable only when restricted to specific classes of fields and, in fact, is completely resolved only in a few rather special cases. Trivial examples of Witt equivalence include the case of quadratically closed fields, which are all Witt equivalent, their Witt ring being just $\mathbb{Z}/2\mathbb{Z}$, and real closed fields, their Witt ring being $\mathbb{Z}$. A slightly more involved, but still approachable by elementary methods, is the case of finite fields, which are all either Witt equivalent to $\mathbb{F}_3$, if their number of elements is $\equiv 3 \bmod 4$, or to $\mathbb{F}_5$, if their number of elements is $\equiv 1 \bmod 4$ (see, for example, **[E3]**, Theorem 4.3). Local fields are also completely classified with respect to Witt equivalence (see **[E3]**, Theorem 6.1, for a short proof in the non-dyadic case, and [37], Theorem VI.2.29, for the dyadic case) with methods involved in proofs that generally do not exceed the scope of material contained in graduate-level textbooks. The case of global fields is much more involved. Since completions of global fields at their primes are local fields, Witt equivalence of completions of global fields is well-understood. Witt equivalence of global fields was completely resolved by a remarkable local-global principle, whose three different proofs were given by Perlis, Szymiczek, Conner, Litherland [48], and Szymiczek [56], [57], which states that two global fields of characteristic $\neq 2$ are Witt equivalent if and only if their primes can be paired so that corresponding completions are Witt equivalent. Moreover, Baeza and Moresi [6] showed that any two global fields of characteristic 2 are Witt equivalent, and it is not difficult to see that a global field of characteristic 2 is never Witt equivalent to a global field of characteristic different from 2. As a consequence of the local-global principle, it is also possible to provide a complete list of invariants of Witt equivalence for number fields, as shown by Carpenter [8]. Finally, as global fields are either number fields or function fields over finite fields in one variable, in recent years a considerable effort has been made in order to investigate if methods for global fields can be applied to study Witt equivalence of general function fields. The case of function fields in one variable over algebraically closed fields is rather easy (see, for example, **[E3]**, Theorem 9.1), and the case of algebraic function fields in one variable over a real closed field has been relatively recently resolved by Koprowski [33] and Grenier-Boley with Hoffmann [22]. As the next three pieces of puzzless, and somewhat motivated by the author reviewing the paper [22] for Zentralblatt, he and Murray Marshall embarked on the project of investigating function fields over local and global fields, which, so far, resulted in publishing the three papers **[E1]**, **[E2]** and **[E3]** (unfortunately, already after the second author passing in 2015). We shall now discuss their content in some detail.

2.1 The paper [E1].

This is the opening paper in the whole sequence that contains most of the theory and techniques developed, and for that reason will be discussed here most thoroughly. For a field F we adopt the standard notation from valuation theory: if v is a valuation on F, Γ_v denotes the value group, A_v the valuation ring, M_v the maximal ideal, U_v the unit group, F_v the residue field, and $\pi = \pi_v$: $A_v \to F_v$ the canonical homomorphism, i.e., $\pi(a) = a + M_v$. We say v is discrete rank one if $\Gamma_v \cong \mathbb{Z}$.

Next, recall that an *ordering* of a field F is a subset P of $F^\times$ such that $F^\times = P \,\dot{\cup}\, -P$ (disjoint union), $P \cdot P \subseteq P$, $P + P \subseteq P$, where $-P = \{-a| \, a \in P\}$. If P is an ordering of F then $F^{\times 2} \subseteq P$ and P is a subgroup of $F^\times$. Orderings of a field F with $\operatorname{char} F \neq 2$ are in bijective correspondence with hyperfield morphisms $Q(F) \to Q_2$ (recall that Q_2 denotes the three-element hyperfield mentioned earlier), and hence orderings of two Witt equivalent fields are in bijective correspondence as well (see **[E3]**, Theorem 7.1, for an easy proof of this generally well-known fact).

Unfortunately, this is not the case for valuations: although true for particular kinds of fields, including global fields of of characteristic $\neq 2$, simple counterexamples can be produced at hand (see, for example, **[E3]**, Example 7.3). The main result of **[E1]** is an extension of the local-global principle by Perlis, Szymiczek, Conner and Litherland (see **[E1]**, Theorem 7.5) stating, that if function fields F and E over global fields are Witt equivalent, then the corresponding isomorphism of quadratic hyperfields $Q(F)$ and $Q(E)$ induces, in a canonical way, a bijection between the Abhyankar valuations of F and E, whose residue fields are neither finite, nor of characteristic 2. Recall that if F is a function field over k and v is a valuation on F, the Abhyankar inequality asserts that

$$\operatorname{trdeg}(F\colon k) \geq \operatorname{rk}_{\mathbb{Q}}(\Gamma_v/\Gamma_{v|k}) + \operatorname{trdeg}(F_v\colon k_{v|k})$$

where $v|k$ denotes the restriction of v to k. For any abelian group Γ, $\mathrm{rk}_{\mathbb{Q}}(\Gamma) := \dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q})$. We will say the valuation v is *Abhyankar* (relative to k) if $\geq$ in the Abhyankar inequality is replaced with $=$. In this case it is well known that $\Gamma_v / \Gamma_{v|k} \cong \mathbb{Z} \times \ldots \times \mathbb{Z}$ (with $\mathrm{rk}_{\mathbb{Q}}(\Gamma_v / \Gamma_{v|k})$ factors) and F_v is a function field over $k_{v|k}$. Moreover, if v is Abhyankar (relative to k) then $\Gamma_v \cong \mathbb{Z} \times \ldots \times \mathbb{Z}$ (with $\mathrm{rk}_{\mathbb{Q}}(\Gamma_v)$ factors) and F_v is either a function field over a global field or a finite field.

The exact formulation of **[E1]**, Theorem 7.5, and, in particular, the precise explanation of how the abovementioned canonical correspondence is built, involves quite a number of technicalities that are probably too meticulous to include in this summary: these are extensively discussed in **[E1]**. The main tool used in the proof is a combination of **[E1]**, Propositions 4.1 – 4.3, which are suitably built generalizations to hyperfields of a classical theorem by Springer [53], and **[E1]**, Proposition 4.6, which, in turn, is a carefully designed generalization of a method of constructing valuations from certain subgroups of the multiplicative group of a field that is due to Arason, Elman and Jacob ([1], Theorem 2.16).

Although, at a first glance, **[E1]**, Theorem 7.5 may seem rather weak, as it only provides a necessary condition for Witt equivalence, it is a surprisingly useful result due to its applications. For any field F, we define the *nominal transcendence degree* of F by

$$\mathrm{ntd}(F) = \begin{cases} \mathrm{trdeg}(F\colon \mathbb{Q}), & \text{if}\ \mathrm{char}\, F = 0, \\ \mathrm{trdeg}(F\colon \mathbb{F}_p) - 1, & \text{if}\ \mathrm{char}\, F = p. \end{cases}$$

Let F be a function field in n variables over a global field. For $0 \leq i \leq n$ denote by $\nu_{F,i}$ the set of Abyankar valuations v on F with $\mathrm{ntd}(F_v) = i$. Observe that

$$\nu_{F,i} = \nu_{F,i,0} \,\dot{\cup}\, \nu_{F,i,1} \,\dot{\cup}\, \nu_{F,i,2},$$

where

1. $\nu_{F,i,0}$ is the set of valuations of $\nu_{F,i}$ such that $\mathrm{char}\, F_v = 0$,

2. $\nu_{F,i,1}$ is the set of valuations of $\nu_{F,i}$ such that $\mathrm{char}\, F_v \neq 0, 2$,

3. $\nu_{F,i,0}$ is the set of valuations of $\nu_{F,i}$ such that $\mathrm{char}\, F_v = 2$.

Of course, some of the sets $\nu_{F,i,j}$ may be empty. Specifically, if $\mathrm{char}(F) = p$ for some odd prime p then $\nu_{F,i,j} = \emptyset$ for $j \in \{0, 2\}$, and if $\mathrm{char}(F) = 2$ then $\nu_{F,i,j} = \emptyset$ for $j \in \{0, 1\}$. The correspondence of **[E1]**, Theorem 7.5 preserves the sets $\nu_{F,i,j}$. To be more specific, one has the following:

Theorem 2.4. *(**[E1]**, Corollary 8.1) Suppose F, E are function fields in n variables over global fields which are Witt equivalent via a hyperfield isomorphism α: $Q(F) \to Q(E)$. Then for each $i \in \{0, 1, \ldots, n\}$ and each $j \in \{0, 1, 2\}$ there is a uniquely defined bijection between $\nu_{F,i,j}$ and $\nu_{E,i,j}$ such that, if $v \leftrightarrow w$ under this bijection, then α maps $(1 + M_v)\, F^{\times 2} / F^{\times 2}$ onto $(1 + M_w)\, E^{\times 2} / E^{\times 2}$ and $U_v\, F^{\times 2} / F^{\times 2}$ onto $U_w\, E^{\times 2} / E^{\times 2}$.*

In particular, considering the bijection between $\nu_{F,0,0}$ and $\nu_{E,0,0}$ yields the following result:

Theorem 2.5. *(**[E1]**, Corollary 8.2) Let $F \sim E$ be function fields over number fields, with fields of constants k and ℓ respectively. If there exists $v \in \nu_{F,0,0}$ with $F_v = k$ and $w \in \nu_{E,0,0}$ with $E_w = \ell$ then $k \sim \ell$.*

Combining Theorem 2.5 with some standard arguments from algebraic geometry, one can show, in particular, that if F and E are algebraic function fields with global fields of constants k and ℓ of characteristic $\neq 2$ such that F and E have no rational points, then $F \sim E$ implies $k \sim \ell$.

The correspondence of **[E1]**, Theorem 7.5 also yields some interesting quantitive results. If k is a number field, every ordering of k is archimedean, i.e., corresponds to a real embedding $k \hookrightarrow \mathbb{R}$. Let r_1 be the number of real embeddings of k, and r_2 the number of conjugate pairs of complex embeddings of k. Thus $[k\colon \mathbb{Q}] = r_1 + 2\, r_2$. Let

$$V_k = \{r \in k^{\times} \,|\, (r) = \mathfrak{a}^2 \text{ for some fractional ideal } \mathfrak{a} \text{ of } k\}.$$

Clearly V_k is a subgroup of $k^{\times}$ and $k^{\times 2} \subseteq V_k$. In this case the local-global principle for function fields over global fields can be improved in the following sense:

Theorem 2.6. *(**[E1]**, Theorem 8.6) Suppose $F = k(x_1, \ldots, x_n)$ and $E = \ell(x_1, \ldots, x_n)$ where $n \geq 1$ and k and ℓ are number fields, and $\alpha\colon Q(E) \to Q(F)$ is a hyperfield isomorphism. Then*

(1) $r \in k^{\times}/k^{\times 2}$ iff $\alpha(r) \in \ell^{\times}/\ell^{\times 2}$.

(2) The map $r \mapsto \alpha(r)$ defines a hyperfield isomorphism between $Q(k)$ and $Q(\ell)$.

(3) α maps $V_k/k^{\times 2}$ to $V_\ell/\ell^{\times 2}$.

(4)The 2-ranks of the ideal class groups of k and ℓ are equal.

If ℓ is a number field, $[\ell\colon \mathbb{Q}]$ even, and $\ell \neq \mathbb{Q}(\sqrt{-1})$, then, for each integer $t \geq 1$, there exists a number field k such that $k \sim \ell$ and the 2-rank of the class group of k is $\geq t$ [58]. Combining this with Theorem 2.6 yields the following:

Corollary 2.7. *(**[E1]**, Corollary 8.8) For a fixed number $n \geq 1$ and a fixed number field ℓ, $[\ell\colon \mathbb{Q}]$ even, $\ell \neq \mathbb{Q}(\sqrt{-1})$, there are infinitely many Witt inequivalent fields of the form $k(x_1, \ldots, x_n)$, k a number field with $k \sim \ell$.*

The case when $[\ell\colon \mathbb{Q}]$ is odd remains open. Likewise, it is not known, if, for arbitrary fields F and E, $F(x) \sim E(x)$ implies $F \sim E$, or if the assumption in Theorem 2.6 that F is purely transcendental over k is really necessary. Attempts to answer these questions are in the scope of interests of the author.

2.2 The paper [E2].

In this work the authors extend the results of **[E1]** to function fields of curves defined over local fields. The main result of this article is a local counterpart of Theorem 2.4, which states that Witt equivalence of two function fields in one variable over local fields of characteristic $\neq 2$ induces a canonical bijection between certain subsets of Abhyankar valuations of the corresponding fields.

More specifically, let F be any field, and let T be a subgroup of $F^{\times}$. Adopting the well-known terminology from the algebraic theory of quadratic forms, we say that $x \in F^{\times}$ is *T-rigid* if $T + T x \subseteq T \cup T x$, and denoting by

$$B(T) = \{x \in F^{\times} \,|\, \text{either } x \text{ or } -x \text{ is not } T-\text{rigid}\}$$

we will refer to the elements of $B(T)$ as to the *T-basic elements*. If $\pm T = B(T)$, and either $-1 \in T$ or T is additively closed, we shall say that the subgroup T is *exceptional*.

Let F be a function field in one variable over a local field k of characteristic $\neq 2$. Let

1. $\mu_{F,0}$ be the set of valuations v of F such that $(F^{\times}\colon U_v\, F^{\times 2}) = 2$, $2^3 \leq (U_v\, F^{\times 2}\colon (1 + M_v)\, F^{\times 2}) < \infty$ and $B\,((1 + M_v)\, F^{\times 2}) = U_v\, F^{\times 2}$,

2. $\mu_{F,1}$ be the set of valuations v on F such that $(F^{\times}\colon U_v\, F^{\times 2}) = 2$, $(U_v\, F^{\times 2}\colon (1 + M_v)\, F^{\times 2}) = \infty$ and $B\,((1 + M_v)\, F^{\times 2}) = U_v\, F^{\times 2}$,

3. $\mu_{F,2}$ be the set of valuations v on F such that $(F^{\times}: U_v F^{\times 2}) = 4$, $(U_v F^{\times 2}: (1+M_v) F^{\times 2}) = 2$ and $B((1+M_v) F^{\times 2}) = U_v F^{\times 2}$,

4. $\mu_{F,3}$ be the set of valuations v on F such that $(F^{\times}: U_v F^{\times 2}) = 4$, $(U_v F^{\times 2}: (1+M_v) F^{\times 2}) = 2$ and $B((1+M_v) F^{\times 2}) = (1+M_v) F^{\times 2}$.

Of course, some of the sets $\mu_{F,i}$ may be empty. Specifically, $\mu_{F,0} \neq \emptyset$ iff k is dyadic, $\mu_{F,1} \neq \emptyset$ iff k is p-adic, $\mu_{F,2} \cup \mu_{F,3} \neq \emptyset$ iff k is p-adic, $p \neq 2$. Observe that

$$\mu_{F,0} \cup \mu_{F,1} \cup \mu_{F,2} \cup \mu_{F,3}$$

is the set of all Abhyankar valuations of F over k. With these remarks and notation out of our way, we are in position to state the following:

Theorem 2.8. *(**[E2]**, Theorem 3.5) Suppose F, E are function fields in one variable over local fields of characteristic $\neq 2$ which are Witt equivalent via a hyperfield isomorphism $\alpha: Q(F) \to Q(E)$. Then for each $i \in \{0, 1, 2, 3\}$ there is a uniquely defined bijection between $\mu_{F,i}$ and $\mu_{E,i}$ such that, if $v \leftrightarrow w$ under this bijection, then α maps $(1 + M_v) F^{\times 2} / F^{\times 2}$ onto $(1 + M_w) E^{\times 2} / E^{\times 2}$ and $U_v F^{\times 2} / F^{\times 2}$ onto $U_w E^{\times 2} / E^{\times 2}$ for $i \in \{0, 1, 2\}$ and such that α maps $(1 + M_v) F^{\times 2} / F^{\times 2}$ onto $(1 + M_w) E^{\times 2} / E^{\times 2}$ for $i = 3$.*

Contrary to the intuition that one might have developed based on the necessary and sufficient conditions for Witt equivalence of local and global fields, the case of function fields of curves over local fields is in no way easier to settle than the case of function fields of curves over global fields.

Theorem 2.8 is then applied to show that, under certain assumptions, Witt equivalence of two function fields of curves over local fields k and ℓ implies Witt equivalence of k and ℓ. This extends Theorem 2.5 to the local case. More specifically:

Theorem 2.9. *(**[E2]**, Theorem 3.6) Let $F \sim E$ be function fields in one variable over local fields of constants k and ℓ, respectively. Then $k \sim \ell$ except possibly when k, ℓ are both dyadic local fields. In the latter case if there exists $v \in \mu_{F,0}$ with $F_v = k$ and $w \in \mu_{E,0}$ with $E_w = \ell$ then $k \sim \ell$.*

Note that the abovestated theorem provides a partial answer to one of the open problems of **[E1]**.

2.3 The paper [E3].

The paper **[E3]**, despite its title, not only surveys the results of **[E1]**, **[E2]** and [21], but foremostly provides new shorter proofs of some classical theorems of the quadratic form theory obtained using the hyperfield approach. This, by the way, illustrates the strength of the new approach. However, the main reason it appears as one of the publications constituting the scientific achievement of the author is that it outlines the work of [21], which, at the time of preparing this summary, was still under review. Therefore, mostly the results of **[E3]** quoting [21] will be discussed here.

Namely, we deal with Witt equivalence of function fields of conic sections over a field k, $\operatorname{char} k \neq 2$. These are of the form $k_{a,b}$, where $k_{a,b}$ denotes the quotient field of the ring $k[x, y] / (ax^2 + by^2 - 1)$. A slightly more detailed version of Theorem 2.6, tailored for the specific case of function fields of conic sections, can be proven in the following form:

Theorem 2.10. *(**[E3]**, Theorem 10.3, or [21], Theorem 4.4) Suppose F and E are function fields of genus zero curves over number fields with fields of constants k and ℓ respectively, and α: $Q(F) \to Q(E)$ is a hyperfield isomorphism. Then*

1. $r \in k^* / k^{*2}$ iff $\alpha(r) \in \ell^* / \ell^{*2}$;

2. *α induces a bijection between orderings P of k which extend to F and orderings Q of ℓ which extend to E via $P \leftrightarrow Q$ iff α maps P^*/k^{*2} to Q^*/ℓ^{*2};*

3. *α maps V_k/k^{*2} to V_ℓ/ℓ^{*2};*

4. *$[k\colon \mathbb{Q}] = [\ell\colon \mathbb{Q}]$;*

5. *F is purely transcendental over k iff E is purely transcendental over ℓ. In this case, the map $r \mapsto \alpha(r)$ defines a hyperfield isomorphism between $Q(k)$ and $Q(\ell)$, and the 2-ranks of the ideal class groups of k and ℓ are equal.*

In the spirit of Corollary 2.7, we are interested in learning if there are infinitely many Witt inequivalent fields of the form $k_{a,b}$, where k is a number field. Combining Theorem 2.10 with some classical arguments from number theory, as well as old results that were known already to Witt, one gets the following:

Theorem 2.11. *(**[E3]**, Theorem 10.5, or [21], Theorem 4.7) Let k be a number field, r the number of orderings of k, w the number of Witt inequivalent fields of the form $k_{a,b}$, $a, b \in k^\times$. Then*

$$w \geq \begin{cases} 2, & \text{if} -\overline{1} \in D_k(\langle \overline{1}, \overline{1} \rangle), \\ 3, & \text{if} -\overline{1} \notin D_k(\langle \overline{1}, \overline{1} \rangle) \text{ and } k \text{ is not formally real}, \\ r+3, & \text{if } k \text{ is formally real.} \end{cases}$$

Likewise, motivated by Theorem 2.5, we would like to learn when $k_{a,b} \sim \ell_{c,d}$ implies $k \sim \ell$. With this regard, we are able to establish the following:

Theorem 2.12. *(**[E3]**, Theorem 10.6, or [21], Proposition 4.9) Suppose $\alpha\colon Q(\mathbb{Q}_{a,b}) \to Q(\mathbb{Q}_{c,d})$ is a hyperfield isomorphism. Then, for each prime integer p, $\alpha(p) = \pm q$ for some prime integer q, and $p = 2 \Rightarrow q = 2$.*

In fact, using the results obtained for function fields over local fields, one is able to obtain slightly more general results:

Theorem 2.13. *(**[E3]**, Theorem 10.9, or [21], Theorem 4.12) Suppose k, ℓ are local fields of characteristic $\neq 2$, $a, b \in k^*$, $c, d \in \ell^*$. Then $k_{a,b} \sim \ell_{c,d} \Rightarrow k \sim \ell$.*

Theorem 2.14. *(**[E3]**, Theorem 10.10, or [21], Theorem 4.13) Suppose k is a local field of characteristic $\neq 2$, $a, b, c, d \in k^*$. Then $k_{a,b} \sim k_{c,d} \Rightarrow \left(\frac{a,b}{k}\right) = \left(\frac{c,d}{k}\right)$ except possibly in the case when k is p-adic of level 1, for some odd prime p.*

Questions pertaining to Witt equivalence of fields are still vastly open and are definitely in the author's scope of currect scientific interests. In addition to some of the problems mentioned in the above discussion, the author is currently working on extending the results of **[E3]** and [21] from function fields of genus 0 curves to function fields of elliptic curves – due to their elegant, yet complicated, arithmetic, this is a challenging and highly motivating undertaking. It is believed that the use of hyperfields might prove fruitful in settling these questions.

Likewise, no non-trivial examples of two fields, one of characteristic 2 and another of characteristic $\neq 2$, that are Witt equivalent are known as of today. The generalizations of Springer theorem obtained in **[E1]** seem to provide a simple way of describing information contained Witt rings of fields of iterated power series in characteristic $\neq 2$. The author believes that similar methods might be developed for characteristic 2 case, conciveably providing the examples in question.

Finally, it would be desirable to find not only necessary, but also sufficient conditions for Witt equivalence of function fields over global and local fields. This seems to be a tremendously hard problem, but the author believes that it might be possible to be settled for weaker forms of Witt equivalence such as symbol equivalence between fields. As the first step towards achieving this goal, the author would like to provide a hyperfield-theoretic characterization of symbol equivalence.

3 Multirings, hyperfields and orderings of higher level.

The celebrated Hilbert's 17th Problem asked whether a polynomial in n variables with coefficients in $\mathbb{R}$ that is nonnegative on $\mathbb{R}^n$ is necessarily a sum of squares of rational functions in n variables with coefficients in $\mathbb{R}$. A complete solution of this question due to Artin and Schreier [3] laid foundations for what is now called real algebra, and their groundbreaking results have been generalized in a plethora of directions. We recall some basic terminology: for a field F, $\operatorname{char} F \neq 2$, a *preordering* is a subset T of F satisfying

$$T+T \subseteq T, TT \subseteq T, \text{and } a^2 \in T \text{ for all } a \in F.$$

Let $\sum F^2$ denote the set consisting of all finite sums $\sum a_i^2$, $a_i \in F$. It is the unique smallest preordering of F. A preordering T is *proper*, if $-1 \notin T$. An *ordering* of F is a subset P of F satisfying

$$P+P \subseteq P, PP \subseteq P, P \cup -P = F, \text{and } P \cap -P = \{0\}.$$

Every ordering is a preordering. A field is called *formally real* if $-1 \notin \sum F^2$. The fundamental facts of the classical theory of ordered fields can be summarized as follows:

1. if T is a proper preordering, $a \notin T$, and P is a preordering maximal subject to the conditions that $T \subseteq P$ and $a \notin P$, then P is an ordering; the set of all orderings containing a preordering T will be denoted by X_T, and the set$X_{\sum F^2}$ of all orderings of F will be denoted by X_F;

2. for every proper preordering T, one has $T = \bigcap_{P \in X_T} P$;

3. a field F is formally real $\Leftrightarrow$ F admits a proper preordering $\Leftrightarrow$ F admits an ordering.

Corresponding notions to preorderings and orderings exist also for commutative rings with 1 such that 2 is a unit (that, from now on, will be just called rings). Let A be such a ring. *Preorderings* in A are defined exactly in the same way as for fields, i.e. as subsets T of A such that

$$T+T \subseteq T, TT \subseteq T, \text{and } a^2 \in T \text{ for all } a \in A,$$

and *orderings* are subsets P of A such that

$$P+P \subseteq P, PP \subseteq P, P \cup -P = F, \text{and } P \cap -P \text{ is a prime ideal in } A \text{ called the } \textit{support} \text{ of } P.$$

Formally real rings are defined just like formally real fields, and the properties 1. – 3. of preorderings and orderings of fields carry over to rings. The set of all orderings of a ring A is called the *real spectrum* of A and denoted by $\operatorname{Sper}(A)$, and the set of all orderings of A containing a preordering T is denoted by $\operatorname{Sper}_T(A)$. For an element $a \in A$, the *sign* function $\operatorname{sgn}_a\colon \operatorname{Sper}(A) \to \{-1, 0, 1\}$ is defined by

$$\operatorname{sgn}_a(P) = \begin{cases} 1, & \text{if } a \notin -P, \\ 0, & \text{if } a \in P \cap -P, \\ -1, & \text{if } a \notin P. \end{cases}$$

An abstract generalization of Hilbert's 17th Problem, commonly known as *Positivstellensatz*, can be now formulated as follows (see, for example, [44], Theorem 2.5.2):

Theorem 3.1. *Let A be a commutative ring with 1 and invertible 2, let T be a preordering of A, and let $a \in A$. Then*

$$\mathrm{sgn}_a(P) \geq 0, \ \textit{for all } P \in \mathrm{Sper}_T(A) \Leftrightarrow pa = a^{2m} + q, \ \textit{for some } p, q \in A, m \in \mathbb{N}.$$

As previously remarked, preorderings and orderings have been generalized in numerous directions. We shall focus on one of them, namely preorderings and orderings of higher level, that are essentially due to Becker [7]. Here, sums of squares are replaced by sums of 2^n-th powers: more specifically, a *preordering of level* n is a subset T of F such that:

$$T + T \subseteq T, TT \subseteq T, \text{and } a^{2^n} \in T \text{ for all } a \in F,$$

and an *ordering of level* n is a subset P of F such that

$$P + P \subseteq P, P^\times \text{ is a subgroup of } F^\times, P \cup -P = F, \text{ and } F^\times / P^\times \text{ is cyclic with } |F^\times / P^\times| \,|\, 2^n.$$

If $|F^\times / P^\times| = 2^n$, we say that P is of *exact level* n. Likewise, an *n-formally real* field is one where -1 is not a sum of 2^n-th powers. The fundamental properties 1. – 3. of preorderings and orderings carry to preorderings and orderings of level n, and, readily, the theory of orderings of level n with $n = 1$ yields the usual theory of orderings.

Preorderings and orderings of level n can be also defined for rings. The definitions of a preordering of level n for rings and n-formally real rings coincide with the ones for fields, whereas an *ordering of level* n in a ring A is a subset $P \subseteq A$ such that

i. $P + P \subseteq P, PP \subseteq P, \text{and } a^{2^n} \in P \text{ for all } a \in A,$

ii. $P \cap -P = \mathfrak{p}$ is a prime ideal of A,

iii. if $ab^{2^n} \in P$, then $a \in P$ or $b \in P$,

iv. the set

$$\overline{P} = \left\{ \sum_{i=1}^{k} a_i^{2^n} \overline{p}_i |\, a_1, \ldots, a_k \in k(\mathfrak{p}), p_1, \ldots, p_k \in P, k \in \mathbb{N} \right\}$$

is an ordering of level n in the field of fractions $k(\mathfrak{p})$ of the ring $A/\mathfrak{p}$. Here $\overline{p}_i = p_i + \mathfrak{p} \in A/\mathfrak{p}$, $i \in \{1, \ldots, k\}$.

The corresponding properties 1. – 3. for preorderings and orderings of level n, as well as a Positivstellensatz, can be established in this setting.

The theory of orderings is strongly related to the theory of quadratic forms due to the role played by sums of squares in both theories. The latter one, as we have already seen, is closely tied with hyperfields. It is thus natural to ask, if preorderings and orderings can be introduced to hyperfields, multirings and hyperrings, and, in particular, if the properties 1. – 3., as well as the Positivstellensatz, have their counterparts in such a conceivable theory. This is, indeed, the case, and has been done by Marshall in [43]. In the concluding remarks of his paper it is suggested that it would be desirable to construct a theory of orderings of level n parallel to the one by Becker [7] for algebras with multivalued addition. This suggestion motivated a project that resulted in papers **[O1]**–**[O3]**. We shall now discuss them in some detail.

3.1 The paper [O1].

This is the opening paper for the whole sequence, where key definitions are introduced along with basic theorems that correspond to the properties 1. – 3. and the Positivstellensatz discussed above. The terminology used in the paper has slightly changed since the time of its publication, and here we shall stick to the one used nowadays: in particular, what is called a *multifield* in **[O1]**, is now a hyperfield. The definitions of preorderings and orderings of level n for hyperfields and multirings follow closely the ones for fields and rings: if H is a hyperfield, a *preordering* of level n is a subset T of H such that

$$T+T\subseteq T, TT\subseteq T, \text{and } a^{2^n}\in T \text{ for all } a\in H,$$

which is *proper* if $-1\notin T$, an *ordering of level* n is a subset P of H such that

$$P+P\subseteq P, P^{\times} \text{ is a subgroup of } H^{\times}, P\cup -P=H, \text{ and } H^{\times}/P^{\times} \text{is cyclic with } |H^{\times}/P^{\times}|\,|\,2^n,$$

which is of *exact level* n if $|F^{\times}/P^{\times}|=2^n$, and a hyperfield is *$n$-formally real* when -1 is not in a sum of 2^n-th powers. The following two results corresponding to the properties 1. – 3. are given:

Theorem 3.2. *(**[O1]**, Theorem 1) Let H be a hyperfield. The following conditions are equivalent:*

1. *H is formally $n-$ real,*
2. *H admits an ordering of level n,*
3. *H admits a proper preordering of level n.*

Theorem 3.3. *(**[O1]**, Theorem 2) Let H be a hyperfield, $T\subset H$ a preordering of level n. If T is proper, then $T=\bigcap_{P\in X_T}P$.*

The proofs of the abovestated theorems are modifications of the proofs available in the field case. The main obstacle in "translating" these results was that in the field case always $1-1=0$, whereas for hyperfields all that we know is $0\in 1-1$: however, at least in the above two theorems, it was always possible to find a path circumventing this inconvenience.

It begins, however, to be a more serious problem when it comes to considering multirings: the comonly used in the ring case assumption that $2=1+1$ is invertible does not make sense here, as now $1+1$ is a set. Nevertheless, the definitions of preorderings and orderings of level n for multirings can be stated in, more or less, the same form as for rings: a multiring is *n-formally real* when -1 is not in a sum of 2^n-th powers, a *preordering of level* n of a multiring A is a subset T of A such that

$$T+T\subseteq T, TT\subseteq T, \text{and } a^{2^n}\in T \text{ for all } a\in A,$$

which is *proper* if $-1\notin T$, an *ordering of level* n of a multiring A is a subset P of A such that

i. $P+P\subseteq P, PP\subseteq P, \text{and } a^{2^n}\in P \text{ for all } a\in A$,

ii. $P\cap -P=\mathfrak{p}$ is a prime ideal of A,

iii. if $ab^{2^n}\in P$, then $a\in P$ or $b\in P$,

iv. the set

$$\overline{P}=\bigcup \{a_1^{2^n}\overline{p}_1+\ldots+a_k^{2^n}\overline{p}_k|\, a_1,\ldots,a_k\in k(\mathfrak{p}), p_1,\ldots,p_k\in P, k\in\mathbb{N}\}$$

is an ordering of level n in the hyperfield of fractions $k(\mathfrak{p})$ of the multiring $A/\mathfrak{p}$.

Here $\overline{p}_i = p_i + \mathfrak{p} \in A/\mathfrak{p}$, $i \in \{1, \ldots, k\}$, and the notions of ideals, prime ideals, quotients and hyperfields of fractions are defined just like for usual rings, but not without certain setbacks: for example, the canonical morphism from a multiring to its hyperfield of fractions $a \mapsto \frac{a}{1}$ need not be injective.

As notions corresponding to the properties 1. – 3. and the Positivstellensatz above, we have the following two results:

Theorem 3.4. *(**[O1]**, Theorem 4) Let A be a multiring. The following conditions are equivalent:*

1. *A is formally $n-$real with $A = \Sigma A^{2^n} - \Sigma A^{2^n}$,*
2. *A admits an ordering P of level n such that $A = P - P$,*
3. *A admits a proper preordering T of level n such that $A = T - T$.*

Theorem 3.5. *(**[O1]**, Theorem 5) Let A be a multiring, $T \subset A$ a preordering of level n. If T is proper and such that $A = T - T$, then the following conditions are equivalent:*

1. *$a \in \bigcap_{P \in X_T} P$,*
2. *$a\,t \in a^{2nk} + t'$, for some $t, t' \in T$, $k \in \mathbb{N}$.*

Unfortunately, the author was only able to prove the abovestated theorems under the additional assumption that the proper preorderings T under consideration also satisfied the condition $A = T - T$. In the ring case, $A = T - T$ can be easily shown to be equivalent to T being proper, and the proof uses the following arithmetical identity (see [26], Théoréme 8.2.2):

$$k!x = \sum_{h=0}^{k-1} (-1)^{k-1-h} \binom{k-1}{h} [(x+h)^k - h^k],$$

which, clearly, does not hold for multirings, thus disables us from transfering the argument from the ring case to the multiring one.

3.2 The paper [O2].

Unsatisfactory results of the second half of **[O1]** motivated the author to seek for possible ways of eliminating the additional assumption $T - T = A$ in Theorems 3.4 and 3.5. This issue was successfully resolved jointly with Marshall in the paper **[O2]**. For a multiring (or a hyperfield) the *characteristic* is the least n such that $0 \in \underbrace{1 + \ldots + 1}_{n}$, or 0 if no such n exists. The authors managed to establish the following:

Theorem 3.6. *(**[O2]**, Theorems 3.2 and 3.5)*

1. *Let H be a hyperfield, $\operatorname{char} H = 0$, let $n \geq 0$. Then $H = \sum H^{2^n} - \sum H^{2^n}$.*
2. *Let A be a multiring such that for each maximal ideal $\mathfrak{m}$ of A and each $s \in A \setminus \mathfrak{m}$*

$$\left(\bigcup_{k \geq 2} \underbrace{s + \ldots + s}_{k} \right) \cap \mathfrak{m} = \emptyset,$$

let $n \geq 0$. Then $A = \sum A^{2^n} - \sum A^{2^n}$.

The proof is complicated and entirely independent of the field/ring case, but follows a usual routine commonly found in number-theoretical considerations of first establishing the result for hyperfields, then for local multirings, and eventually proceeding to the general case. It is expected that the assumption that char $H = 0$ can be weakened.

For a preordering T of level n a $T-$*module* is a subset $M \subset A$ such that

$$M + M \subseteq M, TM \subseteq M, 1 \in M.$$

If, in addition, $-1 \notin M$, we call M a *proper* $T-$module. As an intermediate step in proving the Positivstellensatz one first shows that a T-module M maximal subject to the condition that $-1 \notin M$ satisfies $M \cup -M = A$. This was done in **[O1]** under the assumption that $A = T - T$, and in **[O2]** the authors manages to prove the following:

Theorem 3.7. *(**[O2]**, Theorem 5.2) Suppose A is a multiring, T is a proper preordering of A of level n, and M is a T-module of A which is maximal subject to $-1 \notin M$. Then $M \cap -M$ is a prime ideal of A and $M \cup -M = A$.*

Using this result, it is possible to provide a Positivstellensatz without the extra assumption of Theorem 3.5 in the following form: for a preordering T of level n of A, we define an equivalence relation $\sim$ on A, called T-*equivalence*, by

$$a \sim b \Leftrightarrow \text{for every } P \in X_T \text{ with } \mathfrak{p} = P \cap -P \text{ either } a, b \in P \text{ or } a, b \notin P \text{ and } \frac{a + \mathfrak{p}}{b + \mathfrak{p}} \in \overline{P},$$

where $\overline{P}$ is the induced ordering of the hyperfield $k(\mathfrak{p})$. We denote the equivalence class of a by $\bar{a}$, so $\bar{a} = \bar{b}$ iff $a \sim b$. We refer to $\bar{a}$ as the *sign* of a on X_T. Write $\bar{a} = 0$ (resp., $\bar{a} \geq 0$, resp., $\bar{a} > 0$) at $(\mathfrak{p}, \mathfrak{P})$ to mean that the image of a in ff$(A/\mathfrak{p})$ is zero, resp., in P, resp., in P but not zero.

Theorem 3.8. *(**[O2]**, Corollary 7.3)*

1. *$\bar{a} = 0$ on X_T iff $-a^{2^\ell k} \in T$ for some $k \geq 0$.*

2. *$\bar{a} > 0$ on X_T iff $-1 \in T - \sum A^{2^\ell} a$.*

3. *$\bar{a} \geq 0$ on X_T iff $-a^{2^\ell k} \in T - \sum A^{2^\ell} a$ for some $k \geq 0$.*

4. *Fix $a \in b^{2^\ell} + c^{2^\ell}$. Then $\bar{b} = \bar{c}$ on X_T iff $-a^{2^\ell k} \in T - \sum A^{2^\ell} b\, c^{2^\ell - 1}$ for some $k \geq 0$.*

In addition to the above, more properties for orderings of higher level are settled in **[O2]**. Firstly, the authors explain how results concerning real ideals extend to real ideals of higher level in multirings (**[O2]**, Propositions 8.1 – 8.5). Secondly, the authors construct a functor (a reflection)

$$A \rightsquigarrow Q_{n-\mathrm{red}}(A)$$

from the category of multirings A satisfying $-1 \notin \sum A^{2^n}$ onto a certain (full) subcategory, called the category of n-real reduced multirings, and characterize n-real reduced multirings as non-zero multirings satisfying the following simple axioms:

1. $a^{2^n + 1} = a$,

2. $a + a\, b^{2^n} = \{a\}$,

3. $a^{2^n}+b^{2^n}$ contains a unique element.

In fact, a little bit more is achieved and the authors construct an n-real reduced multiring $Q_T(A)$ for each proper preordering T of level n of A, and $Q_{n-\mathrm{red}}(A)$ is the multiring obtained from this construction when $T=\sum A^{2^n}$ (**[O2]**, Theorem 9.7). Again, the argument is quite involved and relies heavily on Theorem 3.7. For a hyperfield satisfying axiom 1., it may me proven that axioms 2. and 3. reduce to the single axiom

4. $1+1=\{1\}$

(**[O2]**, Proposition 9.2). As is explained in [43], 1-real reduced hyperfields correspond to spaces of orderings (which will be discussed later in this summary), so it is natural to wonder if n-real reduced hyperfields correspond to the spaces of signatures introduced in [46], [50], [51]. The authors produce an example showing that, in fact, this is not the case, and mention one additional axiom, a certain symmetry property:

$$\text{for all odd integers } 1\leq k\leq 2^n,\ a\in b+c\Rightarrow a^k\in b^k+c^k,$$

which is satisfied by spaces of signatures but is not true for general n-real reduced hyperfields (**[O2]**, Example 10.3 and Proposition 10.4).

3.3 The paper [O3].

The last paper in the sequence deals with the notion of root selections, that is somewhat tangential to preorderings and orderings. Taking into consideration the multiplicative group $F^{\times 2}$ of squares of F, it is somewhat natural to ask when it is possible to define a square root function that behaves reasonably well, that is which is a homomorphism ϕ: $F^{\times 2}\rightarrow F^{\times}$ which maps a square c^2 to c multiplied by a “sign”, that is such that $\phi(c^2)=\omega\, c$, where $\omega^2=1$. This question was first addressed by Waterhouse in [62] and it turns out that the existence of such a homomorphism is closely related to the existence of orderings. Firstly, a homomorphism $\phi\colon F^{\times 2}\rightarrow F^{\times}$ such that $\phi(c^2)=\omega\, c$, where $\omega^2=1$, exists if and only if there is a subgroup R of $F^{\times}$ called *root selection* such that every element of $F^{\times}$ can be uniquely represented as $\omega\, r$ with $\omega^2=1$ and $r\in R$ ([62], Lemma, p. 235). Secondly, a root selection exists if and only if -1 is not a square in F ([62], Theorem 1); since, by classical theorems due to Artin and Schreier [3], an ordering in F exists if and only if -1 is not a sum of squares in F, it follows that root selections exist in every ordered field (but, of course, also outside of them, the simplest example being $\mathbb{F}_q$ with $q\equiv 3 \bmod 4$, so that $-1=q-1\notin\mathbb{F}_q^{*2}$). Therefore a root selection can be perceived as a generalizations of an ordering. Hence a small but neat theory of fields with root selections can be built somewhat parallel to the theory of ordered fields, where issues such as existence of root selections (that we have just briefly outlined), extensions of fields with root selections, and structure of maximal root selection fields (somewhat corresponding to real closed fields) are discussed.

All of this was essentially done by Waterhouse in [62], and the results of his paper were presented by an author's colleague, Prof. Andrzej Sładek, during the last meeting of the Algebra Seminar at the University of Silesia prior to his retirement. Towards the end of his talk, Prof. Sładek encouraged audience to develop a theory of root selections of higher level, and the author embarked on such a project. Indeed, most of the results of [62] generalize in an elegant way to the multiplicative group $F^{\times 2^p}$ of 2^p-th powers of F and lead to the consideration of the existence of a reasonably well behaved 2^p-th root function. In a miniature note [20] by the author, that at the moment of completing this summary is still under review, it has been shown that, for a field F containing the 2^p-th primitive root of unity ω_{2^p}, a homomorphism $\phi\colon F^{\times 2^p}\rightarrow F^{\times}$ such that $\phi(c^{2^p})=\omega_{2^p}^k\, c$, for some $k\in\{1,\ldots,2^p\}$ exists if and only if there is a multiplicative subgroup R of $F^{\times}$, called *2^p-th root selection*, such that every element of $F^{\times}$ can be uniquely represented as $\omega_{2^p}^k\, r$ with $k\in\{1,\ldots,2^p\}$ and $r\in R$ ([20], Lemma 2.1), and that 2^p-th root selections exist if and only if -1 is not a 2^p-th power in F ([20], Theorem 2.4). Therefore a tiny theory parallel to the one of orderings of higher level can be built, and, in particular, questions relevant to the existence of 2^p-th root selections, or to extensions of fields with 2^p-th root selections, or to the structure of maximal 2^p-th root selection fields can be addressed.

In the paper **[O3]** the author adds one more piece to the abovedescribed puzzle and defines root selections and 2^p-th root selections over hyperfields and continues to investigate what parts of the theory for fields can be carried to the hyperfield case. In order to make our presentation more compact, we state all of the results already for 2^p-th root selections, and obtain respective definitions and theorems for "ordinary" root selections as special cases. The result opening the discussion in **[O3]** is the following one:

Theorem 3.9. *(**[O3]**, Lemma 2.1) Let H be a hyperfield and assume that H contains the 2^p-th primitive root of unity ω_{2^p}. A multiplicative homomorphism ϕ from the group $H^{\times 2^p}$ of nonzero 2^p-th powers of H to $H^\times$ such that $\phi(c^{2^p}) = \omega_{2^p}^k c$, for some $k \in \{1, \dots, 2^p\}$, exists if and only if there exists a multiplicative subgroup R of H such that for every element $a \in H^\times$ there exist a unique element $r \in R$ and a unique integer $k \in \{1, \dots, 2^p\}$ such that $a = \omega_{2^p}^k r$.*

A multiplicative subgroup R of $H^\times$ (where H is assumed to contain a 2^p-th primitive root of unity ω_{2^p}) such that for every element $a \in H^\times$ there exist a unique element $r \in R$ and a unique integer $k \in \{1, \dots, 2^p\}$ such that $a = \omega_{2^p}^k r$ shall be called a *2^p-th root selection for H*. In case when $p = 1$ we shall simply call it a *root selection*. The existence of 2^p-th root selections is granted by the following, slightly more general result:

Theorem 3.10. *(**[O3]**, Theorem 2.1) Let H be a hyperfield and assume that H contains the 2^p-th primitive root of unity ω_{2^p}. Let $T \subset H^\times$ be a set of nonzero elements of H. Then there exists a 2^p-th root selection for H containing T if and only if the subgroup $H^{\times 2^p}[T] < H^\times$ generated by T and the group of all 2^p-th powers does not contain -1.*

A necessary and sufficient condition for a 2^p-th root selection to exist now easily follows:

Theorem 3.11. *(**[O3]**, Theorem 2.2) Let H be a hyperfield and assume that H contains the 2^p-th primitive root of unity ω_{2^p}. A 2^p-th root selection for H exists if and only if H does not contain a 2^p-th root of -1.*

For the remaining part of the paper **[O3]**, the author attempts to draw more analogies between the theory of root selections of higher level and the higher level Artin-Schreier theory. First of all, the following observation can be made:

Theorem 3.12. *(**[O3]**, Proposition 3.1) Let H be a formally p-real hyperfield, let P be a proper ordering of exact level p. Then H contains the 2^p-th primitive root of unity ω_{2^p} and $P^\times$ is a 2^p-th root selection.*

As a rather direct corollary from Theorem 3.10 one get the following:

Theorem 3.13. *(**[O3]**, Theorem 3.3) Let H be a hyperfield and assume that H contains the 2^p-th primitive root of unity ω_{2^p}. Let $a \in H^\times$ and assume that H does not contain a 2^p-th root of -1. Then there exists a 2^p-th root selection R such that $a \in R$ if and only if $-a^k \notin H^{\times 2^p}$, for all $k \in \{1, \dots, 2^p - 1\}$.*

In particular:

Theorem 3.14. *(**[O3]**, Theorem 3.4) Let H be a hyperfield and assume that H contains the 2^p-th primitive root of unity ω_{2^p}. Let $a \in H^\times$ and assume that H does not contain a 2^p-th root of -1. Then a belongs to all 2^p-th root selections in H if and only if $a \in H^{\times 2^p}$.*

The papers **[O1]** – **[O3]** contain a number of open problems suggesting directions of further research. Of particular interest is the paper **[O3]** and the notion of root selections. In the field case, root selections are examples of subgroups R of index 2 of the multiplicative group $F^{\times}$ of a field F with $-1 \notin R$. The importance of arbitrary subgroups of index 2 for geometry was observed already a long time ago by Sperner [52], and since then they run under the name *half-orderings*. The author believes that root selections in hyperfields might play a similar role in the recently developed by Jun [28] algebraic geometry over hyperfields. This circle of ideas will be discussed in an upcoming paper.

4 Presentable fields and axiomatizations of quadratic forms.

Already in the mid-1970s algebraists working in the field of quadratic forms made first attempts to approach the Witt theory from axiomatic point of view – as a result, a new branch of the algebraic theory of quadratic forms was born, namely the axiomatic theory of quadratic forms. Objects such as quadratic form schemes by Cordes [14], [15], later investigated also by Carson and Marshall [9], Szczepanik [55], [54], and others, became of interest to mathematicians working in the field. Over the years numerous other axiomatizations of the algebraic theory of quadratic forms were also considered: without going into too many technical details, we remark on the notion of quaternionic maps due to Carson and Marshall [9], abstract Witt rings due to Knebusch, Rosenberg and Ware [31], [32], as well as Marshall [41], strongly represational Witt rings due to Kleinstein and Rosenberg [30], and the theory of special groups by Dickmann and Miraglia [18]. Quadratic form schemes, quaternionic maps, strongly representational Witt rings, and special groups are equivalent descriptions of the same thing. Several equivalent definitions of quadratic form schemes are known, and provided in the paper [36], where the relationships between axioms used in these definitions are also discussed, but essentially a quadratic form scheme is a hyperfield H satisfying the conditions

i. $a^2 = 1$, for all $a \in H^{\times}$, and

ii. if $a \neq -1$, then $1 + a$ is a subgroup of $H^{\times}$.

In particular, by **[E3]**, Proposition 3.2, a quadratic hyperfield $Q(F)$ of a field F is an example of a quadratic form scheme. Since the discovery of all the abovementioned axiomatic theories, it remains an open question if they are not too general – in the language used in this summary, this means if it is possible to find an example of a hyperfield H satisfying conditions i. and ii. above which is not isomorphic to a quadratic hyperfield $Q(F)$, for any field F. It is generally believed that such examples do exist, and some attempts to find them will be discussed later. On the other hand, it is interesting to learn how much of the algebraic theory of quadratic forms over fields can be carried to the abstract case. In particular, the spectacular advances in quadratic forms such as resolving the Milnor conjecture by Voevodsky and his collaborators, have no analogue in the abstract setting. In fact, all the abovementioned theories are lacking basic tools required to approach such problems: there is, for instance, no developed theory of cohomological invariants of abstract quadratic forms.

In author's opinion, it is vastly due to the complicated language used in exitsing axiomatizations, as well as to inability to transfer notions typical for fields to the abstract case: for example, Milnor's K-theory relies heavily on a field arithmetic, which is largely lost when advancing to the abstract setting. For this reason it seems worthwhile to seek for new axiomatizations of quadratic forms, and the paper **[P1]** is the first one in an upcoming sequence on that theme. Here, the authors propose a new way of axiomatizing the theory of quadratic forms, essentially taking as a starting point partial orders that satisfy some additional properties. For that reason the notion of what is called *presentable posets* is introduced. Roughly speaking, the axioms of presentable posets are intended to reflect the behaviour of a pointed powerset ordered by inclusion, where every non-empty set is reached as a supremum of underlying singletons, and idea here is to build an axiomatic theory of quadratic forms by describing the behaviour of value sets of quadratic forms. We shall now describe it in some detail.

4.1 The paper [P1].

Let A be a poset. We shall write $\bigsqcup X$ for the supremum of $X \subseteq A$ and $x \sqcup y$ for $\bigsqcup \{x, y\}$. Let $\mathcal{S}_A$ be the set of A's minimal elements. We shall write $\mathcal{S}_a \stackrel{\text{def.}}{=} \downarrow a \cap \mathcal{S}_A$ for the set of all minimal elements below $a \in A$, and $\mathcal{S}_X \stackrel{\text{def.}}{=} \bigcup_{x \in X} \mathcal{S}_x$ for the set of minimal elements below $X \subseteq A$. A poset $(A, \leqslant)$ is *presentable* if

i. every non-empty subset $X \subseteq A$ admits a supremum;

ii. $\mathcal{S}_a$ is non-empty and $a = \bigsqcup \mathcal{S}_a$ for all $a \in A$;

iii. every minimal element $s \in \mathcal{S}_A$ is *compact* in the following sense: if $Y \subseteq A$ is a nonempty subset and $s \leqslant \bigsqcup Y$, then there is an element $y \in Y$ such that $s \leq y$.

The minimal elements of a presentable poset are called *supercompacts*. The authors introduce presentable monoids, groups, rings and fields: a *presentable monoid* $(M, \leq, 0, +)$ is a pointed presentable poset $(M, \leq, 0)$ with a distinguished supercompact 0 and a suprema-preserving binary addition $+ : M \times M \to M$ such that

i. $a + (b + c) = (a + b) + c$ for all $a, b, c \in M$;

ii. $a + 0 = 0 + a = a$ for all $a \in M$;

iii. $a + b = b + a$ for all $a, b \in M$.

A *presentable group* G is a presentable monoid equipped with a suprema preserving involutive homomorphism $- : G \to G$ called *inversion*, verifying

$$(s \leq t + u) \Rightarrow (t \leq s + (-u))$$

for all $s, t, u \in \mathcal{S}_G$. A *presentable ring* R is a presentable group $(R, \leq, 0, +, -)$ consisting of at least two elements as well as a commutative monoid $(R, \cdot, 1)$, such that the element $1 \in R$ is a supercompact, $\cdot$ is compatible with $\leq$ (i.e. $a \leq$ b implies $a \cdot c \leq b \cdot c$, for all $a, b, c \in R$) and $-$(i.e. $a \cdot (-b) = -(a \cdot b)$, for all $a, b \in R$), distributative with respect to $+$, that $0 \cdot a = 0$, for all $a \in R$, and that $\cdot$ verifies

$$\mathcal{S}_{a \cdot b} \;=\; \{s {\cdot} t | s \in \mathcal{S}_a, t \in \mathcal{S}_b\}$$

for all $a, b \in R$. A presentable ring R such that $\mathcal{S}_R^* = \mathcal{S}_R \setminus \{0\}$ is a multiplicative group will be called a *presentable field.*

Powersets (without the empty subset) of abelian groups, commutative rings with identity and fields ordered by inclusion provide examples of presentable groups, rings and fields (**[P1]**, Examples 4, 14, and 18). Likewise, powersets of hypergroups, multirings and hyperfields also provide examples of presentable groups, rings, and fields (**[P1]**, Examples 16 and 21).

The main objective of the paper is a construction of the Witt rings of a suitably choosen presentable field. For this reason, the authors call a presentable field R *pre-quadratically presentable*, if the following conditions hold

i. $a \leq a + b$ for all $a \in \mathcal{S}_R^*, b \in \mathcal{S}_R$;

ii. $(a \leq 1 - b) \wedge (a \leq 1 - c) \Rightarrow (a \leq 1 - b\,c)$ for all $a, b, c \in \mathcal{S}_R$;

iii. $a^2 = 1$ for all $a \in \mathcal{S}_R \setminus \{0\}$.

An example of a pre-quadratically presentable field is the powerset of a quadratic hyperfield of a field (**[P1]**, Proposition 28). Next, a *form* over a pre-quadratically presentable field R is an n-tuple $\langle a_1, ..., a_n\rangle$ of elements of $\mathcal{S}_R^*$. The relation $\cong$ of *isometry* of forms of the same dimension is given by induction:

- $\langle a\rangle \cong \langle b\rangle$ if and only if $a = b$;
- $\langle a_1, a_2\rangle \cong \langle b_1, b_2\rangle$ if and only if $a_1 a_2 = b_1 b_2$ and $b_1 \leq a_1 + a_2$;
- $\langle a_1, ..., a_n\rangle \cong \langle b_1, ..., b_n\rangle$ if and only if there exist $x, y, c_3, ..., c_n \in \mathcal{S}_R^*$ such that
  - i. $\langle a_1, x\rangle \cong \langle b_1, y\rangle$;
  - ii. $\langle a_2, ..., a_n\rangle \cong \langle x, c_3, ..., c_n\rangle$;
  - iii. $\langle b_2, ..., b_n\rangle \cong \langle y, c_3 ..., c_n\rangle$.

The relation $\cong$ is an equivalence on the sets of all unary and binary forms of a pre-quadratically presentable field R (**[P1]**, Proposition 31), and if it happens to be an equivalence relation for forms of arbitrary dimension, then R is called a *quadratically presentable field*. Again, the powerset of a quadratic hyperfield of a field is an example of a quadraticaly presentable field (**[P1]**, Example 33).

Finally, let $\phi = \langle a_1, ..., a_n\rangle$, $\psi = \langle b_1, ..., b_m\rangle$ be two forms. The *orthogonal sum* $\phi \oplus \psi$ is defined as the form

$$\langle a_1, ..., a_n, b_1, ..., b_m\rangle$$

and the *tensor product* $\phi \otimes \psi$ as

$$\langle a_1 b_1, ..., a_1 b_m, a_2 b_1, ..., a_2 b_m, ..., a_n b_1, ..., a_n b_m\rangle$$

We will write $k \times \phi$ for the form $\underbrace{\phi \oplus ... \oplus \phi}_{k\, times}$.

Theorem 4.1. *(**[P1]**, Proposition 35)*

1. *Let R be a pre-quadratically presentable field. The orthogonal sum and the tensor product of isometric forms are isometric.*
2. *(Witt cancellation) Let R be a quadratically presentable field. If $\phi_1 \oplus \psi \cong \phi_2 \oplus \psi$, then $\phi \cong \psi$.*

Two forms ϕ and ψ will be called *Witt equivalent*, denoted $\phi \sim \psi$, if, for some integers $m, n \geq 0$:

$$\phi \oplus m \times \langle 1, -1\rangle \cong \psi \oplus n \times \langle 1, -1\rangle$$

It is easily verified that $\sim$ is an equivalence relation on forms over R, compatible with the isometry. One also easily checks that Witt equivalence is a congruence with respect to orthogonal sum and tensor product of forms. Denote by $W(R)$ the set of equivalence classes of forms over R under Witt equivalence, and by $\bar{\phi}$ the equivalence class of ϕ. With the operations

$$\bar{\phi} + \bar{\psi} = \overline{\phi \oplus \psi} \qquad \bar{\phi} \cdot \bar{\psi} = \overline{\phi \otimes \psi}$$

$W(R)$ is a commutative ring, having as zero the class $\overline{\langle 1, -1\rangle}$, and $\overline{\langle 1\rangle}$ as multiplicative identity. Not surprisingly, $W(R)$ is called the *Witt ring of* R, and, as one might expect, Witt ring of a powerset of a quadratic hyperfield of F is isomorphic to the Witt ring of F:

Theorem 4.2. *(**[P1]**, Theorem 39) For a field F, $W(\mathcal{P}^*(Q(F)))$ is just the usual Witt ring $W(F)$ of non-degenerate symmetric bilinear forms of F.*

As remarked above, the paper **[P1]** is only the first one in an upcoming sequence devoted to the axiomatization of quadratic forms by means of presentable posets. Currently the authors work on establishing basic category-theoretical properties of presentable algebras, in particular on building tools necessary to develop a version of homological algebra for presentable structures. This is a rather delicate task, as major obstacles occur already at the stage of defining exact sequences of presentable monoids: for instance, there is no apparent characterization of monomorphisms in terms of kernels.

At the same time, the author continues to explore new examples of structures that can be expressed in terms of presentable posets. In particular, hyperfields provide a convenient tool in studying fuzzy structures, and the author believes, that hyperfields can be replaced with presentable fields, presumably leading to simplifying already existing results and obtaining new ones.

Below we list and briefly discuss other papers by the paper that were not included in the list of publications constituting the achievement indicated in this summary. These are divided into four groups. The papers **[PP1]** – **[PP4]** are concerned with the pp conjecture in the theory of spaces of orderings, and constituted the core of the author's PhD dissertation. The papers **[SO1]** – **[SO3]** is a sequence of works devoted to the study of structure of spaces of orderings. The paper **[SL1]** is a joint project with Karim Becher dealing with connections between symbol length and stability index, and, finally, the paper **[FP1]** provides some insights into the recently discovered type of fixed point theorems.

5. Other scientific research achievements.

5a. List of other publications.

[PP1]. P. Gładki, M. Marshall, *The pp conjecture for spaces of orderings of rational conics*, J. Algebra Appl. **6** (2007), 245 – 257.

[PP2]. P. Gładki, M. Marshall, *The pp conjecture for the space of orderings of the field* $\mathbb{R}(x, y)$, J. Pure Appl. Algebra **212** (2008), 197 – 203.

[PP3]. P. Gładki, M. Marshall, *On families of testing formulae for a pp formula,* in Quadratic Forms – Algebra, Arithmetic, and Geometry, 181 – 188, Contemp. Math. **493**, Amer. Math. Soc., Providence, RI, 2009.

[PP4]. P. Gładki, *A note on the pp conjecture for sheaves of spaces of orderings,* Commun. Math. **24** (2016), 1 – 5.

[SO1]. P. Gładki, B. Jacob, *On profinite spaces of orderings,* J. Pure Appl. Algebra **216** (2012), 2608 – 2613.

[SO2]. P. Gładki, B. Jacob, *On quotients of the space of orderings of the field* $\mathbb{Q}(x)$, in Algebra, Logic and Number Theory, 63 – 84, Banach Center Publ. **108**, Inst. Math., Polish Acad. Sci., Warszawa, 2016.

[SO3]. P. Gładki, M. Marshall, *Quotients of index two and general quotients in a space of orderings*, Fund. Math. **229** (2015), 255 – 275.

[SL1]. P. Gładki, K. Becher, *Symbol length and stability index*, J. Algebra **354** (2012), 71 – 76.

[FP1]. P. Gładki, *A note on Kuhlmann's fixed point theorems*, Fixed Point Theory **18** (2017), 565 – 568.

5b. Description of the abovementioned papers and obtained results.

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5 The pp conjecture in the theory of spaces of orderings.

Spaces of orderings have already appeared in this summary in discussion between the lines: for a formally real field F consider a proper preordering T. Observe that $T^{\times}$ is a subgroup of the multiplicatie group $F^{\times}$, since if $t\in T^{\times}$, then $\frac{1}{t}=(\frac{1}{t})^2 t\in T^{\times}$. Denote $G_T=F^{\times}/T^{\times}$. The pair (X_T, G_T) is an example of a space of orderings.

G_T is naturally identified with a subgroup of the group $\{-1,1\}^{X_T}$ of all functions from X_T to $\{-1, 1\}$, with the multiplication defined pointwise: $a\in F^{\times}$ gives a rise to the function $X_T\ni P\mapsto a(P)\in\{-1,1\}$, where

$$a(P)=\begin{cases}1, & \text{if } a\in P\\ -1, & \text{if } a\in -P.\end{cases}$$

This correspondence is a homomorphism with kernel equal to $T^{\times}$.

A *space of orderings* is a pair (X,G), where X is a non-empty set, G is a subgroup of $\{-1,1\}^X$ containing the constant function -1, and satisfying the following three axioms. First of all:

i. $\forall x,y\in X\,[(x\neq y)\Rightarrow\exists a\in G\,(a(x)\neq a(y))]$.

We can view elements of X as characters on G: a natural embedding of X into the character group $\chi(G)$ is obtained by identifying $x\in X$ with the character $a\mapsto a(x)$. Forms in the context of spaces of orderings will be just formal n-tuples of elements of G, and if $a,b\in G$, we define the value set $D(a,b)$ as follows:

$$D(a,b)=\{c\in G\colon\forall x\in X\,(c(x)=a(x)\vee c(x)=b(x))\}.$$

With those remarks we can state the remaining two axioms:

ii. If $x\in\chi(G)$ satisfies $x(-1)=-1$, and if

$$\forall a,b\in\ker x\ \ (D(a,b)\subset\ker x),$$

then x is in the image of the natural embedding $X\hookrightarrow\chi(G)$.

iii. For $a_1,a_2,a_3\in G$, if $b\in D(a_1,c)$ for some $c\in D(a_2,a_3)$, then $b\in D(d,a_3)$ for some $d\in D(a_1, a_2)$.

In X we introduce a natural topology, called the *Harrison topology,* as the weakest topology such that the functions $a\colon X\to\{-1,1\}$, $a\in G$, are continuous, given that $\{-1,1\}$ has the discrete topology. In other words, the sets

$$U(a)=\{x\in X\colon a(x)=1\},\qquad a\in G,$$

are clopen and form a subbasis for the topology on X, and the sets

$$U(a_1,\ldots,a_n)=\bigcap_{i=1}^{n}U(a_i)$$

form a basis for the topology on X. A subset $Y\subset X$ will be called a *subspace* of (X,G), if Y is expressible in the form $\bigcap_{a\in S}U(a)$ for some, not necessarily finite, subset $S\subset G$. For any subspace Y we will denote by $G|_Y$ the group of all restrictions $a|_Y$, $a\in G$. Not surprisingly, the pair $(Y, G|_Y)$ is a space of orderings itself. The *stability index* of (X,G), denoted $\mathrm{stab}(X,G)$, is the least integer k such that every basic set $V\subset X$ (in the Harrison topology) is expressible as $V=U(a_1,\ldots, a_k)$ for some $a_1,\ldots,a_k\in G$.

Recall that a first-order formula in the language L with parameters $\underline{a} = (a_1, \ldots, a_k)$ is said to be *positive primitive* (pp for short), if it is of the form $\exists \underline{t}\, \Psi(\underline{t}, \underline{a})$, where $\underline{t} = (t_1, \ldots, t_n)$, and $\Psi(\underline{t}, \underline{a})$ is a finite conjunction of atomic formulae. For a space of orderings (X, G), a pp formula $P(\underline{a})$ with n quantifiers and k parameters in G is expressible as

$$\exists \underline{t} \bigwedge_{j=1}^{m} p_j(\underline{t}, \underline{a}) \in D(1, q_j(\underline{t}, \underline{a})),$$

where $\underline{t} = (t_1, \ldots, t_n)$, $\underline{a} = (a_1, \ldots, a_k)$, for $a_l \in G$, $l \in \{1, \ldots, k\}$, and $p_j(\underline{t}, \underline{a})$, $q_j(\underline{t}, \underline{a})$ are $\pm$ products of some of the t_i's and a_l's, $i \in \{1, \ldots, n\}$, $l \in \{1, \ldots, k\}$. Numerous important notions in the theory of spaces of orderings are expressible in terms of pp formulae, including the formula "two forms are isometric", or "an element is represented by a form". Suppose that $P(\underline{a})$ is one of these two pp formulae. In both cases the following "local-global principle" is true: if $P(\underline{a})$ holds true in every finite subspace of (X, G), then $P(\underline{a})$ holds true in the whole space (X, G) (while speaking of the formula $P(\underline{a})$ in a subspace Y, we mean the formula obtained from $P(\underline{a})$ by replacing each atom $p \in D(1, q)$ by $p|_Y \in D_Y(1, q|_Y)$). In view of the above observations, it is natural to ask the following question, now known as the *pp conjecture*:

For a space of orderings (X, G), is it true that a pp formula $P(\underline{a})$ with parameters $\underline{a}$ in G, which holds in every finite subspace of (X, G), necessarily holds in (X, G)?

The series of papers **[PP1], [PP2], [PP3]** and **[PP4]** is concerned with the pp conjecture for various examples of spaces of orderings. We shall discuss them in some detail.

5.1 The paper [PP1].

The papers **[PP1], [PP2], [PP3]** constituted the core of author's doctoral dissertation. The first paper in the sequence, **[PP1]**, is perhaps the most interesting and provides a negative solution to the pp conjecture by giving an example of a space of orderings and a pp formula which is satisfied in every subspace, but fails in general. More specifically, the following theorem is proven:

Theorem 5.1. *(**[PP1]**, Theorem 6) Let $(X_{\sum F^2}, G_{\sum F^2})$ be the space of orderings of the function field F of a conic section $\mathcal{C}\colon ax^2 + by^2 + c = 0$, $a, b, c \in \mathbb{Q}$, with $a > 0$, $b > 0$, $c < 0$, which contains no points with rational coordinates. There exists a pp formula that holds in every finite subspace of $(X_{\sum F^2}, G_{\sum F^2})$, but fails in $(X_{\sum F^2}, G_{\sum F^2})$.*

The pp formula under consideration is given explicitely. Let $\mathcal{C}\colon ax^2 + by^2 + c = 0$ be the conic section of Theorem 5.1 (for example, take $x^2 + y^2 - 3 = 0$) and consider the six linear irreducibles $p_1, \ldots,$ p_6 of the coordinate ring of $\mathcal{C}$ that intersect with $\mathcal{C}$ as follows:

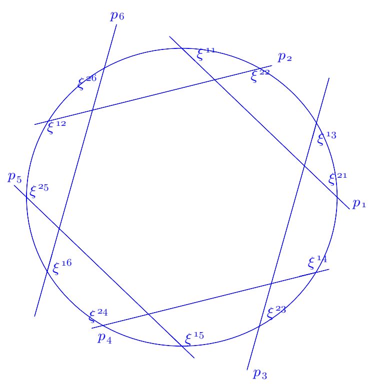


Fig. 1

Here ξ^{1i}, ξ^{2i} denote the two real points of intersection of p_i with $\mathcal{C}$, $i \in \{1, \ldots, 6\}$, and are arranged in the above order. Replacing p_i by $-p_i$ we may assume that every p_i is positive at the origin. Let $a_1 = p_1 p_6$, $a_2 = p_1 p_4$, $d = -p_1 p_2 p_3 p_5$. The formula that disproves the pp conjecture is then:

$$P(\underline{a}) = \exists t_1 \exists t_2 \left(t_1 \in D(1, a_1) \wedge t_2 \in D(1, a_2) \wedge d\, t_1 t_2 \in D\left(1, a_1 a_2\right)\right).$$

The proof is very technical and quite involved. Moreover, by suitably modifying the above example, it is also possible to show that the pp conjecture fails to hold for the space of orderings of the function field of two parallel lines over $\mathbb{Q}$ with no rational points:

Theorem 5.2. *(**[PP1]**, Theorem 9) Let $(X_{\sum F^2}, G_{\sum F^2})$ be the space of orderings of the function field F of a conic section $\mathcal{C}$: $x^2 + c = 0$, $c \in \mathbb{Q}$, with $c < 0$, which contains no points with rational coordinates. There exists a pp formula that holds in every finite subspace of $(X_{\sum F^2}, G_{\sum F^2})$, but fails in $(X_{\sum F^2}, G_{\sum F^2})$.*

For every space of orderings of a conic section other than the ones mentioned in Theorems 5.2 and 5.2 the pp conjecture holds. Thus, the paper **[PP1]** provides a complete classification of conic sections with respect to the pp conjecture.

5.2 The paper [PP2].

This is a continuation of the work of **[PP1]**. The main theorem is:

Theorem 5.3. *(**[PP2]**, Theorem 1) The pp conjecture fails for some formula in the space of orderings $(X_{\sum F^2}, G_{\sum F^2})$, where $F = \mathbb{R}(x, y)$.*

Again, the pp formula is given explicitely, and the technique of the proof mimics the one of Theorem 5.2. For $\epsilon > 0$ consider the subspaces

$$X_\epsilon = U\left(x^2 + y^2 - 1\right) \cap U\left(1 + \epsilon - x^2 - y^2\right)$$

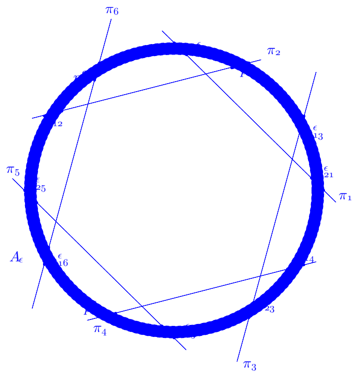
and let $G_\epsilon = G_{\sum \mathbb{R}(x,y)^2}|_{X_\epsilon}$. Define the subspace

$$X = \bigcap_{\epsilon > 0} X_\epsilon$$

and let $G = G_{\sum \mathbb{R}(x,y)^2}|_X$. One shows that it is sufficient to show that the conjecture fails in the space (X, G). For $\epsilon > 0$ denote

$$A_\epsilon = \{(a, b) \in \mathbb{R}^2 \colon 1 < a^2 + b^2 < 1 + \epsilon\}$$

and let $\pi_1, \ldots, \pi_6 \in \mathbb{R}(x, y)$ be linear irreducibles which, for ϵ small enough, intersect with rings A_ϵ as in the following diagram:



Here p_{1i}^{ϵ}, p_{2i}^{ϵ} denote the two connected components of $\mathcal{Z}(\pi_i) \cap A_{\epsilon}$, $i \in \{1, ..., 6\}$, $\epsilon > 0$, and are arranged in the above order, where $\mathcal{Z}(\pi_i)$ is the set of real zeros of π_i. Replacing π_i by $-\pi_i$ we may assume that every π_i is positive at the origin. For two adjacent line segments $p_{i_1 j_1}^{\epsilon}$ and $p_{i_2 j_2}^{\epsilon}$, $i_1, i_2 \in \{1, 2\}$, $j_1, j_2 \in \{1, ..., 6\}$, denote also by $A_n^{i_1 j_1, i_2 j_2}$ the ring sector starting at $p_{i_1 j_1}^{\epsilon}$ and, when moving clockwise along A_{ϵ}, ending at $p_{i_2 j_2}^{\epsilon}$.

Let $a_1 = \pi_1 \pi_6$, $a_2 = \pi_1 \pi_4$ and $d = -\pi_1 \pi_2 \pi_3 \pi_5$. Consider the following pp formula:

$$P(a_1, a_2, d) = \exists t_1 \exists t_2 \, (t_1 \in D(1, a_1) \wedge t_2 \in D(1, a_2) \wedge d\, t_1 t_2 \in D\,(1, a_1 a_2)).$$

One then shows that $P(a_1, a_2, d)$ fails to hold in the space (X, G).

5.3 The paper [PP3].

In this paper, for a pp formula $P(\underline{y})$, a special family of formulae $\mathcal{F}_P$ is defined and it is shown how this family can be used to test whether the pp formula fails on a finite subspace of every space of orderings. We work with a fixed pp formula

$$P(\underline{y}) = \exists \underline{t} \bigwedge_{j=1}^{m} \theta_j(\underline{t}, \underline{y}),$$

where θ_j are atomic formulae and $\underline{y} = (y_1, ..., y_k)$, $\underline{t} = (t_1, ..., t_n)$ are tuples of individual variables. Denote

$\mathbb{K}_P = \{(Y, H, \underline{b}) | \, (Y, H)$ is a finite subspace $, \underline{b} \in H^k, P(\underline{b})$ fails in (Y, H) and holds in every proper subspace of $(Y, H)\}$.

Let $\underline{x} = (x_1, ..., x_l)$ be a tuple of free variables. We build a new formula $P_l(\underline{y}, \underline{x})$ by induction on l. If $l = 1$, we define $P_1(\underline{y}, x_1)$ by replacing each atomic formula $z_1 \in D(z_2, z_3)$ in $P(\underline{y})$ with

$$\exists s_1 \exists s_2 \, [(s_1 \in D(1, x_1)) \wedge (s_2 \in D(1, x_1)) \wedge (z_1 \in D\,(s_1 z_2, s_2 z_3))].$$

If $l \geq 2$, we define $P_l(\underline{y}, (x_1, ..., x_l))$ by performing the above action on $P_{l-1}(\underline{y}, (x_1, ..., x_{l-1}))$.

One sees that for each space of orderings (X, G) and for each subspace of the form $U(b_1, ..., b_l)$, $b_1, ..., b_l \in G$, if $\underline{a} \in G^k$ then $P_l(\underline{a}, \underline{b})$ holds in (X, G) if and only if $P(\underline{a})$ holds in the subspace $U(b_1, ..., b_l)$.

Let $\lambda \geq 1$ be an integer. We shall construct a sequence of formulae $P_{\lambda}^{(i)}(\underline{y})$, $i \geq 0$, by induction. For $i = 0$ we define $P_{\lambda}^{(0)}(\underline{y}) = P(\underline{y})$. For $i = 1$, we define

$$P_{\lambda}^{(1)} = \exists z_0 ... \exists z_{\lambda} \bigwedge_{j=1}^{\lambda} \, [(z_{j-1} \in D(1, z_j)) \wedge P_1\,(\underline{y}, z_{j-1} z_j)],$$

and for $i \geq 2$ we define $P_{\lambda}^{(i)}(\underline{y})$ by performing the above action on $P_{\lambda}^{(i-1)}(\underline{y})$ instead of $P(\underline{y})$. Note that this construction depends on λ.

Trivially, for every space of orderings $(X,\ G)$ and every $\underline{a} \in G^k$, $P(\underline{a}) \Rightarrow P_{\lambda}^{(1)}(\underline{a})$ (by taking $z_0 = z_1 = ... = z_{\lambda} = 1$) and, consequently, $P(\underline{a}) \Rightarrow P_{\lambda}^{(1)}(\underline{a}) \Rightarrow ... \Rightarrow P_{\lambda}^{(i)}(\underline{a})$.

Define the number

$$c_P = \max \{\mathrm{cl}(X, G) \colon (X, G, \underline{a}) \in \mathbb{K}_P\},$$

where $\mathrm{cl}(X, G)$ denoted the *chain length* of the space (X, G), that is the biggest integer d such that there exists a chain $U(a_0) \subsetneq U(a_1) \subsetneq ... \subsetneq U(a_d)$, $a_0, a_1, ..., a_d \in G$, or ∞ if no such finite integer exists.

One shows that it is well-defined. The main result of **[PP3]** is then:

Theorem 5.4. *(**[PP3]**, Theorem 2) Let $\lambda > c_P$, let (X, G) be a space of orderings, let $\underline{a} \in G^k$. The following two conditions are equivalent:*

1. *$P(\underline{a})$ fails in some finite subspace of (X, G);*
2. *for every $i \geq 0$ the formula $\neg P_\lambda^{(i)}(\underline{a})$ holds in (X, G).*

The paper also contains a concrete example of how these formulae are being built, starting with the pp formula that was used to disprove the pp conjecture (**[PP3]**, Example 1).

5.4 The paper [PP4].

As the last paper dealing with questions pertinent to the pp conjecture, the author published the miniature note **[PP4]** that contains one neat result. Eventhough the pp conjecture has been disproven, it is still interesting to investigate numerous examples where it is valid. One of such examples is a sheaf of spaces of orderings: assume (X_i, G_i) are spaces of orderings for each $i \in I$, where I is a Boolean space. Assume further that $X = \dot{\bigcup}_{i \in I} X_i$, the disjoint union of X_i's, is equipped with a topology such that

i. X is a Boolean space,

ii. the inclusion map $\iota_i: X_i \hookrightarrow X$ is continuous, for each $i \in I$,

iii. the projection map $\pi: X \to I$ is continuous, and

iv. if $(i_\lambda)_{\lambda \in D}$ is any net in I converging to $i \in I$ and if $\sigma_1^\lambda, \sigma_2^\lambda, \sigma_3^\lambda, \sigma_4^\lambda$ is a 4-element fan in X_{i_λ} such that σ_j^λ converges to $\sigma_j \in X_i$ for each $j = 1, 2, 3, 4$, then $\sigma_1 \sigma_2 \sigma_3 \sigma_4 = 1$.

Here a *fan* is a pair (X, G), where G is a group of exponent 2 containing the distinguished element -1, and X is the set of all characters x of the group G such that $x(-1) = -1$.

Furthermore, let

$$G := \{\phi \in \mathrm{Cont}(X, \{\pm 1\}) : \phi|_{X_i} \in G_i \, \forall \, i \in I\}$$

where G_i is identified with its image in $\mathrm{Cont}(X_i, \{\pm 1\})$, $i \in I$, under the natural embedding. Then (X, G) is a space of orderings called the *sheaf of spaces* (X_i, G_i) over the Boolean space I. The main result of **[PP4]** is the following:

Theorem 5.5. *(**[PP4]**, Theorem 1) If the pp conjecture holds in (X_i, G_i), for all $i \in I$, then it also holds in the sheaf (X, G) of (X_i, G_i) over the Boolean space I.*

This theorem has been already proven by Astier [5] who used some rather advances tools from model theory, whereas the proof given in **[PP4]** is strictly elementary.

6 The structure of spaces of orderings.

The sequence of papers **[SO1]**, **[SO2]** and **[SO3]** deals with inverse limits and quotients of spaces of orderings. We shall start with explaining these concepts. By a *morphism* F from a space of orderings (X_1, G_1) to a space of orderings (X_2, G_2) we understand a function $F: X_1 \to X_2$ such that

$$\forall b \in G_2 \, (b \circ F \in G_1).$$

A morphism F: $(X_1, G_1) \rightarrow (X_2, G_2)$ defines a group homomorphism F^*: $G_2 \rightarrow G_1$ given by $F^*(b) = b \circ F$ which also satisfies the condition

$$\forall b_1, b_2, b_3 \in G_2 \, [(b_1 \in D_{X_2}(b_2, b_3)) \Rightarrow (F^*(b_1) \in D_{X_1}(F^*(b_2), F^*(b_3)))].$$

Clearly, a bijective morphism will be called an *isomorphism*, and we shall write $(X_1, G_1) \cong (X_2, G_2)$ to indicate that the two spaces of orderings are isomorphic.

An *inverse system* of spaces of orderings is a triple consisting of:

i. a directed set $(I, \succeq)$,

ii. spaces of orderings (X_i, G_i), $i \in I$, and

iii. morphisms F_{ij}: $(X_i, G_i) \rightarrow (X_j, G_j)$ defined for $i \succeq j$, $i, j \in I$, such that

  (a) $F_{ij}(X_i) = X_j$, which implies that F_{ij}^*: $G_j \rightarrow G_i$ is injective, and

  (b) $F_{ik} = F_{jk} \circ F_{ij}$, for $i \succeq j \succeq k$, $i, j, k \in I$.

Clearly, an inverse system $(I, (X_i, G_i), F_{ij})$ of spaces of orderings automatically defines both a direct system of groups (I, G_i, F_{ij}^*), and an inverse system of character sets (I, X_i, F_{ij}). Further, if we let $G = \varinjlim G_i$, and $X = \lim X_i$, then (X, G) is a space of orderings that is called the *inverse limit* of the given inverse system and denoted by $\lim(X_i, G_i)$. For a fixed $j \in I$ we will denote by π_j the projection π_j: $X \rightarrow X_j$ such that $\pi_j = F_{ij} \circ \pi_i$, for $i \succeq j$, $i \in I$, and by γ_j the injection γ_j: $G_j \rightarrow G$ such that $\gamma_j = \gamma_i \circ F_{ij}^*$, for $i \succeq j$, $i \in I$. Since, in fact, $G = \bigcup_{i \in I} G_i$, we will use the same symbol a for an element $a \in G_i$ and its image $a \in \gamma_i(G_i) \subset G$. A space of orderings which is an inverse limit of finite spaces of orderings will be called *profinite*.

If (X, G) is a space of orderings, and G_0 is a subgroup of G, we denote by X_0 the set of all characters from X restricted to G_0. In the case when (X_0, G_0) is a space of orderings, we call it a *quotient space* of (X, G) – otherwise, in general, we call it a *quotient structure*. Determining when a quotient structure is a quotient space is, in general, hard.

We have already remarked before about the stability index and fans. The stability index of (X, G) can be equivalently defined as the maximal integer n such that is (X, G) exists a subspace that is a fan with 2^n elements (or ∞ if no such number exists).

We have also mentioned the realization problem for axiomatic theories of quadratic forms. It is also pertinent to the case of spaces of orderings, and up to this day it is not known if there are examples of spaces of orderings that are not isomorphic to spaces $(X_{\sum F^2}, G_{\sum F^2})$, where F is a field (although it is generally believed that such examples do exist). The constructions of quotient spaces and inverse limits provide a conceivable tool in building such examples. Namely, in author's opinion, it seems that the tool that might be used in determining whether a space of orderings is realizable or not is the following question: is it true that for a space of orderings (X, G) the equality

$$W(X, G) \cap \mathcal{C}(X, 2^n\mathbb{Z}) = I^n(X, G)$$

holds true? Here $\mathcal{C}(X, 2^n\mathbb{Z})$ denotes all continuous functions $X \rightarrow 2^n\mathbb{Z}$, and $I^n(X, G)$ is the $n-$th power of the fundamental ideal $I(X, G)$ of the Witt ring of the space (X, G) (which, in turn, resembles the usual Witt ring, we will, however, omit the technical details of its construction). This question is usually referred to as Lam's Open Problem B. It can be easily verified when $n = 1$ or $n = 2$, and for all spaces of orderings (X, G) such that $\mathrm{sta}(X, G) \leq 3$ [42, Proposition 3.1]. Furthermore, the question has an affirmative answer for realizable spaces of orderings, which has been recently proven by Dickmann and Miraglia in [19] using the celebrated results by Orlov, Vishik and Voevodsky [47], [61].

Therefore, in order to prove that there is a space of orderings, that is not realizable, one would like to construct a space of orderings (X, G) which has at least $16-$element fans, and a quadratic form $\phi = ((a_1, b_1)) \oplus \ldots \oplus ((a_s, b_s))$, $s \in \mathbb{N}$, such that $\phi(x) \equiv 0 \mod 8$, for all $x \in X$, although

$$\phi \neq c_1((d_1, e_1, f_1)) \oplus \ldots \oplus c_t((d_t, e_t, f_t)),$$

for all possible choices of $t \in \mathbb{N}$, and $d_i, e_i, f_i \in G$, $i \in \{1, \ldots, t\}$.

This observation sparked the author's interest in methods of "blowing up" existing fans in well understood spaces of orderings, and thus obtaining examples of new spaces of higher stability index, where Lam's problem needs to be verified. This resulted in the series of papers **[SO1]**, **[SO2]** and **[SO3]**. We shall discuss them in some detail now.

6.1 The paper [SO1].

The opening paper of the whole sequence contains two results that are worth mentioning here. Firstly, the following is established:

Theorem 6.1. *(**[SO1]**, Theorem 1) The space of orderings* $(X_{\sum \mathbb{Q}(x)^2}, G_{\sum \mathbb{Q}(x)^2})$ *is profinite.*

It was somewhat of an open problem, and settling it was quite rewarding, however full strength of this result is revealed in its proof, which is purely geometric and gives a deep insight into the structure of the space of orderings of the field $\mathbb{Q}(x)$.

Secondly, the following theorem is proven:

Theorem 6.2. *(**[SO1]**, Theorem 2) If* $(X, G) = \lim(X_i, G_i)$*, for some inverse system of spaces of orderings* $(I, (X_i, G_i), F_{ij})$*, and if, for all* $i \in I$*, the pp conjecture holds in* (X_i, G_i)*, then it also holds in* (X, G)*.*

This theorem had been previously proven by Astier and Tressl [4], who used rather complicated methods from model theory, whereas the proof given here is elementary.

6.2 The paper [SO2].

The entire paper is devoted to the proof of one single theorem:

Theorem 6.3. *(**[SO2]**, Theorem 9) Let* $G_0 \subset G_{\sum \mathbb{Q}(x)^2}$ *be a subgroup of* $G_{\sum \mathbb{Q}(x)^2}$ *with* $-1 \in G_0$*, and let* $(G_{\sum \mathbb{Q}(x)^2} : G_0) = 2$*. If* (X_0, G_0) *is a space of orderings, where* $X_0 = X|_{G_0}$*, then* (X_0, G_0) *is profinite.*

The proof is a modified version of the method used for the proof of Theorem 6.1. It is extremely complicated and takes the reader through 20 pages of tedious geometric considerations, although fairly elementary in their nature. This ordeal convinced the authors that the methods developed for the proof of Theorem 6.1 are too complicated to apply to more advanced examples.

6.3 The paper [SO3].

The search for new methods in establishing which quotient structures are quotient spaces resulted in the paper **[SO3]**. A few interesting observations have been made that, in principle, generalize the results of **[SO1]** and **[SO2]**. Let (X, G) be a space of orderings. For $S \subseteq G$, $\langle S \rangle$ denotes the subgroup of G generated by S. For $S \subseteq \chi(G)$,

$$S^{\perp} := \{g \in G : \sigma(g) = 1 \forall \sigma \in S\} \text{ and } \langle S \rangle := \chi(G / S^{\perp}),$$

the closed subgroup of $\chi(G)$ generated by S. We say that (X, G) is the *direct sum* of the spaces of orderings (X_i, G_i), $i \in \{1, \ldots, n\}$, denoted $(X, G) = \coprod_{i=1}^{n} (X_i, G_i) = (X_1, G_1) \sqcup \ldots \sqcup (X_n, G_n)$, if X is the disjoint union of the sets $X_1, \ldots, X_n$, and G consists of all functions $a : X \to \{-1, 1\}$ such that $a|_{X_i} \in G_i$, $i \in \{1, \ldots, n\}$. In this case $G = G_1 \oplus \ldots \oplus G_n$, with the role of the distinguished element -1 played by $(-1, -1, \ldots, -1)$. Further, we say that (X, G) is a *group extension* of the space of orderings $(\bar{X}, \bar{G})$, if G is a group of exponent 2, $\bar{G}$ is a subgroup of G, and $X = \{x \in \chi(G) : x|_{\bar{G}} \in \bar{X}\}$. Since G decomposes as $G = \bar{G} \times H$, we shall also write $(X, G) = (\bar{X}, \bar{G}) \times H$ to denote group extensions. Both direct sums and group extensions are spaces of orderings.

Let (X, G) be a space of orderings and (X_0, G_0) a quotient structure of (X, G). We search for necessary and sufficient conditions on G_0 for (X_0, G_0) to be a quotient space of (X, G). Assume now that G_0 is a subgroup of index 2 in G, $-1 \in G_0$. Since G_0 has index 2 and $-1 \in G_0$, G_0 is determined by a character on $G/\{\pm 1\}$, i.e., there exists a unique $\gamma \in \chi(G)$, $\gamma(-1) = 1$ such that $G_0 = \ker(\gamma)$. The result opening the discussion in **[SO3]** is the following one:

Theorem 6.4. *(**[SO3]**, Theorem 2.2)*

(i) Suppose (X_0, G_0) is a quotient space of (X, G), $(Y, G/Y^{\perp})$ is a subspace of (X, G), $\gamma \in \langle Y \rangle$ and $Y_0 = Y|_{G_0}$. Then $(Y_0, G_0/Y^{\perp})$ is a quotient space of $(Y, G/Y^{\perp})$.

(ii) Suppose (X, G) is a group extension of (X', G') and $\gamma' = \gamma|_{G'}$.

If $\gamma' = 1$, then (X_0, G_0) is a quotient space of (X, G).

If $\gamma' \neq 1$, (X_0, G_0) is a quotient space of (X, G) iff (X_0', G_0') is a quotient space of (X', G'). Here, $G_0' := \ker(\gamma')$, $X_0' := X'|_{G_0'}$.

With this result, one is able to show the following:

Theorem 6.5. *(**[SO3]**, Theorem 2.4) A necessary condition for (X_0, G_0) to be a quotient of (X, G) is that $\gamma \in X^4$.*

Here $X^k := \{\prod_{i=1}^k \sigma_i : \sigma_i \in X, i = 1, \ldots, k\}$. Since the σ_i are not required to be distinct, $\{1\} \subseteq X^2 \subseteq X^4$. If (X, G) is a fan then $X^4 = X^2 = \{\gamma \in \chi(G) : \gamma(-1) = 1\}$. It turns out, that there are cases when the above criterion is sufficient:

Theorem 6.6. *(**[SO3]**, Theorem 2.5) If $\operatorname{stab}(X, G) = 1$, then the necessary condition in Theorem 6.5 is also sufficient.*

For a space of orderings (X, G) we define the *connectivity* relation $\sim$ as follows: if $x_1, x_2 \in X$, then $x_1 \sim x_2$ if and only if either $x_1 = x_2$ or there exists a four element fan V in (X, G) such that x_1, $x_2 \in V$. The equivalence classes with respect to $\sim$ are called the *connected components* of (X, G). It is known that if (X, G) is a finite space of orders and $X_1, \ldots, X_n$ are its connected components, then $(X, G) = (X_1, G|_{X_1}) \sqcup \ldots \sqcup (X_n, G|_{X_n})$. Moreover, the spaces $(X_i, G|_{X_i})$, are either one element spaces or proper group extensions.

To simplify things we assume from now on that the space of orderings (X, G) contains no infinite fans. This is the case, for example, if the stability index of (X, G) is finite. For $\delta \in \chi(G)$ we denote $X_\delta := \{\sigma \in X : \sigma\delta \in X\} = X \cap \delta X$. It is possible to show that since (X, G) has no infinite fans, every connected component of (X, G) is either singleton or has the form X_δ for some $\delta \in \chi(G)$, $\delta \neq 1$, $|X_\delta| \geq 4$.

The requirement of Theorem 6.5 that $\gamma \in X^4$ can be substantially refined as follows:

Theorem 6.7. *(**[SO3]**, Theorem 2.8) A necessary condition for (X_0, G_0) to be a quotient of (X, G) is that $\gamma = \prod_{i=1}^k \sigma_i$, $\sigma_i \in X$, $k = 2$ or $k = 4$ and $\gamma \notin X^2$, and, in the case where not all σ_i are in the same connected component of (X, G) and the connected components of the σ_i in (X, G) are not all singleton, either $k = 2$ and exactly one of the connected components of the σ_i is not singleton, or $k = 4$, $\gamma \notin X^2$ and, after reindexing suitably, the connected component of σ_3 and σ_4 is $X_{\sigma_3\sigma_4}$ and either the connected component of σ_1 and σ_2 is $X_{\sigma_1\sigma_2}$ or the connected component of σ_i is singleton for $i = 1, 2$.*

It is natural to wonder if the necessary conditions of Theorem 6.7 are sufficient when (X, G) has stability index two. We are unable to prove this in general. We are however able to prove the following:

***Theorem* 6.8.** *([SO3], Theorem 2.9) If (X, G) has stability index two and just finitely many non-singleton connected components, then the necessary conditions of Theorem 6.7 are sufficient.*

In fact, a much more surprising result can be established:

Theorem 6.9. *([SO3], Theorem 3.3) The following are equivalent:*

1. *(X_0, G_0) is a quotient space of (X, G).*

2. *γ satisfies the necessary conditions of Theorem 6.7.*

3. *(X_0, G_0) is a profinite space of orderings.*

For the remaining part of the paper, general quotients are considered. The first obtained result is the following one:

Theorem 6.10. *([SO3], Theorem 5.1) A necessary condition for the quotient structure (X_0, G_0) of (X, G) to be a space of orderings is that S generates $\chi(G/G_0)$ as a topological group, i.e., $\chi(G/G_0)$ is the closure of the subgroup of $\chi(G/G_0)$ generated by S, i.e., $S^{\perp} = G_0$.*

For each $\gamma \in S$, $\gamma \neq 1$, γ has some (not necessarily unique) minimal expression $\gamma = \prod_{i=1}^{k} \sigma_i$, $\sigma_i \in X$, $k = 2$ or 4. Denote by $(Y, G/\Delta) = (Y, G/Y^{\perp})$ the subspace of (X, G) generated by the connected components of the various σ_i, $i = 1, \ldots, k$, γ running through $S \setminus \{1\}$, and let Y_0 denote the set of restrictions of elements of Y to G_0.

***Theorem* 6.11.** *([SO3], Theorem 5.2) A necessary condition for the quotient structure (X_0, G_0) of (X, G) to be a space of orderings is that S generates $\chi(G/G_0)$ as a topological group and the quotient structure $(Y_0, G_0/\Delta)$ of $(Y, G/\Delta)$ is a space of orderings.*

The subspace $(Y, G/\Delta)$ of (X, G) defined above will be referred to as the *core* of the space of orderings (X, G) with respect to the quotient structure (X_0, G_0). Again it is natural to wonder if the necessary conditions for a quotient structure to be a quotient space given by Theorem 6.10 are sufficient. Although we are unable to prove this, we are able to show it is true in certain cases.

***Theorem* 6.12.** *([SO3], Theorem 5.4) For a space of orderings (X, G) with finitely many non-singleton connected components and no infinite fans and a quotient structure (X_0, G_0) of (X, G) of finite index, the following are equivalent:*

1. *(X_0, G_0) is a space of orderings.*

2. *$X^4 \cap \chi(G/G_0)$ generates $\chi(G/G_0)$ and the quotient structure $(Y_0, G_0/\Delta)$ of the core $(Y, G/\Delta)$ is a space of orderings.*

Finally, in the special case of the space of orderings of the field $\mathbb{Q}(x)$, the above theorem can be substantially strengthened:

Theorem 6.13. *([SO3], Theorem 5.5) For the space of orderings $(X, G) = (X_{\sum \mathbb{Q}(x)^2}, G_{\sum \mathbb{Q}(x)^2})$ and a quotient structure (X_0, G_0) of (X, G) of finite index, the following are equivalent:*

1. *(X_0, G_0) is a space of orderings.*

2. *$X^4 \cap \chi(G/G_0)$ generates $\chi(G/G_0)$ and the quotient structure $(Y_0, G/\Delta)$ of the core $(Y, G/\Delta)$ is a space of orderings.*

3. (X_0, G_0) *is a profinite space of orderings.*

We thus obtain a surprisingly spectacular refinement of the main result of **[SO2]** by a completely different technique.

7 Symbol length.

7.1 The paper [SL1].

Let F be a field of characteristic $\neq 2$, let $n \in \mathbb{N}$, and let $k_n(F)$ denote the Milnor K-group $K_n(F)$ modulo 2. $k_n(F)$ is additivel generated by "symbols" $\{a_1, ..., a_n\}$, $a_i \in F^\times$, and by the Milnor conjecture it is known that $k_n(F)$ is isomorphic via cannonical homomorphism to $I^n(F)/I^{n+1}(F)$ and to the n-th Galois cohomology group of F with coefficients in $\mathbb{Z}/2\mathbb{Z}$, $H^n(F, \mathbb{Z}/2\mathbb{Z})$.

The *n-symbol length* $\lambda_n(F)$ is the smallest nonnegative integer m such that each element of $k_n(F)$ can be written as a sum of m symbols, or ∞ if no such integer exists. The purpose of this paper was relating $\operatorname{stab}(X_{\sum F^2}, G_{\sum F^2})$ with $\lambda_n(F)$. The main result is the following:

Theorem 7.1. *(**[SL1]**, Theorem 3.6) Let F be a Pythagorean field and $s = \operatorname{stab}(X_{\sum F^2}, G_{\sum F^2}) < \infty$.*

1. *If $s \leq 2$, then either $\lambda_2(F) = s$, or F is uniquely ordered, $s = 0$, and $\lambda_2(F) = 1$.*

2. *If $s \geq 3$, then $\frac{s+1}{2} \leq \lambda_2(F) \leq 2^{s-1}(2^{s-2} - 1)$.*

8 Fixed point theorems.

8.1 The paper [FP1].

Kuhlmann and Kuhlmann have recently proven [35] a new type of fixed point theorems, that they refer to as the fixed point theorems for ball spaces. Their results are stated in a very general language of spaces which are neither metric nor even topological, but instead utilize the notion of so called balls, that is an arbitrary family of some subsets. Let X be a nonempty set. A *ball space* is a pair $(X, \mathcal{B})$, where $\mathcal{B}$ is a fixed family of nonempty subsets of X called *balls*. A *nest of balls* is any nonempty chain $\mathcal{N} \subset \mathcal{B}$ ordered by inclusion. If $f: X \to X$ is a mapping, a ball $B \in \mathcal{B}$ is called *f-contracting* if it is either a singleton consisting of a fixed point of f, or if $f(B) \subsetneq B$. The results in [35] are the following ones:

Theorem 8.1. *If $(X, \mathcal{B})$ is a ball space and $f: X \to X$ a mapping such that:*

1. *there exists an f-contracting ball,*

2. *$f(B)$ contains an f-contracting ball, if $B \in \mathcal{B}$ is an f-contracting ball,*

3. *for every nest of f-contracting balls $\mathcal{N}$, the intersection $\bigcap \mathcal{N}$ contains an f-contracting ball,*

then f has a fixed point.

Theorem 8.2. *If $(X, \mathcal{B})$ is a ball space and $f: X \to X$ a mapping such that:*

1. *X is an f-contracting ball,*

2. *$f(B)$ is an f-contracting ball, if $B \in \mathcal{B}$ is an f-contracting ball,*

3. for every nest of f-contracting balls $\mathcal{N}$, the intersection $\bigcap \mathcal{N}$ is an f-contracting ball,

then f has a unique fixed point.

The main results of the miniature note **[FP1]** by the author are two theorems (**[FP1]**, Theorems 4 and 6) showing that Theorems 8.1 and 8.2 can be also deduced from Bourbaki-Witt fixed point theorems.

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